Boundary Harnack Principle for Diffusion with Jumps in Lipschitz domain

Jieming Wang

(Beijing Institute of Technology)

The 14th Workshop on Markov Processes and Related Topics

(Joint work with Professor Z.-Q. Chen)
Background and Motivation

- We say that $h$ is a harmonic function with respect to a process $X$ on an open set $D$ if for each open bounded $V \subset \overline{V} \subset D$,

$$h(x) = \mathbb{E}_x h(X_{\tau_V}), \quad x \in V,$$

where $\tau_V := \inf \{ t > 0 : X_t \notin V \}$.

- The statement of the boundary Harnack principle (BHP):

Let $D$ be a domain with a certain geometric property. There exist positive constants $R_0$ and $C$ such that for any $Q \in \partial D$, $r \in (0, R_0]$ any positive harmonic functions $u$ and $v$ w.r.t. $X$ in $D \cap B(Q, r)$ that vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{u(x)}{u(y)} \leq C \frac{v(x)}{v(y)}, \quad x, y \in D \cap B(Q, r/2).$$
Background and Motivation

- We say that $h$ is a harmonic function with respect to a process $X$ on an open set $D$ if for each open bounded $V \subset \overline{V} \subset D$,

$$h(x) = \mathbb{E}_x h(X_{\tau_V}), \quad x \in V,$$

where $\tau_V := \inf\{t > 0 : X_t \notin V\}$.

- The statement of the boundary Harnack principle (BHP):
  Let $D$ be a domain with a certain geometric property. There exist positive constants $R_0$ and $C$ such that for any $Q \in \partial D, r \in (0, R_0]$ any positive harmonic functions $u$ and $v$ w.r.t. $X$ in $D \cap B(Q, r)$ that vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{u(x)}{u(y)} \leq C \frac{v(x)}{v(y)}, \quad x, y \in D \cap B(Q, r/2).$$
The boundary Harnack principle (BHP) for Brownian motion in Lipschitz domain was obtained independently by Ancona (1978), Dahlberg (1977) and Wu (1978).

The result was generalized to elliptic operator by Caffarelli-Fabes- Mortola-Salsa (1981) and Fabes-Carofalo-Marin-Malave and Salsa (1988).

In 1989, Bass-Burdzy developed a quite different probabilistic method (also called the ”box” method) to prove the BHP of Brownian motion in Lipschitz domain and extended it to more general non-smooth domain.

Kim-Song (2007) established the BHP for diffusion with measure valued drifts which belong to some Kato class in Lipschitz domains.
The boundary Harnack principle (BHP) for Brownian motion in Lipschitz domain was obtained independently by Ancona (1978), Dahlberg (1977) and Wu (1978).

The result was generalized to elliptic operator by Caffarelli-Fabes-Mortola-Salsa (1981) and Fabes-Carofalo-Marin-Malave and Salsa (1988).

In 1989, Bass-Burdzy developed a quite different probabilistic method (also called the "box" method) to prove the BHP of Brownian motion in Lipschitz domain and extended it to more general non-smooth domain.

Kim-Song (2007) established the BHP for diffusion with measure valued drifts which belong to some Kato class in Lipschitz domains.
The boundary Harnack principle (BHP) for Brownian motion in Lipschitz domain was obtained independently by Ancona (1978), Dahlberg (1977) and Wu (1978).

The result was generalized to elliptic operator by Caffarelli-Fabes- Mortola-Salsa (1981) and Fabes-Carofalo-Marin-Malave and Salsa (1988).

In 1989, Bass-Burdzy developed a quite different probabilistic method (also called the ”box” method) to prove the BHP of Brownian motion in Lipschitz domain and extended it to more general non-smooth domain.

Kim-Song (2007) established the BHP for diffusion with measure valued drifts which belong to some Kato class in Lipschitz domains.
The boundary Harnack principle (BHP) for Brownian motion in Lipschitz domain was obtained independently by Ancona (1978), Dahlberg (1977) and Wu (1978).

The result was generalized to elliptic operator by Caffarelli-Fabes-Mortola-Salsa (1981) and Fabes-Carofalo-Marin-Malave and Salsa (1988).

In 1989, Bass-Burdzy developed a quite different probabilistic method (also called the ”box” method) to prove the BHP of Brownian motion in Lipschitz domain and extended it to more general non-smooth domain.

Kim-Song (2007) established the BHP for diffusion with measure valued drifts which belong to some Kato class in Lipschitz domains.
Pure Discontinuous Processes

- A stochastic process $Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ is called a rotationally symmetric $\alpha$-stable process with $\alpha \in (0, 2)$ on $\mathbb{R}^d$ if it is a Lévy process such that its characteristic function is given by

$$\mathbb{E}_x \left[ e^{i \xi \cdot (Z_t - Z_0)} \right] = e^{- |\xi|^{\alpha}} \quad \text{for every } x \in \mathbb{R}^d.$$ 

- The infinitesimal generator for rotationally symmetric $\alpha$-stable process

$$\Delta^{\alpha/2} f(x) = A(d, -\alpha) \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{dz}{|z|^{d+\alpha}}$$

- Truncated symmetric $\alpha$-stable process is the symmetric $\alpha$-stable process with large jumps more than 1 removed. Denote by $\overline{\Delta}^{\alpha/2}$ the operator of truncated symmetric $\alpha$-stable process, then

$$\overline{\Delta}^{\alpha/2} f(x) = A(d, -\alpha) \int_{\{|z| \leq 1\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \right) \frac{dz}{|z|^{d+\alpha}}$$
A stochastic process $Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ is called a rotationally symmetric $\alpha$-stable process with $\alpha \in (0, 2)$ on $\mathbb{R}^d$ if it is a Lévy process such that its characteristic function is given by

$$
\mathbb{E}_x \left[ e^{i \xi \cdot (Z_t - Z_0)} \right] = e^{-t |\xi|^\alpha}
$$

for every $x \in \mathbb{R}^d$.

