Fluctuations and moderate deviations for a catalytic Fleming-Viot branching system in nonequilibrium

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Outline

Background and Model

Fluctuations

Moderate deviations
The catalytic Fleming-Viot branching system

- The catalytic Fleming-Viot branching system is a jump diffusion process describing a system of diffusing particles (see Grigorescu [3]).
- The hydrodynamic limit for the empirical measure is the solution to a generalized semilinear (reaction-diffusion) equation, with nonlinearity given by a quadratic operator.
\( \text{d-dimensional unit torus } \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d. \)

\( V(x) \) is a bounded continuous function on \( \mathbb{T}^d. \)

For \( x = (x_1, x_2, \ldots, x_N) \in (\mathbb{T}^d)^N, x_{ij} \in (\mathbb{T}^d)^N \) is the vector where the component \( i \) has been deleted and replaced with the component \( j \) for all \( 1 \leq i \neq j \leq N. \)

\( h \in C^{1,2}([0, \infty) \times \mathbb{T}^d), H(t, x) = \sum_{i=1}^{N} h(t, x_i), \) and

\[
p_{ij}^{N,h}(t, x) = p_{ij} = \frac{1}{N-1} e^{H(t,x_{ij})-H(t,x)}, \quad j \neq i, \quad p_{ii}^{N,h} = p_{ii} = 0. \quad (1.1)
\]
\(\zeta(dx)\) is a probability measure on \((\mathbb{T}^d)^N\).

\(P_{\zeta,H}^N\) is a probability measure on \(D([0, \infty), (\mathbb{T}^d)^N)\) such that under \(P_{\zeta,H}^N\), the coordinate process

\[\{\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_N(t)), t \geq 0\}\]

is a Feller process with the generator \(L_t^{N,h}\) defined by

\[
L_t^{N,h}f(t, \mathbf{x}) = \sum_{i=1}^{N} \left( \frac{1}{2} \triangle x_i f(t, \mathbf{x}) + \nabla x_i H(t, \mathbf{x}) \cdot \nabla x_i f(t, \mathbf{x}) \right) + \sum_{i=1}^{N} \int_{0}^{t} \sum_{j \neq i} p_{ij}^{N,h}(t, \mathbf{x})(f(t, \mathbf{x}^j) - f(t, \mathbf{x})) V(x_i),
\]

(1.2)

for \(f \in C^{1,2}([0, \infty) \times (\mathbb{T}^d)^N)\), where \(x \cdot y\) denotes the inner product.
The process $\{x(t), P^N_{\zeta, h}\}$ exists, and it is the solution of the following martingale problem: for any $f \in C^{1,2}([0, \infty) \times (\mathbb{T}^d)^N)$,

$$M_{t}^{N, h, f} = f(t, x(t)) - f(0, x(0)) - \int_0^t \left( \partial_s f(s, x(s)) + \mathcal{L}_t^{N, h} f(s, x(s)) \right) ds \quad (1.3)$$

is a $P$-martingale with

$$\langle M^{N, h, f} \rangle_t = \frac{1}{2} \int_0^t \left( \sum_{i=1}^N |\nabla x_i f(s, x(s))|^2 + \sum_{j \neq i} p_{ij}^{N, h}(s, x(s)) \right. \left. \times (f(s, x_{ij}(s)) - f(s, x(s))) V(x_i(s)) \right) ds.$$

The process $\{\{x(t), t \geq 0\}, P^N_{\zeta, h}\}$ is called a catalytic Fleming-Viot branching system.
The catalytic Fleming-Viot branching system with uniform redistribution mechanism, i.e., \( h = 0 \).

\( P^N_{\gamma} \) denotes the law of the process starting at

\[
\zeta(dx) = \bigotimes_{j=1}^N \gamma(dx_j), \quad \gamma(dx) = \gamma(x)dx,
\]

with bounded initial density \( \gamma(x) \).

The expectation with respect to \( P^N_{\gamma} \) is denoted by \( E^N_{\gamma} \).

