Strong convergence of propagation of chaos for McKean-Vlasov SDEs with singular interactions

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(This is a joint work with Zimo Hao and Michael Röckner.)

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Introduction: Kac's Chaos

- Let E be a Polish space and $\mu \in \mathcal{P}(E)$ a probability measure on E.
- Let $(\mu^N)_{N\in\mathbb{N}}$ be a sequence of symmetric probability measures on the product space E^N, where symmetric means that for any permutation $(x_{i_1}, \cdots, x_{i_N})$ of $(x_1, \cdots, x_N)$,

$$\mu^N(\mathrm{d}x_{i_1}, \cdots, \mathrm{d}x_{i_N}) = \mu^N(\mathrm{d}x_1, \cdots, x_N).$$

- One says that $(\mu^N)_{N\in\mathbb{N}}$ is μ-chaotic if for any $k \in \mathbb{N}$,

$$\mu^{N,k} \text{ weakly converges to } \mu^{\otimes k} \text{ as } k \leqslant N \to \infty, \tag{1.1}$$

where $\mu^{N,k}(\mathrm{d}x_1, \cdots, \mathrm{d}x_k) = \mu^N(\mathrm{d}x_1, \cdots, \mathrm{d}x_k, E, \cdots, E)$ is the k-fold marginal distribution of μ^N. Note that (1.1) holds if and only if (1.1) holds for only $k = 2$.

Introduction: Kac's Chaos

- Let $\xi^N := (\xi^{N,1}, \cdots, \xi^{N,N})$ be a family of E-valued random variables. If the law of ξ^N is symmetric and μ-chaotic, one says that ξ^N is μ-chaotic. It is also equivalent to a law of large numbers

$$\eta_{\xi^N}(\mathrm{d}y) := \tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{\xi^{N,j}}(\mathrm{d}y) \in \mathcal{P}(E) \text{ converges to } \mu \text{ in law.} \quad (1.2)$$

More precisely, for any $\varphi \in C_b(E)$,

$$\eta_{\xi^N}(\varphi) := \frac{1}{N}\sum_{j=1}^N \varphi(\xi^{N,j}) \to \mu(\varphi) := \int_E \varphi(x)\mu(\mathrm{d}x), \quad \text{in law.}$$

- ξ^N can be regarded as N-random particles in state space E. Kac's chaos means that if one observes the distribution of any k-particles, then they become statistically independent as N goes to infinity.

- Hauray M. and Mischler, S.: On Kac's chaos and related problems. JFA 266 (2014), no. 10, 6055-6157.
- Sznitman A.S.: Topics in propagation of chaos. LNM, 1464, 1991.

Introduction: Propagation of chaos

- Let $(\boldsymbol{\xi}_t^N)_{t\geqslant 0} := (\xi_t^{N,1}, \cdots, \xi_t^{N,N})_{t\geqslant 0}$ be a family of E^N-valued continuous stochastic processes, which can be thought of as the evolution of N-particles.
- Let $(\xi_t)_{t\geqslant 0}$ be a limit E-valued continuous stochastic process defined on the same probability space.
- Let μ_t^N be the law of $\boldsymbol{\xi}_t^N$ in E^N and μ_t be the law of ξ_t in E.
- Suppose that μ_0^N is μ_0-chaotic at time 0. One says that propagation of chaos holds if for any time $t > 0$, μ_t^N is μ_t-chaotic.
- This is a basic assumption in the derivation of Bolzmann equations. Appears also in data science, mean-field games and the training of neural networks.

Introduction: Propagation of chaos

- As the evolution of particle distributions, the probability measures μ_t^N and μ_t satisfy Fokker-Planck-Kolmogorov equation in the weak sense. Therefore, it can be studied by purely PDE's method.
- As stochastic processes, one would like to ask the following stronger convergence in a probabilistic sense: for each $t > 0$,

$$\lim_{N\to\infty} \mathbb{E}|\xi_t^{N,1} - \xi_t| = 0,$$

or in the functional path sense

$$\lim_{N\to\infty} \mathbb{E}\left(\sup_{s\in[0,t]} |\xi_s^{N,1} - \xi_s|\right) = 0. \tag{1.3}$$