The infinitesimal generator for rotationally symmetric $\alpha$-stable process

$$
\Delta^{\alpha/2} f(x) = A(d, -\alpha) \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{dz}{|z|^{d+\alpha}}
$$

Truncated symmetric $\alpha$-stable process is the symmetric $\alpha$-stable process with large jumps more than 1 removed. Denote by $\Delta^{\alpha/2}$ the operator of truncated symmetric $\alpha$-stable process, then

$$
\overline{\Delta}^{\alpha/2} f(x) = A(d, -\alpha) \int_{\{|z| \leq 1\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \right) \frac{dz}{|z|^{d+\alpha}}
$$
**Pure Discontinuous Processes**

- A stochastic process \( Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d) \) is called a rotationally symmetric \( \alpha \)-stable process with \( \alpha \in (0, 2) \) on \( \mathbb{R}^d \) if it is a Lévy process such that its characteristic function is given by

\[
\mathbb{E}_x \left[ e^{i \xi \cdot (Z_t - Z_0)} \right] = e^{-t |\xi|^\alpha} \quad \text{for every } x \in \mathbb{R}^d.
\]

- The infinitesimal generator for rotationally symmetric \( \alpha \)-stable process

\[
\Delta^{\alpha/2} f(x) = A(d, -\alpha) \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{dz}{|z|^{d+\alpha}}
\]

- Truncated symmetric \( \alpha \)-stable process is the symmetric \( \alpha \)-stable process with large jumps more than 1 removed. Denote by \( \overline{\Delta}^{\alpha/2} \) the operator of truncated symmetric \( \alpha \)-stable process, then

\[
\overline{\Delta}^{\alpha/2} f(x) = A(d, -\alpha) \int_{\{ |z| \leq 1 \}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \right) \frac{dz}{|z|^{d+\alpha}}
\]
Bogdan(1997) obtained BHP for rotationally symmetric $\alpha$-stable process in bounded Lipschitz domains.

Bogdan(1999) adapted the "box" method developed by Bass-Burdzy to obtain a probabilistic proof for the BHP for the symmetric stable process in Lipschitz domain.

The BHP result was extended to $\kappa$-fat open sets by Song-Wu (1999) and finally to arbitrary open sets by Bogdan-Kulczycki-Kwasnicki (2008).

The result has been further extended to more pure jump processes, including truncated stable processes by Kim-Song, pure jump subordinate Brownian motions by Kim-Song-Vondracek, etc.
Bogdan (1997) obtained BHP for rotationally symmetric $\alpha$-stable process in bounded Lipschitz domains.

Bogdan (1999) adapted the "box" method developed by Bass-Burdzy to obtain a probabilistic proof for the BHP for the symmetric stable process in Lipschitz domain.

The BHP result was extended to $\kappa$-fat open sets by Song-Wu (1999) and finally to arbitrary open sets by Bogdan-Kulczycki-Kwasnicki (2008).

The result has been further extended to more pure jump processes, including truncated stable processes by Kim-Song, pure jump subordinate Brownian motions by Kim-Song-Vondracek, etc.
Bogdan (1997) obtained BHP for rotationally symmetric $\alpha$-stable process in bounded Lipschitz domains.

Bogdan (1999) adapted the “box” method developed by Bass-Burdzy to obtain a probabilistic proof for the BHP for the symmetric stable process in Lipschitz domain.

The BHP result was extended to $\kappa$-fat open sets by Song-Wu (1999) and finally to arbitrary open sets by Bogdan-Kulczycki-Kwasnicki (2008).

The result has been further extended to more pure jump processes, including truncated stable processes by Kim-Song, pure jump subordinate Brownian motions by Kim-Song-Vondracek, etc.
Bogdan(1997) obtained BHP for rotationally symmetric $\alpha$-stable process in bounded Lipschitz domains.

Bogdan(1999) adapted the "box" method developed by Bass-Burdzy to obtain a probabilistic proof for the BHP for the symmetric stable process in Lipschitz domain.

The BHP result was extended to $\kappa$-fat open sets by Song-Wu (1999) and finally to arbitrary open sets by Bogdan-Kulczycki-Kwasnicki (2008).

The result has been further extended to more pure jump processes, including truncated stable processes by Kim-Song, pure jump subordinate Brownian motions by Kim-Song-Vondracek, etc.
Chen-Kim-Song-Vondracek (2012) established the boundary Harnack principle for $\mathcal{L} = \Delta + b\Delta^{\alpha/2}$ with $b \in (0, M]$ in $C^{1,1}$ domain and get the explicit boundary decay rates.

The result was extended to subordinate Brownian motions with Gaussian component by Kim-Song-Vondracek (2013).

Chen-W. (2018+) established the BHP with explicit boundary decay rate for a class of diffusion with jumps on $C^{1,1}$ domain under some conditions on the jump density function.
Chen-Kim-Song-Vondracek (2012) established the boundary Harnack principle for $\mathcal{L} = \Delta + b\Delta^{\alpha/2}$ with $b \in (0, M]$ in $C^{1,1}$ domain and get the explicit boundary decay rates.

