Extensions, relatively projective modules and an approach to finitistic dimension conjecture

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Outline

I. The finitistic dimension conjecture
II. A little bit background
III. Interest in the conjecture
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V. Approach by extensions
VI. Approach by relative global dimensions

Notations

R:	commutative Artin ring,
A:	Artin algebra over R, (special examples: finite-dimensional algebras over fields);
A-mod :	category of all finitely generated left A-modules ;
pd(M) :	projective dimension of M $\in$ A-mod;
I$\trianglelefteq_l$ A:	I is a left ideal of A;
add(M) :	dir. summands of dir. sums of f. m. copies of M

$\mathscr{P}^{\infty}(A)$: $= \{$M $\in$ A-mod $\mid$ pd(M) $< \infty\}$;

The finitistic dimension of A is defined by

$$\text{fin.dim}(A) := \sup\{\text{pd}(M) \mid M \text{ in } \mathscr{P}^{\infty}(A)\}.$$

The finitistic dimension conjecture says:

A: Artin algebra over R $\Longrightarrow$ fin.dim(A)$< \infty$.

This is a conjecture of over 50 years old. In 1960, H.Bass wrote this conjecture as a question of Rosenberg and Zelinsky in one of his papers.

Note: The finite ring $\mathbb{Z}/(4)$ is an Artin algebra, but it is *not* a finite-dimensional algebra over any field.

The first homological result might be the Hilbert's syzygy theorem (1890):

$$\text{gl.dim } \mathsf{k}[x_1, ..., x_n] = n.$$

Thus any module over $\mathsf{k}[x_1, ..., x_n]$ can be resolved as an long exact sequence of length at most n of free modules.

A little bit background

In 1940's, homological algebra which stemmed from algebraic topology became popular and was widely applied to the study of rings and algebras. Since then, there have been many people who exploit homological methods to investigate algebras and representations. For example, H.Cartan, S.Eilenberg, S.MacLane, M.Auslander, D.Buchsbaum, M.Nagata, T.Nakayama, ... Also, one may discern a little flavor of abstract algebra and homological algebra in that years from a statement of Hermann Weyl(1885-1955):

" In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics. "

A little bit background

In algebraic geometry, the polynomial ring k$[x_1, ..., x_n]$ and its factor rings are some basic elements . The nice relationship between geometry and homological algebra can be seen from one beautiful result of Auslander-Buchsbaum-Serre in 1955.

Auslander-Buchsbaum-Serre theorem(1955):
V: algebraic variety over a field $k = \bar{k}$,
A: coordinate ring of V. Then:
V is smooth $\iff$ gl.dim$(A) < \infty$.

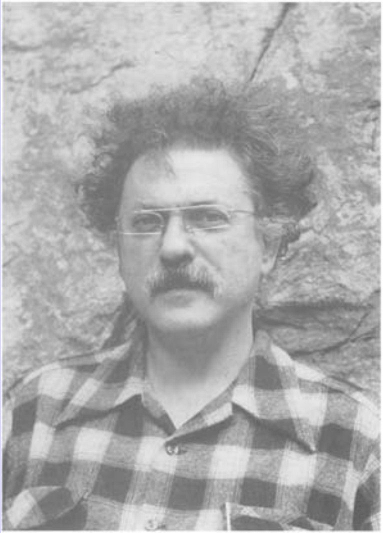
On the other hand, one often faces geometric objects whose coordinate rings have infinite global dimension. To investigate algebras and modules with infinity dimension, the finitistic dimension was then introduced. Of course, a major task here is to understand the finitistic dimension.

Interests in the conjecture

The conjecture attracts many people in the areas of representation theory of algebras, and of homological algebra.

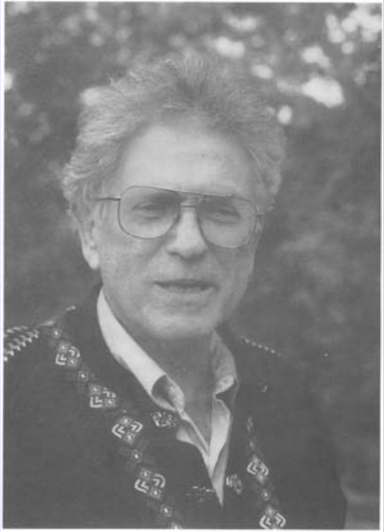
One of them is Maurice Auslander(1926-1994):

Interests in the conjecture



MAURICE AUSLANDER
August 3, 1926–November 18, 1994

Interests in the conjecture



(Photograph courtesy of Gordana Todorov.)

MAURICE AUSLANDER
August 3, 1926–November 18, 1994

He was "one of the founders of the modern aspects of the representation theory of artin algebras ". And

"one of his main interests in the theory of artin algebras was the finitistic dimension conjecture and related homological conjectures" .