- (Rate of Convergence) For any $T > 0$ and some C_T and $\gamma > 0$,

$$\mathbb{E}\left(\sup_{s\in[0,T]} |\xi_s^{N,1} - \xi_s|\right) \leqslant C_T N^{-\gamma}.$$

Introduction: Interaction particle system

- Let $\phi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m$, $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ be Borel measurable functions.
- For a (sub)-probability measure μ over $\mathbb{R}^d$, we define

$$b(t, x, \mu) := F(t, x, (\phi_t \circledast \mu)(x)),$$

where $\phi_t(x, y) := \phi(t, x, y)$ and

$$(\phi_t \circledast \mu)(x) := \int_{\mathbb{R}^d} \phi_t(x, y)\mu(\mathrm{d}y).$$

- Let $\mathbf{X}_t^N := (X_t^{N,1}, \cdots, X_t^{N,N})$ be the evolution of N-particles governed by stochastic system associated with above b and σ.

Introduction: Interaction particle system

- Consider the following interacting system of N-particles,

$$\mathrm{d}X_t^{N,i} = b\Big(t, X_t^{N,i}, \eta_{\mathbf{X}_t^N}\Big)\mathrm{d}t + \sigma\Big(t, X_t^{N,i}\Big)\mathrm{d}W_t^i, \quad i = 1, \cdots, N, \qquad (1.4)$$

  where $\eta_{\mathbf{X}_t^N}$ stands for the empirical distribution measure of $\mathbf{X}_t^N$, and $\{W^i, i \in N\}$ is a sequence of independent standard Brownian motions on some stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geqslant 0})$.

- The infinitesimal generator of the above system is given by

$$\mathcal{L}_t^N \varphi(\mathbf{x}) = \mathrm{tr}\big(a(t, x^i) \cdot \nabla^2_{x^i}\varphi(\mathbf{x})\big) + F\Big(t, x^i, \frac{1}{N}\sum_{j=1}^N \phi_t(x^i, x^j)\Big) \cdot \nabla_{x^i}\varphi(\mathbf{x}),$$

  where $\mathbf{x} = (x^1, \cdots, x^N) \in (\mathbb{R}^d)^N$ and $a = \frac{1}{2}\sigma\sigma^*$. Here and below we use Einstein's convention for summation.

Introduction: McKean-Vlasov equation

- Formally, it is expected that $\eta_{\mathbf{X}_t^N}$ weakly converges to the distribution μ_{X_t} of the solution to the following distribution-dependent (or McKean-Vlasov) SDE (abbreviated as DDSDE) when $N \to \infty$:

$$\mathrm{d}X_t = b(t, X_t, \mu_{X_t})\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^1. \tag{1.5}$$

- $\mu := (\mu_{X_t})_{t\geqslant 0}$ solves the following nonlinear Fokker-Planck equation in distributional sense:

$$\partial_t \mu = \partial_i \partial_j (a_{ij}\mu) + \mathrm{div}(b(\mu)\mu).$$

- When b is Lipschitz continuous in x and μ, McKean(1986) and Sznitman(1991) showed the following result:

$$\mathbb{E}\left(\sup_{s\in[0,T]} |X_s^{N,1} - X_s|^2\right) \leqslant \frac{C(b, \sigma, T)}{N}. \tag{1.6}$$

Beyond Lipschitz

- Consider the following interaction particle system:

$$\mathrm{d}X_t^{N,i} = \frac{1}{N}\sum_{j\neq i} K\left(X_t^{N,i} - X_t^{N,j}\right)\mathrm{d}t + \nu \mathrm{d}W_t^i,$$

  where $K(x)$ is the interaction kernel.

- Osada (1987) firstly showed the propagation of chaos for the point vortices associated with the 2d Navier-Stokes equation with large viscosity (singular interaction kernel like $K(x) = x^\perp/|x|^2$).

- Fournier, Hauray and Mischler (2014) dropped the assumption of large viscosity by the classical martingale method.

- Osada H.: Propagation of chaos for the two dimensional Navier-Stokes equation. *Probabilistic Methods in Mathematical Physics (Katata Kyoto, 1985)*, 303-334, *Academic Press, Boston, MA*, 1987.
- Fournier, N., Hauray, M. and Mischler, S.: Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* 16 (2014), no. 7, 1423-1466.