The empirical measure process

\[
\mu_t^N(dx) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)} \in D([0, T], M_1(\mathbb{R}^d)), \quad 0 \leq t \leq T.
\]

Consider a linear operator on the space \( D([0, \infty), M_b(\mathbb{R}^d)) \),

\[
a : \mu \rightarrow a(\mu)(t,x) = \langle \mu_t, V \rangle - V(x).
\]
A measure-valued path \( \{\rho_t(dx), t \geq 0\} \) is the unique weak solution of the integro-differential equation

\[
\partial_t \mu = \frac{1}{2} \Delta \mu + \mu a(\mu), \quad \mu_0 = \gamma. \tag{1.6}
\]

The hydrodynamic limit (Grigorescu [3]), i.e., for any \( \phi \in C^{1,2}([0, T] \times \mathbb{T}^d) \), any \( \epsilon > 0 \),

\[
\lim_{N \to \infty} P^N_{\gamma} \left( \sup_{t \in [0, T]} |\langle \mu^N_t(\cdot) - \rho_t(\cdot), \phi(t, \cdot) \rangle| \geq \epsilon \right) = 0. \tag{1.7}
\]

The large deviations for the empirical measure process (Grigorescu [3]).
Our purpose

▶ Fluctuation: The weak convergence of the empirical fluctuation fields $\eta_t^N(dx)$, $N \geq 1$ defined by

$$
\eta_t^N(dx) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\delta_{x_i(t)}(dx) - \rho_t(dx)),
$$

(1.8)

▶ The moderate deviation principle: The large deviation principle of the centralized empirical measure process

$$
\tilde{\eta}_t^N(dx) = \frac{1}{a(N)} \sum_{i=1}^{N} (\delta_{x_i(t)}(dx) - \rho_t(dx)) = \frac{\sqrt{N}}{a(N)} \eta_t^N(dx),
$$

(1.9)

where \( \{a(t), t \geq 0\} \) is a positive function with

$$
\lim_{t \to \infty} \frac{a(t)}{\sqrt{t}} = \infty, \quad \lim_{t \to \infty} \frac{a(t)}{t} = 0,
$$

(1.10)
For every integer $m$, for each $g \in C^\infty(\mathbb{T}^d)$, define

$$\|g\|_m = \left( \sum_{|k| \leq m} \int_{\mathbb{T}^d} |\partial^k g(x)|^2 \, dx \right)^{1/2} < \infty.$$ 

Let $\mathbb{H}^m$ be the complete of $(C^\infty(\mathbb{T}^d), \| \cdot \|_m)$, and $\mathbb{H}^{-m}$ the dual space of $\mathbb{H}^m$.

Let $\frac{1}{2} \triangle$ be the Laplace operator on $\mathbb{T}^d$ and let $U(t)$ be the heat semigroup associated with $\frac{1}{2} \triangle$ on $\mathbb{T}^d$. 
(A0). \( h(t, x) \equiv 0 \), and \( V \) is a non-negative continuous function on \( \mathbb{T}^d \) with partial derivatives up to order \( (5+3D) \), where \( D = \lfloor d/2 \rfloor + 1 \).

Let the condition (A0) hold. For \( m \geq 1 + D \), let \( W \) be the continuous Gaussian martingale process taking its values in \( \mathbb{H}^{-m} \) with mean 0 and variance given by

\[
E (W_t(\varphi)^2) = \int_0^t \left( \frac{1}{2} \langle \rho_s, |\nabla \varphi|^2 \rangle + \langle \rho_s, \varphi^2 \rangle \langle \rho_s, V \rangle - 2 \langle \rho_s, \varphi \rangle \langle \rho_s, \varphi V \rangle + \langle \rho_s, \varphi^2 V \rangle \right) ds
\]

for every \( \varphi \in \mathbb{H}^m \) and \( t \in [0, T] \).