The result was extended to subordinate Brownian motions with Gaussian component by Kim-Song-Vondracek (2013).

Chen-W. (2018+) established the BHP with explicit boundary decay rate for a class of diffusion with jumps on $C^{1,1}$ domain under some conditions on the jump density function.
Chen-Kim-Song-Vondracek (2012) established the boundary Harnack principle for $\mathcal{L} = \Delta + b\Delta^{\alpha/2}$ with $b \in (0, M]$ in $C^{1,1}$ domain and get the explicit boundary decay rates.

The result was extended to subordinate Brownian motions with Gaussian component by Kim-Song-Vondracek (2013).

Chen-W. (2018+) established the BHP with explicit boundary decay rate for a class of diffusion with jumps on $C^{1,1}$ domain under some conditions on the jump density function.
The goal

Our aim is to obtain the BHP for a class of diffusion with jumps in Lipschitz domain.
The setting

Suppose \( d \geq 3 \) and \( 0 < \alpha < 2 \). Define

\[
\mathcal{L}f(x) := \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} f(x) \right) + b(x) \cdot \nabla f(x) + S^\kappa f(x), \quad f \in C^2_b(\mathbb{R}^d)
\]

where

\[
S^\kappa f(x) := \int_{\mathbb{R}^d} (f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}}) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz.
\]
Assumptions:

(1) Assume that for each \( x \in \mathbb{R}^d \), \( a_{ij}(x) \) is a symmetric matrix and satisfies the uniform ellipticity condition and the Hölder continuity condition, i.e.

\[
L^{-1}I_{d \times d} \leq a_{ij}(x) \leq LI_{d \times d}.
\]

\[
|a_{ij}(x_1) - a_{ij}(x_2)| \leq c|x_1 - x_2|^{\gamma}, \quad x_1, x_2 \in \mathbb{R}^d.
\]

(2) We assume that \( b(\cdot) \) belongs to Kato class \( K_{d}^{1} \). Here we say that a function \( b \in K_{d}^{1} \) if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |y - x|^{1-d} |b|(y) = 0.
\]

(3) \( \kappa(x, z) \) is a nonnegative real-valued bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying \( \kappa(x, z) = \kappa(x, -z) \) for every \( x, z \in \mathbb{R}^d \) and there exist \( \rho > 0 \) and \( A \geq 1 \) such that

\[
A^{-1} \leq \kappa(x, z) \leq A, \quad |z| \leq \rho.
\]
Assumptions:

(1) Assume that for each \( x \in \mathbb{R}^d \), \( a_{ij}(x) \) is a symmetric matrix and satisfies the uniform ellipticity condition and the Hölder continuity condition, i.e.

\[
L^{-1}I_d \leq a_{ij}(x) \leq LI_d.
\]

\[
|a_{ij}(x_1) - a_{ij}(x_2)| \leq c|x_1 - x_2|^\gamma, \quad x_1, x_2 \in \mathbb{R}^d.
\]

(2) We assume that \( b(\cdot) \) belongs to Kato class \( K^1_d \). Here we say that a function \( b \in K^1_d \) if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |y - x|^{1-d}|b'(y)| = 0.
\]

(3) \( \kappa(x,z) \) is a nonnegative real-valued bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying \( \kappa(x,z) = \kappa(x,-z) \) for every \( x, z \in \mathbb{R}^d \) and there exist \( \rho > 0 \) and \( A \geq 1 \) such that

\[
A^{-1} \leq \kappa(x,z) \leq A, \quad |z| \leq \rho.
\]
Assumptions:

(1) Assume that for each \( x \in \mathbb{R}^d \), \( a_{ij}(x) \) is a symmetric matrix and satisfies the uniform ellipticity condition and the Hölder continuity condition, i.e.

\[
L^{-1}I_{d \times d} \leq a_{ij}(x) \leq LI_{d \times d}.
\]

\[
|a_{ij}(x_1) - a_{ij}(x_2)| \leq c|x_1 - x_2|^{\gamma}, \quad x_1, x_2 \in \mathbb{R}^d.
\]

(2) We assume that \( b(\cdot) \) belongs to Kato class \( K^1_d \). Here we say that a function \( b \in K^1_d \) if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |y - x|^{1-d} |b|(y) = 0.
\]

(3) \( \kappa(x,z) \) is a nonnegative real-valued bounded function on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying \( \kappa(x,z) = \kappa(x,-z) \) for every \( x, z \in \mathbb{R}^d \) and there exist \( \rho > 0 \) and \( A \geq 1 \) such that

\[
A^{-1} \leq \kappa(x,z) \leq A, \quad |z| \leq \rho.
\]
Remarks:

1. By a similar argument as Z.-Q. Chen-E. Hu-L. Xie-X. Zhang (2016), the fundamental solution of $\mathcal{L}$ exists. Associated with it is a conservative Feller process $X$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ having Lévy system $(J^\kappa(x, y)dy, dt)$, where $J^\kappa(x, y) = \kappa(x, y - x)/|y - x|^{d+\alpha}$.

2. In general, the operator $\mathcal{L}$ is a nonsymmetric operator. It covers the two examples as belowing:

   (1) $\mathcal{L} = \Delta + b \cdot \nabla + a \Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$.

   (2) $\mathcal{L} = \Delta + b \cdot \nabla + a \overline{\Delta}^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1_{|z|\leq 1}(z)$.