Beyond Lipschitz

- Jabin and Wang (2018) obtained a first quantitative convergence rate about the relative entropy between the law of particle system and the tensorized limit law with kernels $K \in W^{-1,\infty}$ and $K(x) = -K(-x)$.
- Serfaty (2020) showed the propagation of chaos for first order system with Coulomb potential or a super-Coulombic Riesz potential with $|K(x)| \leqslant C|x|^{-\alpha}$, where $d-2 < \alpha < d$ and $d \geqslant 2$.
- Bao and Huang (2021) obtained the rate of the propagation of chaos for Hölder $K(x)$ by Zvonkin's transformation.
- Lacker D. (2021) showed the optimal rate of the propagation of chaos for bounded $K(x)$ by BBGKY method.

- Bao J. and Huang X.: Approximations of Mckean-Vlasov Stochastic Differential Equations with irregular coefficients. *J. Theoret. Probab.* (2021): 1-29.
- Jabin P.-E. and Wang Z.: Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.* 214 (2018), no. 1, 523-591.
- Serfaty S.: Mean field limit for Coulomb-type flows. *Duke Math. Journal*, Vol. 169, No. 15, 2887-2935(2020).
- Lacker D.: Hierarchies, entropy, and quantitative propagation of chaos for mean field diffusions. Available in arXiv:2105.02983.

Beyond Lipschitz

- Röckner and Zhang (2021) showed the strong well-posedness for DDSDE with $K \in L^p(\mathbb{R}^d)$ for some $p > d$.
- Tomašević (2020) uses the partial Girsanov transform to derive the propagation of chaos under $K \in L^p(\mathbb{R}^d)$ for some $p > d$ and the extra assumption that the set of discontinuous points of the interaction kernel has Lebesgue measure zero.
- Hoeksema, Holding, Maurelli and Tse (2020) showed a large deviation result for a particle system with L^p-singular interaction kernels.
- Liu, Wu, Zhang(2021, CMP), Wang, Zhao, Zhu(2021)...

- Röckner M. and Zhang X.: Well-posedness of distribution dependent SDEs with singular drifts. *Bernoulli* 27 (2021), no. 2, 1131-1158.
- Tomašević M.: Propagation of chaos for stochastic particle systems with singular mean-field interaction of L^q-L^p type. 2020. hal-03086253
- Hoeksema J., Holding T., Maurelli M., Tse O. : Large deviations for singularly interacting diffusions. Available at arXiv: 2002.01295.

Difficulties

- Consider the following SDE in $\mathbb{R}^{3d}$:

$$\begin{cases} \mathrm{d}X_t^1 = \Big[\phi\Big(X_t^1, X_t^2\Big) + \phi\Big(X_t^1, X_t^3\Big)\Big]\mathrm{d}t + \mathrm{d}W_t^1, \\ \mathrm{d}X_t^2 = \Big[\phi\Big(X_t^2, X_t^1\Big) + \phi\Big(X_t^2, X_t^3\Big)\Big]\mathrm{d}t + \mathrm{d}W_t^2, \\ \mathrm{d}X_t^3 = \Big[\phi\Big(X_t^3, X_t^1\Big) + \phi\Big(X_t^3, X_t^2\Big)\Big]\mathrm{d}t + \mathrm{d}W_t^3, \end{cases} \tag{1.7}$$

  where $|\phi(x,y)| \leqslant h(x-y)$ and $h \in L^p$ with $p > d$. For $i = 1,2,3$, let $\phi_i(x_1,x_2,x_3) := \sum_{j\neq i} \phi(x_i, x_j)$.
- As a function of (x_1,x_2,x_3) in $\mathbb{R}^{3d}$, one only has

$$\phi_i \in L^\infty_{x_i^*} L^p_{x_i} \subset L^p_x \text{ locally}, \quad i = 1,2,3, \tag{1.8}$$

  where x_i^* stands for the remaining variables except for x_i.
- It does not satisfy the conditions in Krylov-Röckner's work.