Let \( F_t \) be the operator on \( \mathbb{H}^m \) defined by

\[
F_t \varphi(x) = \langle \rho_t, V \rangle \varphi(x) + \langle \rho_t, \varphi \rangle V(x) - V(x)\varphi(x).
\]
Theorem 2.1 (Fluctuation Theorem)

Assume that the condition \((A0)\) holds. Then under \(P_N^\gamma\), the sequence \(\{\eta^N, N \geq 1\}\) converges in law to the generalized Ornstein-Uhlenbeck process \(\eta\) with catalyst \(V\) in \(D([0, T], \mathbb{H}^{-(4+2D)})\). i.e., for any \(\varphi \in \mathbb{H}^{4+2D}\),

\[
\langle \eta_t, \varphi \rangle = \langle \eta_0, U(t)\varphi \rangle + \int_0^t \langle \eta_s, F_s U(t-s)\varphi \rangle ds + \int_0^t \langle U(t-s)\varphi, dW_s \rangle,
\]

(2.3)
For any $t \in [0, T]$ and $\varphi \in C^2$, applying (1.3) to $f(t, x(t)) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varphi(x_i(t))$, we have

$$\langle \eta_t^N, \varphi \rangle = \langle \eta_0^N, \varphi \rangle + \int_0^t \langle \eta_s^N, \frac{1}{2} \Delta \varphi \rangle ds + \int_0^t \langle \eta_s^N, F_s^N \varphi \rangle ds + M_t^N(\varphi),$$

(2.4)

where

$$F_s^N \varphi(x) = \frac{N}{N - 1} \langle \mu_s^N, V \rangle \varphi(x) + \langle \rho_s, \varphi \rangle V(x) - V(x)\varphi(x),$$

(2.5)

$M_t^N(\varphi)$ is a square integrable martingale with

$$\langle M^N(\varphi) \rangle_t = \int_0^t \langle \mu_s^N, (\nabla \varphi)^2 \rangle ds + \frac{N}{N - 1} \int_0^t \left( \langle \mu_s^N, \varphi^2 \rangle \langle \mu_s^N, V \rangle - 2 \langle \mu_s^N, \varphi \langle \mu_s^N, \varphi V \rangle + \langle \mu_s^N, \varphi^2 V \rangle \right) ds.$$

(2.6)
Informally,

- $M^N(\varphi) \to W(\varphi)$ follows from $\mu^N \to \rho$,
- and so if $\eta^N$ has a limit point $\eta$, then $\eta$ satisfies (2.3).

In order to give a rigorous proof, we need some moment estimates.

The sequence

$$\left\{ \left\{ (M^N_t, \eta^N_t), t \in [0, T] \right\}, N \geq 1 \right\}$$

is tight in $D([0, T], \mathbb{H}^{-4-2D} \times \mathbb{H}^{-4-2D})$.

All limit points of the sequence $\{\mathcal{L}((M^N, \eta^N)), N \geq 1\}$ charge only in $C([0, T], \mathbb{H}^{-4-2D} \times \mathbb{H}^{-4-2D})$.

Let $(M, \eta)$ be a weak limit point of the sequence $\{(M^N, \eta^N), N \geq 1\}$ in $D([0, T], \mathbb{H}^{-4-2D} \times \mathbb{H}^{-4-2D})$. Then $M$ has the same law as $W$, and $(W, \eta)$ solves the equation (2.3).
Theorem 3.1 (Moderate deviations)
Assume that the condition \((A0)\) holds. Then the sequence \(\\{\tilde{\eta}_N, N \geq 1\}\) satisfies a large deviation principle on \(D([0, T], \mathbb{H}^{-(5+3D)})\) with the speed \(a^2(N)/N\) and the good rate function \(I\) defined by

\[
I(\nu) = \sup_{\psi \in C(T^d)} \left\{ \langle \nu_0, \psi \rangle - \frac{1}{2} \left( \int_{T^d} |\psi(x)|^2 \gamma(x)dx - \left( \int_{T^d} \psi(x)\gamma(x)dx \right)^2 \right) \right\}
\]

\[+ \sup_{\phi \in C^\infty([0, T] \times T^d)} \left\{ \ell_\phi(\nu) - \frac{1}{2} \int_0^T \langle \rho_s, |\nabla \phi(s)|^2 \rangle ds \right. \]

\[- \frac{1}{2} \int_0^T \int_{T^d} \int_{T^d} (\phi(s, y) - \phi(s, x))^2 V(x)\rho_s(dy)\rho_s(dx)ds \right\}
\]

\[:= I_0(\nu_0) + I_{\text{dyn}}(\nu). \tag{3.1} \]
\[ \ell_\phi(\nu) = \langle \nu_T, \phi(T) \rangle - \langle \nu_0, \phi(0) \rangle - \int_0^T \left( \langle \nu_s, \partial_s \phi(s) + \frac{1}{2} \triangle \phi(s) \rangle \right) ds \]

\[ - \int_0^T \left( \langle \rho_s, V \rangle \langle \nu_s, \phi(s) \rangle + \langle \rho_s, \phi(s) \rangle \langle \nu_s, V \rangle - \langle \nu_s, \phi(s) V \rangle \right) ds. \]  

