SDEs driven by multiplicative stable-like Lévy processes

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Supercritical SDEs driven by multiplicative stable-like Lévy processes

This talk is concerned with strong and weak well posedness of solutions to

\[ dX_t = \sigma(t, X_{t-})dZ_t + b(t, X_t)dt, \quad X_0 = x \in \mathbb{R}^d, \]

where \( Z \) is a \( d \)-dimensional non-degenerate \( \alpha \)-stable-like process with \( \alpha \in (0, 2) \), beyond Lipschitz condition on \( b \).
dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d,

Infinitesimal generator:

\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b(x) \cdot \nabla.

1. \(d = 1\) and \(b = 0\): (Strong solution and PU)
Yamada-Watanabe condition (1971). \(\sigma(x) \in C^{1/2}(\mathbb{R})\).
Counter-example: Barlow (1992) for \(\sigma(x) \in C^\beta(\mathbb{R})\) with \(\beta < 1/2\).

2. \(d \geq 2\): (Strong solution and PU)
Yamada-Watanabe (1971): \(b = 0\). Slightly weaker than Lipschitz with a logarithmic term.
\(\sigma = I\): bounded \(b\): Zvonkin (1974), Veretennikov (1979)
\(\sigma = I_d \times d\): \(b(t, x) \) in \(L^p/L^q\), Krylov and Röckner (2005).
• Weak existence and uniqueness:
\(\sigma(x)\): continuous and uniformly elliptic.

Girsanov: removing and adding drifts.

Bass-C. (2003): \(\sigma(x) = I_{d \times d}\), measure-valued drift \(\vec{\mu}(x) \cdot \nabla\).
SDEs driven by stable processes

\[ dX_t = \sigma(X_{t-})dZ_t + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d, \]

where \( Z \) is an isotropic \( \alpha \)-stable process with \( \alpha \in (0, 2) \).

Infinitesimal generator when \( \sigma = I_{d \times d} \):

\[ \mathcal{L} = -(-\Delta)^{\alpha/2} + b \cdot \nabla. \]

Subcritical: \( \alpha > 1 \); critical: \( \alpha = 1 \); supercritical: \( \alpha < 1 \).

1. \( d = 1 \) and \( \sigma = 1 \) (Strong solution and PU)
   
   Tanaka-Tsuchiya-Watanabe (1974): \( b \in L^\infty(\mathbb{R}) \) when \( \alpha \in (1, 2) \); \( b \in C(\mathbb{R}) \) when \( \alpha = 1 \). Counter-example for \( b \in C^\beta(\mathbb{R}) \) with \( \beta < 1 - \alpha \) when \( \alpha \in (0, 1) \).

2. \( d = 1 \) and \( b = 0 \) (Strong solution and PU)
   
   Komatsu (1982), Bass (2003): \( \sigma \in C^{1/\alpha}(\mathbb{R}) \) for \( \alpha > 1 \). Counter-example: Bass-Burdzy-C. (2004) for \( \sigma \in C^\beta(\mathbb{R}) \) with \( \beta < 1 \wedge (1/\alpha) \) for any \( \alpha \in (0, 2) \).
SDE driven by stable processes

\[ d \geq 2: \sigma = l_{d \times d} \] (Strong solution and PU)

1 Priola (2012): \( Z \) non-degenerate symmetric \( \alpha \)-stable with \( \alpha \in [1, 2) \), \( b(x) \in C^\beta(\mathbb{R}^d) \) with \( \beta > 1 - (\alpha/2) \).
Open problem for supercritical case: \( \alpha \in (0, 1) \).

2 X. Zhang (2013): \( \alpha \in (1, 2) \), \( b(t, x) \) in some fractional Sobolev space.

3 C.-Song-Zhang (2018): \( Z \) a class of Lévy processes and subordinate BMs
  
  • When \( Z \) is an isotropic \( \alpha \)-stable process on \( \mathbb{R}^d \), for \( b \in L^\infty([0, T], C^\beta(\mathbb{R}^d)) \) with \( \beta > 1 - (\alpha/2) \) for \( \alpha \in (0, 2) \).
  • When \( Z \) is a cylindrical \( \alpha \)-stable process on \( \mathbb{R}^d \), for \( b \in L^\infty([0, T], C^\beta(\mathbb{R}^d)) \) with \( \beta > 1 - (\alpha/2) \) for \( \alpha > 2/3 \).
SDEs driven by stable-like Lévy processes

Weak existence and weak uniqueness.