Main Results

- Let $d \in \mathbb{N}$. For a multi-index $\boldsymbol{p} = (p_1, \cdots, p_d) \in (0, \infty]^d$ and any permutation $\mathbf{x} \in \mathscr{X}$, the mixed $\mathbb{L}^{\boldsymbol{p}}_{\mathbf{x}}$-space is defined by

$$\|f\|_{\mathbb{L}^{\boldsymbol{p}}_{\mathbf{x}}} := \left[\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \cdots, x_d)|^{p_d} \mathrm{d}x_{i_d} \right]^{\frac{p_{d-1}}{p_d}} \cdots \mathrm{d}x_{i_2} \right]^{\frac{p_1}{p_2}} \mathrm{d}x_{i_1} \right]^{\frac{1}{p_1}}.$$

- When $\boldsymbol{p} = (p, \cdots, p) \in (0, \infty]^d$, the mixed $\mathbb{L}^{\boldsymbol{p}}_{\mathbf{x}}$-space is the usual $L^p(\mathbb{R}^d)$-space. Note that for general $\mathbf{x} \neq \mathbf{x}'$ and $\boldsymbol{p} \neq \boldsymbol{p}'$,

$$\mathbb{L}^{\boldsymbol{p}'}_{\mathbf{x}} \neq \mathbb{L}^{\boldsymbol{p}}_{\mathbf{x}} \neq \mathbb{L}^{\boldsymbol{p}}_{\mathbf{x}'}.$$

- For multi-indices $\boldsymbol{p}, \boldsymbol{q} \in (0, \infty]^d$, we denote

$$\frac{1}{\boldsymbol{p}} := \left(\frac{1}{p_1}, \cdots, \frac{1}{p_d}\right), \quad \left|\frac{1}{\boldsymbol{p}}\right| = \frac{1}{p_1} + \cdots + \frac{1}{p_d},$$

$$\boldsymbol{p} > \boldsymbol{q} \ \ (\text{resp. } \boldsymbol{p} \geqslant \boldsymbol{q};\ \boldsymbol{p} = \boldsymbol{q}) \iff p_i > q_i \ (\text{resp. } p_i \geqslant q_i;\ p_i = q_i).$$

- Let $\chi : \mathbb{R}^d \to [0,1]$ be a smooth cutoff function with $\chi|_{B_1} = 1$ and $\chi|_{B_2^c} = 0$. For fixed $r > 0$, we set

$$\chi_z^r(x) := \chi((x-z)/r), \quad x, z \in \mathbb{R}^d.$$

- For $\boldsymbol{p} \in [1,\infty]^d$, we introduce the following localized $L^{\boldsymbol{p}}$-space:

$$\widetilde{\mathbb{L}}_{\mathbf{x}}^{\boldsymbol{p}} := \Big\{ f \in L^1_{\rm loc}(\mathbb{R}^d), \|f\|_{\widetilde{\mathbb{L}}_{\mathbf{x}}^{\boldsymbol{p}}} := \sup_z \|\chi_z^r f\|_{\mathbb{L}_{\mathbf{x}}^{\boldsymbol{p}}} < \infty \Big\}.$$

- For a finite time interval $\mathrm{I} \subset \mathbb{R}$ and $q \in [1,\infty]$,

$$\widetilde{\mathbb{L}}_{\mathrm{I}}^q(\widetilde{\mathbb{L}}_{\mathbf{x}}^{\boldsymbol{p}}) := \Big\{ f \in L^1_{\rm loc}(\mathrm{I} \times \mathbb{R}^d), \|f\|_{\widetilde{\mathbb{L}}_{\mathrm{I}}^q(\widetilde{\mathbb{L}}_{\mathbf{x}}^{\boldsymbol{p}})} := \sup_z \|\chi_z^r f\|_{\mathbb{L}_{\mathrm{I}}^q(\mathbb{L}_{\mathbf{x}}^{\boldsymbol{p}})} < \infty \Big\},$$

  where for a Banach space $\mathbb{B}$ we set

$$\mathbb{L}_{\mathrm{I}}^q(\mathbb{B}) := L^q(\mathrm{I}; \mathbb{B}).$$

- We introduce the following index sets:

$$\mathscr{I}^o := \left\{(q, \boldsymbol{p}) \in (2, \infty)^{1+d} : |\tfrac{1}{\boldsymbol{p}}| + \tfrac{2}{q} < 1\right\}$$

and

$$\mathscr{X} := \left\{\mathbf{x} = (x_{i_1}, \cdots, x_{i_d}) : \text{any permutation of } (x_1, \cdots, x_d)\right\}.$$