That is, for any closed set \( F \subset D([0, T], H^{-(5+3D)}) \),

\[ \limsup_{N \to \infty} \frac{N}{a^2(N)} \log P_\gamma^N(\eta^N \in F) \leq - \inf_{\nu \in F} I(\nu) \]  

(3.3)

and for any open set \( O \subset D([0, T], H^{-(5+3D)}) \),

\[ \liminf_{N \to \infty} \frac{N}{a^2(N)} \log P_\gamma^N(\eta^N \in O) \geq - \inf_{\nu \in O} I(\nu). \]  

(3.4)
For any $\phi \in C^{1,2}([0, T] \times \mathbb{T}^d)$,

$$\langle \tilde{\eta}^N_t, \phi(t) \rangle = \frac{1}{a(N)} \sum_{i=1}^{N} (\phi(t, x_i(t)) - \langle \rho_t, \phi(t) \rangle).$$

We consider the exponential martingale $Z_{\phi,N}^t$ associated with $\frac{a^2(N)}{N} \langle \tilde{\eta}^N_t, \phi(t) \rangle$. Under the condition (A0) holds, $P_{\gamma}$-martingale $Z_{\phi,N}^t$ has the following approximation:

$$Z_{\phi,N}^T = \exp \left\{ \frac{a^2(N)}{N} \ell_{\phi}(\tilde{\eta}^N) - \frac{a^2(N)}{2N} \int_0^T \left( \langle \mu^N_s, |\nabla \phi(s)|^2 \rangle \right. \right. \right.

+ \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( \phi(s, y) - \phi(s, x) \right)^2 V(x) \mu^N_s(dy) \mu^N_s(dx) \right) \left. ds \right. \right. \right.

+ \frac{a^2(N)}{N} \int_0^T \langle \tilde{\eta}^N_s, \phi(s) \rangle \langle \mu^N_s - \rho_s, V \rangle ds + o \left( \frac{a^2(N)}{N} \right) \right\}. \quad (3.5)
Define

\[ \Lambda_0(\psi) = \frac{1}{2} \left( \int_{\mathbb{T}^d} |\psi(x)|^2 \gamma(x) dx - \left( \int_{\mathbb{T}^d} \psi(x) \gamma(x) dx \right)^2 \right), \]

\[ \Lambda_{\text{dyn}}(\phi) = \frac{1}{2} \int_0^T \langle \rho_s, |\nabla \phi(s)|^2 \rangle ds \]

\[ + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (\phi(s, y) - \phi(s, x))^2 V(x) \rho_s(dy) \rho_s(dx) ds, \]

\[ \Lambda(\psi, \phi) = \Lambda_0(\psi) + \Lambda_{\text{dyn}}(\phi) \]
If \( \int_0^T \langle \eta_s^N, \phi(s) \rangle \langle \mu_s^N - \rho_s, V \rangle ds \to 0 \) in MDP sense, then
\[
Z_{\phi, N}^T = \exp \left\{ \frac{a^2(N)}{N} \left( \ell_{\phi}(\eta^N) - \Lambda_{\text{dyn}}(\phi) \right) + o\left( \frac{a^2(N)}{N} \right) \right\}.
\]

For any \( \nu \in D([0, T], \mathbb{H}^{-(5+3D)}) \) and the ball \( B(\nu, \varepsilon) \), when \( N \to \infty, \varepsilon \to 0 \),
\[
\frac{N}{a^2(N)} \log P_{\gamma}^N(B(\nu, \varepsilon)) \sim -I(\nu)
\]


Thank you!