Quasi-Stationary Distributions for Markov Chains

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Talk in The 16th Workshop on MPRT, July, 16, 2021
Outline of Topics

1. Quasi-stationary distributions (QSDs)

2. QSDs of Markov Chains
A Quasi-Stationary Distribution (in short QSD) for $X$ is a probability measure supported on $(0, \infty)$ satisfying for all $t \geq 0$,

$$P_\nu(X(t) \in A | T > t) = \nu(A), \quad \forall \text{ borel set } A \subseteq (0, \infty).$$

where

$$T = \inf\{t \geq 0, X(t) = 0\}$$
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for some $\mu \in (0, \infty)$. 
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A QSD must be infinitely divisible (D. Vere-Jones 1969)
A probability measure \( \pi \) supported on \((0, \infty)\) is a LCD if there exists a probability measure \( \nu \) on \((0, \infty)\) such that the following limit exists in distribution

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\lim_{t \to \infty} P_\nu(X(t) \in \bullet \mid T > t) = \pi(\bullet).
\]

We also say that \( \nu \) is attracted to \( \pi \) or is in the domain of attraction of \( \pi \) or \( \pi \) is a \( \nu \)-LCD.
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- The $\nu$-LCD is a QSD (Vere-Jones(1969)).
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(i) determination of all QSD’s; and

(ii) solve the domain of attraction problem, namely, characterize all probability measure $\nu$ such that a given QSD $M$ is a $\nu$-LCD.

(iii) The rate of convergence of the transition probabilities of the conditioned process to their limiting values.
Both (i) and (ii) are known only for finite Markov processes, and for the subcritical Markov Branching Process (MBP).
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Proposition (i) if $S = \infty$, then either $\lambda_C$ (Kingman’s decay parameter) $= 0$ and there is no QSD, or $\lambda_C > 0$ and there is a one-parameter family of QSDs, Viz, $\{q_j(x)\}, 0 < x \leq \lambda_C$.

(ii) If $S < \infty$, then $\lambda_C > 0$ and there is precisely one QSD, Viz, $\{q_j(\lambda_C)\}$.
Let $X_t$ be a continuous-time Markov chain in $I = \{0\} \cup \{1, 2, \ldots\}$ such that 0 is an absorbing state. Let $C \equiv \{1, 2, \ldots\}$. Denote by $Q = (q_{ij})$ the $q$-matrix (transition rate matrix) and $P(t) = (P_{ij}(t))$ the transition function. $X_t$ is stochastically monotone if and only if $\sum_{j \geq k} P_{ij}(t)$ is a nondecreasing function of $i$ for every fixed $k \in I$ and $t > 0$. We assume that all states other than 0 form an irreducible class and that $Q$ is totally stable, conservative and regular, that is, $q_i = \sum_{i \neq j} q_{ij} < \infty$, and the minimal process $\{X_t\}_{t \geq 0}$ corresponding to $Q$ is an honest process. We further define $T = \inf\{t \geq 0 : X_t = 0\}$, the absorption time at 0. So $X_t = 0$ for any $t \geq T$. 

Proposition 2 Assume that

\[ \lim_{i \to \infty} P_i(T < t) = 0 \quad \text{for any } t \geq 0 \]

and that \( P_i(T < \infty) = 1 \) for some (and hence all) \( i \). Then a necessary and sufficient condition for the existence of a QSD is that

\[ E_i(e^{\theta T}) < \infty \]

for some \( \theta > 0 \) and some \( i \in C \) (and hence for all \( i \)).
Our result is

**Theorem 1** Assume that

\[
\lim_{i \to \infty} E_i T = \infty,
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for some \( \theta > 0 \) and some \( i \in C \) (and hence for all \( i \)). When it holds, there exists a family of QSDs.
For stochastically monotone Markov chains, we discuss the existence, uniqueness and domain of attraction of QSDs.

**Theorem 2** Assume $Q$ is regular and conservative, and $X_t$ is stochastically monotone,

(i) If $\lim_{i \to \infty} E_i T = \infty$, then there exists a QSD if and only if

$$E_i(e^{\theta T}) < \infty$$

for some $\theta > 0$ and some $i \in C$ (and hence for all $i$).

(ii) If $\lim_{i \to \infty} E_i T < \infty$, and the set $N_0 = \{i \in C : q_{i0} > 0\}$ is finite, then there is a unique QSD. Moreover, the unique QSD $\rho = \{\rho_j, j \in C\}$ attracts all initial distributions that supported in $C$, that is, for any probability measure $\nu = \{\nu_i, i \in C\}$,

$$\rho_j = \lim_{t \to \infty} P_{\nu}(X_t = j|T > t), \quad j \in C.$$
Thank you all for your attention!