$\mathbf{H}^\sigma$ There are $\kappa_0 \geqslant 1$, $\gamma_0 \in (0, 1]$ such that for all $t \geqslant 0$ and $x, x', \xi \in \mathbb{R}^d$,

$$\kappa_0^{-1}|\xi| \leqslant |\sigma(t,x)\xi| \leqslant \kappa_0|\xi|, \|\sigma(t,x) - \sigma(t,x')\|_{HS} \leqslant \kappa_0|x - x'|^{\gamma_0},$$

where $\|\cdot\|_{HS}$ is the usual Hilbert-Schmidt norm of a matrix. Moreover, for some $(q_0, \boldsymbol{p}_0) \in \mathscr{I}^o$ and $\mathbf{x}_0 \in \mathscr{X}$ and any $T > 0$,

$$\|\nabla\sigma\|_{\mathbb{L}^{q_0}_T(\widetilde{\mathbb{L}}^{\boldsymbol{p}_0}_{\mathbf{x}_0})} \leqslant \kappa_0.$$

$\mathbf{H}^b$ Suppose that $\phi_t(x,x)=0$ and for some measurable $h:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}_+$ and $\kappa_1>0$,

$$|F(t,x,r)|\leqslant h(t,x)+\kappa_1|r|,\ \ |F(t,x,r)-F(t,x,r')|\leqslant\kappa_1|r-r'|,$$

and for some $(q,\boldsymbol{p})\in\mathscr{I}^o$ and $\mathbf{x}\in\mathscr{X}$ and for any $T>0$,

$$\|h\|^q_{\mathbb{L}^q_T(\widetilde{\mathbb{L}}^{\boldsymbol{p}}_{\mathbf{x}})}+\int_0^T\sup_{y\in\mathbb{R}^d}\Big[\|\phi_t(\cdot,y)\|^q_{\widetilde{\mathbb{L}}^{\boldsymbol{p}}_{\mathbf{x}}}+\|\phi_t(y,\cdot)\|^q_{\widetilde{\mathbb{L}}^{\boldsymbol{p}}_{\mathbf{x}}}\Big]\,\mathrm{d}t\leqslant\kappa_1. \quad (2.1)$$

Example 1 Let $d\geqslant 2$ and $\phi_t(x,y)=c_t(x,y)/|x-y|^\alpha$, where $c_t(x,y)$ is bounded measurable and $\alpha\in(0,1)$. It is easy to see that (2.1) holds for q close to ∞ and $p\in(d,\frac{d}{\alpha})$ with $\frac{d}{p}+\frac{2}{q}<1$.

Example 2 Let $d\geqslant 1$ and $\phi_t(x,y)=c_t(x,y)/\Pi_{i=1}^d|x_i-y_i|^{\alpha_i}$, where $\alpha_i\in(0,\frac{1}{2})$ satisfies $\alpha_1+\cdots+\alpha_d<1$ and $c_t(x,y)$ is bounded measurable. Note that one can choose q close to ∞ and $p_i>2$ close to $1/\alpha_i$ so that $|\frac{1}{\boldsymbol{p}}|+\frac{2}{q}<1$ and (2.1) holds.

Theorem 1

Under $(\mathbf{H}^\sigma)$ *and* $(\mathbf{H}^b)$*, for any initial values* $\mathbf{X}_0^N$ *and* X_0*, there are unique strong solutions* $\mathbf{X}_t^N$ *and* X_t *to particle system* (1.4) *and DDSDE* (1.5)*, respectively. Moreover, letting* μ_0^N *be the law of* $\mathbf{X}_0^N$ *in* $\mathbb{R}^{dN}$ *and* μ_0 *the law of* X_0 *in* $\mathbb{R}^d$*, we have the following strong convergence results:*

(i) **(Singular kernel)** *Suppose that* μ_0^N *is symmetric and* μ_0*-chaotic,*

$$\lim_{N\to\infty} \mathbb{E}|X_0^{N,1} - X_0|^2 = 0.$$

Then for any $\gamma \in (0,1)$*,*

$$\lim_{N\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) = 0. \tag{2.2}$$

Continue...