3. Kim-Song-Vondracek (2013) presented a counterexample that the BHP does not hold for $\Delta + \overline{\Delta}^{\alpha/2}$ even on half space. Some structures of the jumps are needed to establish the boundary Harnack principle.
Remarks:

1. By a similar argument as Z.-Q. Chen-E. Hu-L. Xie-X. Zhang(2016), the fundamental solution of $\mathcal{L}$ exists. Associated with it is a conservative Feller process $X$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ having Lévy system $(J^\kappa(x, y)dy, dt)$, where $J^\kappa(x, y) = \kappa(x, y - x) / |y - x|^{d+\alpha}$.

2. In general, the operator $\mathcal{L}$ is a nonsymmetric operator. It covers the two examples as belowing:

   (1) $\mathcal{L} = \Delta + b \cdot \nabla + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$.

   (2) $\mathcal{L} = \Delta + b \cdot \nabla + a\overline{\Delta}^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1_{|z| \leq 1}(z)$.

3. Kim-Song-Vondracek(2013) presented a counterexample that the BHP does not hold for $\Delta + \overline{\Delta}^{\alpha/2}$ even on half space. Some structures of the jumps are needed to establish the boundary Harnack principle.
Remarks:

1. By a similar argument as Z.-Q. Chen-E. Hu-L. Xie-X. Zhang(2016), the fundamental solution of $\mathcal{L}$ exists. Associated with it is a conservative Feller process $X$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ having Lévy system $(J^\kappa(x,y)dy, dt)$, where $J^\kappa(x,y) = \kappa(x, y - x)/|y - x|^{d+\alpha}$.

2. In general, the operator $\mathcal{L}$ is a nonsymmetric operator. It covers the two examples as belonging:
   
   (1) $\mathcal{L} = \Delta + b \cdot \nabla + a \Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$.
   
   (2) $\mathcal{L} = \Delta + b \cdot \nabla + a \Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1_{|z| \leq 1}(z)$.

3. Kim-Song-Vondracek(2013) presented a counterexample that the BHP does not hold for $\Delta + \Delta^{\alpha/2}$ even on half space. Some structures of the jumps are needed to establish the boundary Harnack principle.
Remarks:

1. By a similar argument as Z.-Q. Chen-E. Hu-L. Xie-X. Zhang(2016), the fundamental solution of $\mathcal{L}$ exists. Associated with it is a conservative Feller process $X$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ having Lévy system $(J^\kappa(x, y)dy, dt)$, where $J^\kappa(x, y) = \kappa(x, y - x)/|y - x|^{d+\alpha}$.

2. In general, the operator $\mathcal{L}$ is a nonsymmetric operator. It covers the two examples as belowing:

   (1) $\mathcal{L} = \Delta + b \cdot \nabla + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$.

   (2) $\mathcal{L} = \Delta + b \cdot \nabla + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1|z|\leq 1(z)$.

3. Kim-Song-Vondracek(2013) presented a counterexample that the BHP does not hold for $\Delta + \Delta^{\alpha/2}$ even on half space. Some structures of the jumps are needed to establish the boundary Harnack principle.
Remarks:

1. By a similar argument as Z.-Q. Chen-E. Hu-L. Xie-X. Zhang (2016), the fundamental solution of $\mathcal{L}$ exists. Associated with it is a conservative Feller process $X$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ having Lévy system $(J^\kappa(x, y)dy, dt)$, where $J^\kappa(x, y) = \kappa(x, y - x)/|y - x|^{d+\alpha}$.

2. In general, the operator $\mathcal{L}$ is a nonsymmetric operator. It covers the two examples as belowing:
   (1) $\mathcal{L} = \Delta + b \cdot \nabla + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$.
   (2) $\mathcal{L} = \Delta + b \cdot \nabla + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1_{|z| \leq 1}(z)$.

3. Kim-Song-Vondracek (2013) presented a counterexample that the BHP does not hold for $\Delta + \Delta^{\alpha/2}$ even on half space. Some structures of the jumps are needed to establish the boundary Harnack principle.
Assumption (A1):

For each set $F \subseteq \mathbb{R}^d$ and each $u \in F^c$, define

$$\Lambda_{u,F} := \{y \in F : J^\kappa(y,u) > 0\}.$$

For each $x \in \mathbb{R}^d$ and $r > 0$, define

$$\|\kappa\|_{x,r} := \text{esssup}\{\kappa(y,u-y) : y \in B(x,r), u \in B(x,4r)^c\}.$$

Assumption (A1): There exist positive constants $\delta$ and $c$ such that for any $r \in (0, \delta)$, $x \in \mathbb{R}^d$ and $u \in \{u \in B(x,4r)^c : m(\Lambda_{u,B(x,r)}) > 0\},$

$$\frac{1}{r^d} \int_{B(x,2r)} J^\kappa(y,u) \, dy \geq cJ\|\kappa\|_{x,r}(w,u), \quad w \in B(x,r).$$
Assumption \((A1)\):

For each set \(F \subseteq \mathbb{R}^d\) and each \(u \in F^c\), define

\[
\Lambda_{u,F} : = \{ y \in F : J_\kappa(y,u) > 0 \}.
\]

For each \(x \in \mathbb{R}^d\) and \(r > 0\), define

\[
\| \kappa \|_{x,r} : = \text{esssup} \{ \kappa(y,u - y) : y \in B(x,r), u \in B(x,4r)^c \}.
\]