1. **Z**: isotropic $\alpha$-stable processes with $\alpha \in (1, 2)$, $\sigma = I_d \times d$:
   - Portenko (1994 for $d = 1$), Podolynny-Portenko (1995 for $d \geq 2$): $b(x) \in L^p(\mathbb{R}^d)$ with $p > d/(\alpha - 1)$.
   - C.-L. Wang (2016): $b(x)$ in Kato class, including $L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ with $p > d/(\alpha - 1)$.

2. **Z**: isotropic $\alpha$-stable processes with $\alpha \in (0, 1)$, $\sigma = I_d \times d$:
   - Tanaka-Tsuchiya-Watanabe (1974), Tsutsumi (1974): $d = 1$. **Counter-example** for $b \in C^\beta(\mathbb{R})$ with $\beta < 1 - \alpha$;
   - Kulik (2019): $0 < c_1 \leq \sigma(x) \leq c_2$ scalar Hölder, $b(x) \in C^\beta(\mathbb{R}^d)$ with $\beta > (1 - \alpha)^+$.

3. Zhao (2019): a subclass of $\alpha$-stable processes $Z$ with $\alpha \in (0, 1)$, $\sigma = I_d \times d$, $b(x) \in C^\beta(\mathbb{R}^d)$ with $\beta > 1 - \alpha$. 
Our setting

$Z$: Lévy process with $\mathbb{E} \left[ e^{i \xi \cdot (Z_t - Z_0)} \right] = e^{-t \psi(\xi)}$:

$$
\psi(\xi) = \int_{\mathbb{R}^d} \left( 1 - e^{i \xi \cdot z} + i \xi \cdot z \mathbb{1}_{\{|z|<1\}} \right) \nu(dz).
$$

Lévy measure $\nu$: $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$.

For $\alpha \in (0, 2)$, let $\mathbb{L}_{\text{non}}^{(\alpha)}$ be the space of all non-degenerate $\alpha$-stable Lévy measures $\nu^{(\alpha)}$:

$$
\nu^{(\alpha)}(A) = \int_0^{\infty} \left( \int_{S^{d-1}} \mathbb{1}_A(r\theta) \Sigma(d\theta) \right) \frac{dr}{r^{1+\alpha}}, \quad A \in \mathcal{B}(\mathbb{R}^d),
$$

where $\Sigma$ is a finite measure on $S^{d-1}$ with

$$
\int_{S^{d-1}} |\theta_0 \cdot \theta| \Sigma(d\theta) > 0 \quad \text{for every } \theta_0 \in S^{d-1}.
$$
Let $Z$ be a purely discontinuous Lévy process with Lévy measure $\nu$ so that

$$\nu_1(A) \leq \nu(A) \leq \nu_2(A), \quad A \in B(\mathbb{B}(0, 1)).$$

for some $\nu_1, \nu_2 \in \mathbb{L}_{non}^{(\alpha)}$. 

**Example:** $Z$ is the independent sum of $\alpha$-stable and $\beta$-stable processes with $\beta < \alpha$.

Assume

$$\Lambda^{-1} |\xi| \leq |\sigma(t, x)\xi| \leq \Lambda |\xi| \quad \text{and} \quad |b(t, x)| \leq \Lambda.$$
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Our main results

Theorem (C.-Zhang-Zhao, 2021+)

Under the above assumptions, for each \( x \in \mathbb{R}^d \), the SDE

\[
dX_t = \sigma(t, X_{t-})dZ_t + b(t, X_t)dt, \quad X_0 = x_0.
\]

1. has a unique weak solution if \( \sigma(t, x) \) and \( b(t, x) \) are \( C^\beta \) in \( x \) with \( \beta > (1 - \alpha)^+ \);

2. has a unique strong solution if \( \sigma(t, x) \) is Lipschitz in \( x \) and \( b(t, x) \) are \( C^\beta \) in \( x \) with \( \beta > 1 - (\alpha/2) \).