" Shortly before his death Auslander expressed that he was sorry not to live to see the solution of the finitistic dimension conjecture".

Note: Maurice Auslander was a special invited lecturer at International Congress of Mathematicians in 1962 and in 1986. All the above quotations are from p.501 and p.815 of "Selected works of Maurice Auslander" edited by I.Reiten, S.Smalo and O.Solberg, 1999.

Connection to other conjectures

The finitistic dimension conjecture is closely relevant to at least 6 other famous conjectures in representation theory of Artin algebras.

Connection to other conjectures

— **Nakayama conjecture**: If all I_j in a minimal injective resolution of an Artin algebra A, say $0 \to {}_AA \to I_0 \to I_1 \to \ldots$, are projective, then A is self-injective. (1958, T. Nakayama)

— **Generalized Nakayama conjecture**: If $0 \to {}_AA \to I_0 \to I_1 \to \ldots$ is a minimal injective resolution of an Artin algebra A, then any indecomposable injective is a direct summand of some I_j. Equivalently, if M is in A-mod such that add$(A) \subseteq$ add(M) and $\text{Ext}^i_A(M,M) = 0$ for all $i \geq 1$, then M is projective. (1975, Auslander-Reiten)

— **Strong Nakayama conjecture**: If M is a non-zero module over an Artin algebra A, then there is an integer $n \geq 0$ such that $\text{Ext}^n_A(M,A) \neq 0$. (1990, Colpi-Fuller)

—**Gorenstein symmetry conjecture**: If the injective dimension of ${}_AA$ is finite, then the injective dimension of A_A is finite.

— **Tilting complement conjecture**: Every almost tilting module has only finitely many non-isomorphic indecomposable tilting complements.

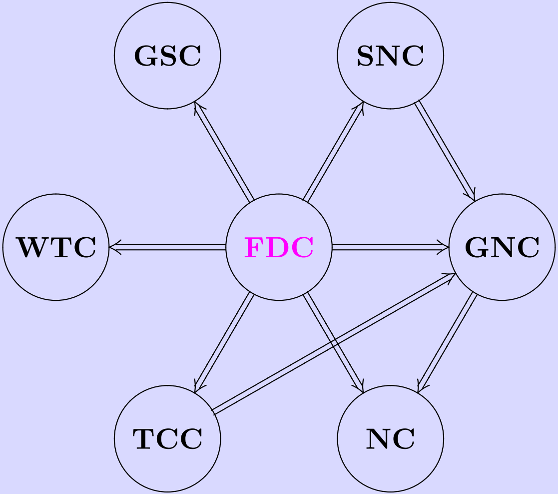
— **Wakamatsu tilting conjecture**: Let T be an A-module with $\mathrm{Ext}_A^i(T,T) = 0$ for all i such that there is an exact sequence

$$(*) \qquad 0 \to {}_AA \to T_0 \to \cdots \to T_n \to \cdots$$

with $T_i \in \mathrm{add}(T)$ and $\mathrm{Hom}_A(*, T)$ is exact. If $\mathrm{pd}(T) < \infty$, then T is a tilting module.

Connection to other conjectures

Note: All these conjectures are open.



Thus, the finitistic dimension possesses a strong homological property and can be far more revealing measures of homological complexity of an algebra at hand, while infinite global dimension often does not reveal much about that complexity.

Some known results

Now, we recall some history of the developments of the finitistic dimension conjecture. Let A and B be Artin algebras.

— 1965: H.Mochizuki:
$\mathrm{rad}^2(A)=0 \Longrightarrow$ fin.dim.conj. is true for A.

— 1991: E.Green & Zimmermann-Huisgen:
$\mathrm{rad}^3(A)=0 \Longrightarrow$ fin.dim.conj. is true for A.

— 1991: E.Green, Kirkman, Kuzmanovich:
A is monomial$\Longrightarrow$fin.dim.conj. is true for A.

An algebra A , given by a quiver with relations, is called a monomial algebra if the relations consist only of paths of length at least two.

Some known results

— 1991: Auslander and Reiten:

$\mathscr{P}^{\infty}$(A) is contravariantly finite in A-mod $\Rightarrow$fin.dim.conj. is true for A.

A subcategory $\mathcal{C}$ of A-mod is called *contravariantly finite* in A-mod if for any module M in A-mod there is an approximation f: C$\longrightarrow M$ with C$\in \mathcal{C}$.

Note: $\mathscr{P}^{\infty}$(A) is not always contravariantly finite in A-mod.