Assumption \((A1)\): There exist positive constants \(\delta\) and \(c\) such that for any \(r \in (0, \delta)\), \(x \in \mathbb{R}^d\) and \(u \in \{ u \in B(x,4r)^c : m(\Lambda_{u,B(x,r)}) > 0 \}\),

\[
\frac{1}{r^d} \int_{B(x,2r)} J_\kappa(y,u) \, dy \geq cJ\| \kappa \|_{x,r}(w,u), \quad w \in B(x,r).
\]
Theorem: (Harnack principle)

Suppose Assumption (A1) holds. There exist two positive constants $\delta = \delta(d, \alpha)$ and $C = C(d, \alpha, \kappa)$ such that for any $x \in \mathbb{R}^d$, $r \in (0, \delta)$ and any nonnegative bounded harmonic function $h$ with respect to $X$ in $B(x, r)$,

$$h(y_1) \leq Ch(y_2), \quad y_1, y_2 \in B(x, r/4).$$
Assumption (A2):

Suppose $D$ is a Lipschitz domain with characteristic $(R, \Lambda)$. Fix a constant $\eta \in (0, 1/4)$. For each $r \in (0, R)$ and $Q \in \partial D$, define the ”interior set” of $D \cap B(Q, r)$

$$\Omega_{\eta, r, Q} := \{ y \in D \cap B(Q, r) : \delta_{D \cap B(Q, r)}(y) > \eta r \}.$$

For each $Q \in \partial D$ and $r \in (0, R)$, define

$$\|\kappa\|_{Q, D, r} := \text{esssup}\{ \kappa(y, u - y) : y \in D \cap B(Q, r), u \in B(Q, 4r)^c \}.$$
Assumption (A2): There exist two positive constants $R_1 \in (0, R)$ and $c > 0$ such that for any $r \in (0, R_1)$, $Q \in \partial D$ and

$$u \in \{ u \in B(Q, 4r)^c : m(\Lambda_{u,D \cap B(Q,r)}) > 0 \},$$

we have

$$\frac{1}{r^d} \int_{\Omega_{\eta,2r,Q}} J^\kappa(y,u) \, dy \geq c J\|\kappa\|_{Q,D,r}(w,u), \quad w \in D \cap B(Q,r).$$

Remark:

1. The condition (A2) is restricted on large jumps of $X$ from small sets near the boundary of $D$.

2. Roughly speaking, the assumption (A2) tells that if $X$ has large jumps from $D \cap B(Q,r)$ to $B(Q,4r)^c$, then it is required that there have enough jumps from the ”interior set” $\Omega_{\eta,2r,Q}$ of $D \cap B(Q,2r)$ to $B(Q,4r)^c$. 
Assumption (A2): There exist two positive constants $R_1 \in (0, R)$ and $c > 0$ such that for any $r \in (0, R_1)$, $Q \in \partial D$ and

$$u \in \left\{ u \in B(Q, 4r)^c : m(\Lambda_{u,D \cap B(Q,r)}) > 0 \right\},$$

we have

$$\frac{1}{r^d} \int_{\Omega_{\eta,2r,Q}} J^\kappa(y,u) \, dy \geq cJ\|\kappa\|_{Q,D,r}(w,u), \quad w \in D \cap B(Q,r).$$

Remark:

1. The condition (A2) is restricted on large jumps of $X$ from small sets near the boundary of $D$.

2. Roughly speaking, the assumption (A2) tells that if $X$ has large jumps from $D \cap B(Q,r)$ to $B(Q,4r)^c$, then it is required that there have enough jumps from the ”interior set” $\Omega_{\eta,2r,Q}$ of $D \cap B(Q,2r)$ to $B(Q,4r)^c$. 
Assumption (A2): There exist two positive constants $R_1 \in (0, R)$ and $c > 0$ such that for any $r \in (0, R_1)$, $Q \in \partial D$ and

$$ u \in \{ u \in B(Q, 4r)^c : m(\Lambda_{u,D \cap B(Q,r)}) > 0 \} , $$

we have

$$ \frac{1}{r^d} \int_{\Omega_{\eta,2r,Q}} J^\kappa (y, u) \, dy \geq cJ\|\kappa\|_{Q,D,r} (w, u), \quad w \in D \cap B(Q, r). $$

Remark:

1. The condition (A2) is restricted on large jumps of $X$ from small sets near the boundary of $D$.

2. Roughly speaking, the assumption (A2) tells that if $X$ has large jumps from $D \cap B(Q, r)$ to $B(Q, 4r)^c$, then it is required that there have enough jumps from the ”interior set” $\Omega_{\eta,2r,Q}$ of $D \cap B(Q, 2r)$ to $B(Q, 4r)^c$. 
Suppose Assumptions (A1) and (A2) hold. For each Lipschitz open set $D$ with characteristics $(R, \Lambda)$, there exists a positive constant $C$ such that for all $Q \in \partial D$, $r \in (0, R]$ and all functions $h_k \geq 0$, $k = 1, 2$ on $\mathbb{R}^d$ that are harmonic with respect to $X$ in $D \cap B(Q, r)$ and vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{h_1(x)}{h_1(y)} \leq C \frac{h_2(x)}{h_2(y)}, \quad x, y \in D \cap B(Q, r/2).$$
Theorem 1: Suppose that $A > 0$. For any Lipschitz domain $D$ with characteristics $(R, \Lambda)$, there exists a positive constant $C = C(d, \alpha, R, \Lambda, A)$ such that for $Q \in \partial D, r \in (0, R], A^{-1} \leq \kappa(x, z) \leq A$ and all functions $h_k \geq 0, k = 1, 2$ on $\mathbb{R}^d$ that are harmonic with respect to $X$ in $D \cap B(Q, r)$ and vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{h_1(x)}{h_1(y)} \leq C \frac{h_2(x)}{h_2(y)}, \quad x, y \in D \cap B(Q, r/2).$$