- Hold for any \( \alpha \)-stable process including cylindrical ones.
- Multiplicative noise; Localization
- Sharp in \( \sigma \) (strong solution); sharp in \( b \) (weak solution).
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Our approach

The infinitesimal generator for SDE is $\mathcal{L}_t + b(t, x) \cdot \nabla$, where

\[
\mathcal{L}_t u(x) := \int_{\mathbb{R}^d} (u(x + \sigma(t, x)z) - u(x) - \mathbb{1}_{|z| \leq 1} \sigma(t, x)z \cdot \nabla u(x)) \nu(dz).
\]

For strong well posedness, we use Zvonkin's change of variable to remove the drift. Key: existence, uniqueness and regularity estimates for

\[
\partial_t u = (\mathcal{L}_t - \lambda) u + b \cdot \nabla u + f \quad \text{with } u(0, x) = 0.
\]

We show when $\alpha + \beta > 1$ and $\sigma(t, x) = \sigma(t)$, for every $p > d/(\alpha + \beta - 1)$,

\[
\|u\|_{L^\infty([0,T]; B^{\alpha+\beta}_{p,\infty})} \leq C \|f\|_{L^\infty([0,T]; B^{\beta}_{p,\infty})},
\]

and for any $\gamma \in (0, \alpha + \beta)$,

\[
\|u\|_{L^\infty([0,T]; B^{\gamma}_{p,\infty})} \leq c_\lambda \|f\|_{L^\infty([0,T]; B^{\beta}_{p,\infty})},
\]

where $B^{\beta}_{p,\infty}$ is the usual Besov space and $c_\lambda \to 0$ as $\lambda \to \infty$. 
Our approach for strong well posedness

For general Hölder $\sigma(t, x)$, we use a localization and a patching-together procedure to establish the above a priori estimate for Lévy measure $\nu$ with bounded support.

Take $f = b$ and $\lambda > 0$ large. By Sobolev embedding, $\|\nabla u\|_\infty < 1/2$. So $\Phi(t, x) := x + u(t, x)$ is 1-1. When $\sigma(t, x)$ is Lipschitz in $x$, $\beta > 1 - (\alpha/2)$ and $\nu$ has bounded support, by Itô’s formula, $Y_t := \Phi(t, X_t)$ satisfies an SDE with Lipschitz coefficients. Thus $Y_t$ is well-posed and so is $X_t$.

General $\nu$: truncation and piecing-together argument.

New feature: we use the Littlewood-Paley theory and some Bernstein’s type inequalities to establish the above a priori estimates for the fractional PDE. This approach allows us to address the open problem affirmatively for any non-degenerate stable processes.

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(i) a substitute for orthogonality arguments in $L^2$ spaces.

(ii) decompose $f$ into a sum of functions $\Pi_j f$ with localized frequencies (between $2^{j-1}$ and $3 \cdot 2^{j-1}$), and use it to characterize the Besov spaces.

(iii) Bernstein’s type inequality: estimates on $\| \nabla^k \Pi_j f \|_q$ and $\| (−\Delta)^{\beta/2} \Pi_j f \|_p$.

(iv) Commutator estimates on $\| [\Pi_j, f] g \|_p$.

(iv) Sobolev embedding theorem for Besov spaces.
(i) Uniqueness for the martingale problem for SDE driven by truncated Lévy process, using solution of fractional PDE.

(ii) Weak existence follows from a weak convergence argument.

(iii) Weak uniqueness for general $\nu$: resurrect at times $\{\tau_k; k \geq 1\}$ when the driving Lévy process $Z$ makes jumps larger than 1 and show the resulting solution satisfies SDE driven by the truncated Lévy process. This gives weak uniqueness on $[0, \tau_1)$, and then on $[0, \tau_2)$, ...
Thanks for watching!