(ii) **(Bounded kernel)** If h and ϕ in **(H^b)** are bounded measurable and

$$\kappa_2 := \sup_N \mathcal{H}\left(\mu_0^N | \mu_0^{\otimes N}\right) < \infty, \tag{2.3}$$

where $\mu_0^{\otimes N} \in \mathcal{P}((\mathbb{R}^d)^N)$ is the N-tensor of μ_0 and $\mathcal{H}$ stands for the relative entropy, then for any $\delta > 2$ and $\gamma \in (0,1)$, there are constants $C_i = C_i(T, \gamma, \delta, \Theta) > 0$, $i = 1, 2$ independent of ϕ and κ_2 such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) \leqslant C_1 \mathrm{e}^{C_2\|\phi\|_\infty^\delta} \left(\mathbb{E}|X_0^{N,1} - X_0|^2 + \frac{\kappa_2 + 1}{N}\right)^\gamma.$$

Example

- Let $d = 1$. Consider the following rank-based interaction:

$$b(t, x, \mu) = F(t, x, \mu(-\infty, x]). \tag{2.4}$$

- The interaction kernel is $\phi(x, y) = 1_{(-\infty,x]}(y) = 1_{x-y\geqslant 0}$, which is bounded and discontinuous.
- If we let $V(x) := \mu((-\infty, x])$, $\sigma(t, x) = \sqrt{2}$ and $F(t, x, r) = g(r)$, then V solves the following Burgers type equation:

$$\partial_t V = \Delta V + \left(\int_0^V g(r) \mathrm{d}r \right)'.$$

- For $g(r) = r$, this is the classical Burgers equation.

Theorem 2

Suppose that $(\mathbf{H}^\sigma)$ *holds, and*

$$|F(t,x,r)| \leqslant \kappa_1, \ \ |F(t,x,r) - F(t,x,r')| \leqslant \kappa_1 |r - r'|, \tag{2.5}$$

and for $\varepsilon_N \in (0,1)$ *with* $\varepsilon_N \to 0$ *as* $N \to \infty$,

$$\phi_t(x,y) = \phi_{\varepsilon_N}(x-y) = \varepsilon_N^{-d}\phi((x-y)/\varepsilon_N),$$

where ϕ *is a bounded probability density function in* $\mathbb{R}^d$ *with support in the unit ball. Then for any initial value* X_0 *with bounded density* ρ_0, *there is a unique strong solution* X *to density-dependent SDE*

$$\mathrm{d}X_t = F(t, X_t, \rho_{X_t}(X_t))\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \tag{2.6}$$

such that for each $t > 0$, X_t *admits a density* ρ_t *with*

$$\|\rho_t\|_\infty \leqslant C(T,\Theta)\|\rho_0\|_\infty, \ t \in [0,T]. \tag{2.7}$$

Cont.

Moreover, under (2.3), for any $T > 0$, $\beta \in (0, \gamma_0)$, $\gamma \in (0, 1)$ and $\delta > 2$, there are constants $C_i = C_i(T, \beta, \gamma, \delta, \Theta) > 0$, $i = 1, 2, 3$ such that for all $N \geqslant 2$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,1} - X_t|^{2\gamma}\right) \leqslant C_1 \mathrm{e}^{C_2 \varepsilon_N^{-\delta d}} \left(\mathbb{E}|X_0^{N,1} - X_0|^2 + \frac{\kappa_2 + 1}{N}\right)^{\gamma} + C_3 \varepsilon_N^{2\beta\gamma}.$$

Method

- Zvonkin's transformation and heat kernel estimates to show the strong well-posedness for particle system.
- Partial Girsanov's transformation to derive some uniform estimate for particle system.
- Martingale approach to show the weak convergence.
- Zvonkin's method to show the strong convergence.
- Open question: The rate of weak convergence!

Thank you for your attention!