— 1994: Y.Wang: rad^{2l+1}(A)$=$0 and A/rad^{l}(A) is rep-finite $\Longrightarrow$ fin.dim.conj. is true for A.

Some known results

— 2000: Agoston,Happel,Lukas and Unger:
A is standardly stratified $\Longrightarrow$ fin.dim.conj. is true for A.

— 2002-2005: Igusa and Todorov:
rep.dim(A)$\leq$3$\Longrightarrow$ fin.dim.conj.is true for A.

The *representation dimension* of A is defined by Auslander:

$$\text{rep.dim}(A)$$

$$= \inf\{\text{gl.dim}\ (\text{End}_A(M)) \mid M \in A\text{-mod},\ A \oplus DA \in \text{add}(M)\}.$$

Note: The representation dimensions of Artin algebras can be any numbers.

Some known results

— 2002: X.:
A is stably hereditary$\Rightarrow$rep.dim(A)$\leq$ 3.

An Artin algebra is called *stably hereditary* if (1) each indecomposable submodule of an indecomposable projective module is either projective or simple, and (2) each indecomposable factor module of an indecomposable injective module is either injective or simple.

— 2003: Hongbo Shi:
For monomial algebras, a graphic algorithm to calculate the finitistic dimension was given.

Some known results

— 2004: Erdmann,Holm,Iyama & Schroeer:
fin.dim.conj. is true for special biserial algebras and string algebras.

— 2009: Huard, Lanzilotta & Mendoza:
A: algebra with radical layer of inf. proj. dimension. If the layer $\leq 3 \Longrightarrow$ fin.dim(A)$< \infty$.

Of course, there are also some other results: Model structure approach, geometric approach, approach by infinitely generated tilting modules, ... I am unable to give a full list!

5.1 General question

The most investigations, mentioned above, on the finitstic dimension conjecture are mainly concentrated on one single algebra. Our philosophy is: to bound the finitistic dimension by studying a chain of algebras. This is motivated by the following fact.

Proposition. For each f. d. algebra A over a field, there is a chain of algebras: $\mathrm{A}=\mathrm{A}_0 \subseteq \mathrm{A}_1 \subseteq \ldots \subseteq \mathrm{A}_s$ such that A_s is rep-finite, and that $\mathrm{rad}(\mathrm{A}_i)$ is left ideal in A_{i+1} for all i.

Approach by extensions of algebras

So, our question is generally as follows:

Assumption: There is given a chain of algebras $A = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_s$ such that $\mathrm{rad}(A_i)$ is a left (or an) ideal in A_{i+1} for all i.

Question: If some of the bigger algebras in the chain have **finite** finitistic dimension, what could we say about the finiteness of the finitistic dimension of the smallest algebra A_0 ?

Important: If the answer is Yes, then fin.dim.conj. holds by the above proposition.

Approach by extensions of algebras

In many cases, given an extension B⊆A, the algebra A may have simpler homological properties or representation theory than B does. So, it make sense to control the small algebra B by using the big algebra A.

Two examples:

1) Any f. d. algebra is a subalgebra of some full matrix algebra.

2) If an algebra, given by a quiver and with relations, contains at least three arrows, then it contains a representation-wild subalgebra.

5.2 Representation distance

Now, let us introduce the notion of representation distances of Artin algebras.

Definition. Given an Artin algebra A, we define the **left** representation distance of A, denoted by lrd(A), to be the following number:

$$\mathrm{lrd(A)} = \inf\{\, s \mid \exists \text{ chain of algebras } \mathrm{A}_0{=}\mathrm{A} \subseteq \mathrm{A}_1 \subseteq \ldots \subseteq \mathrm{A}_s,\ \text{rad }(\mathrm{A}_i) \text{ is a left ideal in } \mathrm{A}_{i+1} \text{ for all } \text{and } \mathrm{A}_s \text{ is representation-finite}\}.$$

Approach by extensions of algebras

The representation distances have the following properties.

- The left representation distance is invariant under Morita equivalences.
- Any finite-dimensional algebra over a field has finite left representation distance.

5.3 Some results on extensions

Theorem 1. A: Artin algebra. If lrd(A)$\leq$ 2, then fin.dim(A)$< \infty$.

That is, if $C \subseteq B \subseteq A$ is a chain of algebras with the same 1 such that rad(C) is a left ideal in B and rad(B) is a left ideal in A and if A is representation-finite, then fin.dim$(C) < \infty$.

Approach by extensions of algebras

Theorem 2. B$\subseteq$ A: extension of Artin algebras, rad(B)$\trianglelefteq_l$A and rad(A)$=$ rad(B)A.

If gl.dim(A)$\leq$ 4, then fin.dim(B)$< \infty$.