Remark:

$L = \Delta + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$. 
Examples

**Theorem 2:** Suppose that $A > 0$. For each Lipschitz open set $D$ with characteristics $(R, \Lambda)$, suppose there exists a constant $c = c(d, R, \Lambda) > 0$ such that for any $r \in (0, R)$, $Q \in \partial D$ and $u \in \{u \in B(Q, 4r)^c : m(H_{u,D_{Q,r}}) > 0\}$,

$$\int_{\Omega_{\kappa,2r,Q}} 1_{|u-y| \leq 1}(y) \, dy \geq cr^d.$$  

Then there exists a positive constant $C = C(d, \alpha, R, \Lambda, A)$ such that for $Q \in \partial D$, $r \in (0, R]$, $A^{-1}1_{|z| \leq 1}(z) \leq \kappa(x, z) \leq A1_{|z| \leq 1}(z)$ for $x \in \mathbb{R}^d$ and all functions $h_k \geq 0$, $k = 1, 2$ on $\mathbb{R}^d$ that are harmonic with respect to $X$ in $D \cap B(Q, r)$ and vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{h_1(x)}{h_1(y)} \leq C \frac{h_2(x)}{h_2(y)}, \quad x, y \in D \cap B(Q, r/2).$$

**Remark:**

$\mathcal{L} = \Delta + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1_{\{|z| \leq 1\}}(z)$. 
Main Method

Bass-Burdzy (1989) developed a probabilistic method (also called the "box" method) to prove the BHP of Brownian motion in Lipschitz domain. Bogdan (1999) adapted this method in the symmetric stable process.

We shall use this method in our case to prove the BHP in Lipschitz domain.
An open set $D$ in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a **Lipschitz domain** if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a Lipschitz function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$
\phi(0) = \nabla \phi(0) = 0, |\phi(z_1) - \phi(z_2)| \leq \Lambda |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}^{d-1}
$$

and an orthonormal coordinate system $CS_Q : y = (y_1, \cdots, y_{d-1}, y_d) =: (\tilde{y}, y_d)$ with its origin at $Q$ such that

$$
B(Q, R) \cap D = \{ y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\tilde{y}) \}.
$$

The pair $(R, \Lambda)$ is called the characteristics of the Lipschitz domain $D$. 
”Box” Method

For each \( Q \in \partial D \) and \( x \in B(Q, R) \), define the vertical distance from \( x \) to \( \partial D \) by
\[
\rho_Q(x) = x^d - \phi_Q(x^1, \cdots, x^{d-1}),
\]
where \((x^1, \cdots, x^d)\) is the coordinate of \( x \) in \( CS_Q \).

For each \( Q \in \partial D \), define the ”box”
\[
\Delta(Q, a, R) = \{ y \in CS_Q : 0 < \rho_Q(y) < a, |(y^1, \cdots, y^{d-1})| < R \}.
\]
\[
\nabla(Q, a, R) = \{ y \in CS_Q : -a < \rho_Q(y) < 0, |(y^1, \cdots, y^{d-1})| < R \}.
\]
"Box" Method

For each $Q \in \partial D$ and $x \in B(Q, R)$, define the vertical distance from $x$ to $\partial D$ by

$$\rho_Q(x) = x^d - \phi_Q(x^1, \cdots, x^{d-1}),$$

where $(x^1, \cdots, x^d)$ is the coordinate of $x$ in $CS_Q$.

For each $Q \in \partial D$, define the "box"

$$\Delta(Q, a, R) = \{ y \in CS_Q : 0 < \rho_Q(y) < a, |(y^1, \cdots, y^{d-1})| < R \}.$$

$$\nabla(Q, a, R) = \{ y \in CS_Q : -a < \rho_Q(y) < 0, |(y^1, \cdots, y^{d-1})| < R \}.$$
Let $Q \in \partial D$. For the ”box” $\Delta(Q, a, La)$, we define

$$S_{\Delta(Q,a,La)} := \Delta(Q, a, (L + 1)a) \setminus \Delta(Q, a, La)$$
$$U_{\Delta(Q,a,La)} := \Delta(Q, 2a, (L + 1)a) \setminus \Delta(Q, a, (L + 1)a)$$
$$W_{\Delta(Q,a,La)} := [\Delta(Q, 2a, (L + 1)a) \cup \nabla(Q, 2a, (L + 1)a)]^c.$$
Let $h$ be a harmonic function in $\Delta(Q, 4r, 10r)$ and vanishes continuously on $\nabla(Q, 4r, 10r)$. Then for $x \in \Delta(Q, r, r)$,

$$h(x) = \mathbb{E}_x h(X_{\tau_{\Delta(Q,2r,8r)}})$$

$$= \mathbb{E}_x [h(X_{\tau_{\Delta(Q,2r,8r)}}); X_{\tau_{\Delta(Q,2r,8r)}} \in S_{\Delta(Q,2r,8r)}]$$

$$+ \mathbb{E}_x [h(X_{\tau_{\Delta(Q,2r,8r)}}); X_{\tau_{\Delta(Q,2r,8r)}} \in U_{\Delta(Q,2r,8r)}]$$

$$+ \mathbb{E}_x [h(X_{\tau_{\Delta(Q,2r,8r)}}); X_{\tau_{\Delta(Q,2r,8r)}} \in W_{\Delta(Q,2r,8r)}]$$
Main goal:

For $x \in \Delta(Q, r, r)$,

$$h(x) \asymp h(x_0) \mathbb{P}_x(X_{\tau_{\Delta(Q, 2r, 8r)}} \in U_{\Delta(Q, 2r, 8r)})$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$. Thus,

$$\frac{h(x)}{h(y)} \asymp \frac{\mathbb{P}_x(X_{\tau_{\Delta}} \in U_{\Delta})}{\mathbb{P}_y(X_{\tau_{\Delta}} \in U_{\Delta})}, \quad x, y \in \Delta(Q, r, r).$$

That is, the BHP holds.
**Proposition:** (Key estimates)

There exists $c > 0$ such that for any $Q \in \partial D$ and $x \in \Delta(Q, r, r)$, we have

$$
\mathbb{P}_x(X_{\tau_{\Delta(Q, 2r, 8r)}} \in S_{\Delta(Q, 2r, 8r)}) \leq c \mathbb{P}_x(X_{\tau_{\Delta(Q, 2r, 8r)}} \in U_{\Delta(Q, 2r, 8r)}).
$$

**Lemma 1:**

There exist positive constants $c_k, k = 0, 1$ such that for every $Q \in \partial D$, $a < \delta$, $L \geq 1$ and $x \in \Delta(Q, a, La/2)$

$$
\mathbb{P}_x(X_{\tau_{\Delta(Q, a, La)}} \in S_{\Delta(Q, a, La)}) \leq c_1 \exp(-c_0 L) + c_1 a^{2-\alpha} (L + 1)^{-(1+\alpha)}.
$$

**Lemma 2:**

There exist positive constants $c$ and $\beta$ such that for every $Q \in \partial D$, $a < \delta$, $L \geq 1$ and $y \in \Delta(Q, a/2, a)$, we have

$$
\mathbb{P}_y(X_{\tau_{\Delta(Q, a, La)}} \in U_{\Delta(Q, a, La)}) \geq c \left(\frac{\delta_D(y)}{a}\right)^\beta + c(\delta_D(y))^{2-\alpha} \left(\frac{\delta_D(y)}{a}\right)^\alpha.
$$
Proposition: (Key estimates)

There exists $c > 0$ such that for any $Q \in \partial D$ and $x \in \Delta(Q, r, r)$, we have

$$\mathbb{P}_x(X_{\tau_{\Delta(Q,2r,8r)}} \in S_{\Delta(Q,2r,8r)}) \leq c\mathbb{P}_x(X_{\tau_{\Delta(Q,2r,8r)}} \in U_{\Delta(Q,2r,8r)}) .$$

Lemma 1:

There exist positive constants $c_k$, $k = 0, 1$ such that for every $Q \in \partial D$, $a < \delta$, $L \geq 1$ and $x \in \Delta(Q, a, La/2)$

$$\mathbb{P}_x(X_{\tau_{\Delta(Q,a,La)}} \in S_{\Delta(Q,a,La)}) \leq c_1 \exp(-c_0L) + c_1a^{2-\alpha}(L + 1)^{-(1+\alpha)} .$$

Lemma 2:

There exist positive constants $c$ and $\beta$ such that for every $Q \in \partial D$, $a < \delta$, $L \geq 1$ and $y \in \Delta(Q, a/2, a)$, we have

$$\mathbb{P}_y(X_{\tau_{\Delta(Q,a,La)}} \in U_{\Delta(Q,a,La)}) \geq c \left( \frac{\delta_D(y)}{a} \right)^\beta + c(\delta_D(y))^{2-\alpha} \left( \frac{\delta_D(y)}{a} \right)^\alpha .$$
Proposition: (Key estimates)

There exists $c > 0$ such that for any $Q \in \partial D$ and $x \in \Delta(Q, r, r)$, we have

$$
P_x(X_{\tau\Delta(Q,2r,8r)} \in S_{\Delta(Q,2r,8r)}) \leq cP_x(X_{\tau\Delta(Q,2r,8r)} \in U_{\Delta(Q,2r,8r)}).
$$

Lemma 1:

There exist positive constants $c_k, k = 0, 1$ such that for every $Q \in \partial D, a < \delta, L \geq 1$ and $x \in \Delta(Q, a, La/2)$

$$
P_x(X_{\tau\Delta(Q,a,La)} \in S_{\Delta(Q,a,La)}) \leq c_1 \exp(-c_0L) + c_1 a^{2-\alpha}(L + 1)^{-(1+\alpha)}.
$$

Lemma 2:

There exist positive constants $c$ and $\beta$ such that for every $Q \in \partial D, a < \delta, L \geq 1$ and $y \in \Delta(Q, a/2, a)$, we have

$$
P_y(X_{\tau\Delta(Q,a,La)} \in U_{\Delta(Q,a,La)}) \geq c \left(\frac{\delta_D(y)}{a}\right)^\beta + c(\delta_D(y))^{2-\alpha} \left(\frac{\delta_D(y)}{a}\right)^\alpha.
$$
Proposition: (Carleson estimate)

There exists a constant $c$ such that for any $0 < r < R, Q \in \partial D$ and any nonnegative harmonic function $h$ in $D \cap B(Q, r)$ which vanishes in $D^c \cap B(Q, r)$, we have

$$h(x) \leq ch(x_0), \quad x \in D \cap B(Q, r/2),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$.

