Hyperhomological Algebra

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Abstract. Examples from ring theory will show the power of classical homological algebra ($\mathbf{H}^1\mathbf{A}$). Hyperhomological algebra ($\mathbf{H}^2\mathbf{A}$) will be presented and its strength will be demonstrated by several examples.

Contents

$\mathbf{H}^0\mathbf{A}$: **Rings** (one lecture). Noetherian rings, Krull dimension, height of ideals, Krull's Principal Ideal Theorem and generalizations, regular local rings, Krull's conjectures: Regularity is stable under localization and implies unique factorization. Gorenstein local rings; do they localize? Depth of rings via regular sequences, Cohen–Macaulay rings; do they localize? Hierarchy: Regularity implies Gorensteinness, which implies Cohen–Macaulayness.

$\mathbf{H}^1\mathbf{A}$: **Homological Algebra** (two lectures). Modules, homomorphism functor, tensor product. Exactness of sequences, functors, projective modules, injective modules, flat modules, homological dimensions. Derived functors, Ext, Tor, homological dimensions revisited. Solutions/answers to the conjectures/questions in $\mathbf{H}^0\mathbf{A}$. Auslander's G–dimension. Bass' conjecture, Kaplansky's conjecture, Serre's intersection theorem, the intersection conjectures. The New Intersection Theorem. Section functors, local cohomology.

$\mathbf{H}^2\mathbf{A}$: **Hyperhomological Algebra** (six lectures). Complexes of modules and their morphisms, modules as complexes, quasi-isomorphisms. Homomorphism-complex of two complexes, tensor product of two complexes, canonical morphisms, canonical isomorphisms. The derived category, semi-projectivity, semi-injectivity, semi-flatness, derived functors, **R**Hom functor, **L**tensor functor, canonical morphisms and isomorphisms revisited. Section functors revisited, completion functors. Solutions to the homological conjectures in $\mathbf{H}^1\mathbf{A}$. Extension of Auslander's G–dimension, Auslander and Bass categories, their equivalence, their relations to G–dimensions. Local homomorphisms. Examples over non-commutative rings.

It turns out that $\mathbf{H}^2\mathbf{A}$ is a much stronger tool than $\mathbf{H}^1\mathbf{A}$ in the sense that it yields *stronger results* and/or *easier proofs*. This is because it deals with a larger class of objects: complexes of modules instead of merely modules (which are just complexes concentrated in degree zero). This can explained as follows.

$\mathbf{H}^1\mathbf{A}$ has *three* steps: (1) Take resolution. (2) Apply functor. (3) Take homology.
$\mathbf{H}^2\mathbf{A}$ has *only two:* (1) Take resolution. (2) Apply functor.
Important information is lost in step (3)!

$\mathbf{H}^3\mathbf{A}$: **Differential Graded Homological Algebra** (one lecture). Differential Graded algebras. Koszul algebras, fiber algebras of local homomorphisms. Differential Graded modules.