Approach by extensions of algebras

One of the ingredients of the proofs is the Igusa-Todorov function [Igusa-Todorov, 2002-2005]:

$\exists$ function Ψ: A-mod$\longrightarrow \mathbb{N}$, such that

$0 \to X \to Y \to Z \to 0$ exact in A-mod, pd$(Z) < \infty$

$\Longrightarrow$

pd$(Z) \leq \Psi(X \oplus Y) + 1$.

Approach by relatively global dimensions

Idea: To approach finitistic dimension conjecture by the relatively homological global dimension.

$B \subseteq A$: extension of algebras

${}_AX$ is relatively projective if ${}_AX$ is dir. summand of $A \otimes_B X$.

Using the class of rel. proj. modules, one defines rel. proj. dimension of A-modules, and the relative global dimension of the extensions, denoted by rel.pd$({}_AX)$, gl.dim(A, B), respectively.

Approach by relatively global dimensions

Basic facts:

1. fin.dim$\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \leq$ fin.dim$(A)+$ fin.dim$(B)+1$.
2. If B$\subseteq$ A with rad(B)$\trianglelefteq_l$A and rad(A)=rad(B)A, then gl.dim(A,B)$\leq$ 1.
3. Every algebra B over perfect field can be embedded in A such that rad(B)$\trianglelefteq_l$A and rad(A)=rad(B)A.

   $B = S \oplus rad(B)$, $rad^{n+1}(B) = 0$, define

$$B \longrightarrow \begin{pmatrix} S & 0 \\ rad(B) & B/rad^n(B) \end{pmatrix}$$

$$b = s + x \mapsto \begin{pmatrix} s & 0 \\ x & \overline{b} \end{pmatrix}$$

Approach by relatively global dimensions

Over a perfect field k, the following are equivalent:

(1) Every f. d. k-algebra has finite finitistic dimension.

(2) Suppose B$\subseteq$ A is an extension of k-algebras with rad(B)$\trianglelefteq_l$A and fin.dim(A)$< \infty$. If gl.dim(A,B)≤ 1, then fin.dim(B)$< \infty$.

gl.dim(A,B)=0

Approach by relatively global dimensions

Over a perfect field k, the following are equivalent:

(1) Every f. d. k-algebra has finite finitistic dimension.

(2) Suppose B$\subseteq$ A is an extension of k-algebras with rad(B)$\trianglelefteq_l$A and fin.dim(A)$< \infty$. If gl.dim(A,B)≤ 1, then fin.dim(B)$< \infty$.

gl.dim(A,B)$=0$

Approach by relatively global dimensions

Theorem

(D.M.Xu +X. 2013)

$B \subseteq A$: extension of Artin algebras, rad(B)$\trianglelefteq_l$ A.
The category of relatively projective A-modules is closed under taking A-syzygies.
$\Longrightarrow$
fin.dim(B)$\leq$ fin.dim(A) + fin.dim($_BA$)+ 3.

fin.dim$({}_BA) = max\{$pd$({}_BX) \mid$
X is a direct summand of ${}_BA$ with pd$({}_BX) < \infty\}$.

The proof of this result does not use Igusa-Todorov function.

Only properties of relatively projective modules are used.

Approach by relatively global dimensions

In particular, if gl.dim(A,B)=0, that is, the extension is semisimple, then (2) holds.

So the next step is to show that (2) holds for gl.dim(A,B)=1. Now we are working on this problem.

gl.dim $= 0$	semisimple algebras
gl.dim $= 1$	hereditary algebras
rel.gl.dim $= 0$	semisimple extensions
rel.gl.dim $= 1$	???

Theorem

(H.X.Chen + X.)
R_1, R_2 *and* R_3 : *rings,* $\exists$ *recollement:*

$$\mathscr{D}(R_1) \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow[\;i^!\;]{} \end{array} \mathscr{D}(R_2) \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^!\;} \\ \xleftarrow[\;j_*\;]{} \end{array} \mathscr{D}(R_3)$$

$\Longrightarrow$

(1) *Suppose that* $j_!$ *restricts to a functor* $\mathscr{D}^b(R_3) \to \mathscr{D}^b(R_2)$. *If* $\mathrm{fin.dim}(R_2) < \infty$, *then* $\mathrm{fin.dim}(R_3) < \infty$.

(2) *Suppose that* $i_*(R_1)$ *is compact in* $\mathscr{D}(R_2)$. *Then*

(a) *If* $\mathrm{fin.dim}(R_2) < \infty$, *then* $\mathrm{fin.dim}(R_1) < \infty$.

(b) *If* $\mathrm{fin.dim}(R_1) < \infty$ *and* $\mathrm{fin.dim}(R_3) < \infty$, *then* $\mathrm{fin.dim}(R_2) < \infty$.