Estimates in small range

Let $\Delta := \Delta(Q, 2r, 8r)$. For $x \in \Delta(Q, r, r)$,

$$\mathbb{E}_x[h(X_{\tau_{\Delta}}), X_{\tau_{\Delta}} \in S_{\Delta}] + \mathbb{E}_x[h(X_{\tau_{\Delta}}), X_{\tau_{\Delta}} \in U_{\Delta}] \leq c_1 h(x_0) (\mathbb{P}_x[X_{\tau_{\Delta}} \in S_{\Delta}] + \mathbb{P}_x[X_{\tau_{\Delta}} \in U_{\Delta}]) \leq c_2 h(x_0) \mathbb{P}_x[X_{\tau_{\Delta}} \in U_{\Delta}].$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$. 
**Proposition: (Carleson estimate)**

There exists a constant $c$ such that for any $0 < r < R$, $Q \in \partial D$ and any nonnegative harmonic function $h$ in $D \cap B(Q, r)$ which vanishes in $D^c \cap B(Q, r)$, we have

$$h(x) \leq ch(x_0), \quad x \in D \cap B(Q, r/2),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$.

---

**Estimates in small range**

Let $\Delta := \Delta(Q, 2r, 8r)$. For $x \in \Delta(Q, r, r)$,

$$\mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in S_\Delta] + \mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in U_\Delta] \leq c_1 h(x_0)(\mathbb{P}_x[X_{\tau_\Delta} \in S_\Delta] + \mathbb{P}_x[X_{\tau_\Delta} \in U_\Delta]) \leq c_2 h(x_0)\mathbb{P}_x[X_{\tau_\Delta} \in U_\Delta].$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$. 
Estimates for large jumps

Suppose $D$ is an open set in $\mathbb{R}^d$ and $h$ is a non-negative function, we have

$$
\mathbb{E}_x[h(X_{\tau_D}), X_{\tau_D} \neq X_{\tau_D}] = \int_{D^c} \int_D G_D^X(x, y) \frac{\kappa(y, u - y)}{|y - u|^{d+\alpha}} dy h(u) du.
$$

Let $\Delta := \Delta(Q, 2r, 8r)$. Hence, for $x \in \Delta(Q, r, r)$,

$$
\mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in W_\Delta] = \int_{W_\Delta} \int_\Delta G_\Delta^X(x, y) \frac{\kappa(y, u - y)}{|y - u|^{d+\alpha}} h(u) dy du.
$$

By the assumption (A2), for $x \in \Delta(Q, r, r)$,

$$
\mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in W_\Delta] \leq c h(x_0) \mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta),
$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$. 

Estimates for large jumps

Suppose $D$ is an open set in $\mathbb{R}^d$ and $h$ is a non-negative function, we have

$$
\mathbb{E}_x[h(X_{\tau_D}), X_{\tau_D} \neq X_{\tau_D}] = \int_{D^c} \int_D G_D^x(x, y) \frac{\kappa(y, u - y)}{|y - u|^{d+\alpha}} dy h(u) du.
$$

Let $\Delta := \Delta(Q, 2r, 8r)$. Hence, for $x \in \Delta(Q, r, r)$,

$$
\mathbb{E}_x[h(X_{\tau_{\Delta}}), X_{\tau_{\Delta}} \in W_\Delta] = \int_{W_\Delta} \int_\Delta G_{\Delta}^x(x, y) \frac{\kappa(y, u - y)}{|y - u|^{d+\alpha}} h(u) dy du.
$$

By the assumption $(A2)$, for $x \in \Delta(Q, r, r)$,

$$
\mathbb{E}_x[h(X_{\tau_{\Delta}}), X_{\tau_{\Delta}} \in W_\Delta] \leq c h(x_0) \mathbb{P}_x(X_{\tau_{\Delta}} \in U_\Delta),
$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$. 

Proof of BHP

1. Let $h$ be a harmonic function in $\Delta(Q, 4r, 10r)$ and vanishes continuously on $\nabla(Q, 4r, 10r)$. Let $\Delta := \Delta(Q, 2r, 8r)$. Then for $x \in \Delta(Q, r, r)$,

$$h(x) = \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in S_\Delta] + \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in U_\Delta]$$

$$+ \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in W_\Delta]$$

$$\leq ch(x_0)\mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$.

2. On the other hand, by the assumption $(A1)$ and the Harnack principle,

$$h(x) \geq \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in U_\Delta]$$

$$\geq ch(x_0)\mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta).$$

3. Hence, $h(x) \asymp h(x_0)\mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta)$. 
Proof of BHP

1. Let $h$ be a harmonic function in $\Delta(Q, 4r, 10r)$ and vanishes continuously on $\nabla(Q, 4r, 10r)$. Let $\Delta := \Delta(Q, 2r, 8r)$. Then for $x \in \Delta(Q, r, r)$,

$$h(x) = \mathbb{E}_x[h(X_{\tau_{\Delta}}); X_{\tau_{\Delta}} \in S_{\Delta}] + \mathbb{E}_x[h(X_{\tau_{\Delta}}); X_{\tau_{\Delta}} \in U_{\Delta}]$$

$$+ \mathbb{E}_x[h(X_{\tau_{\Delta}}); X_{\tau_{\Delta}} \in W_{\Delta}]$$

$$\leq c h(x_0) \mathbb{P}_x(X_{\tau_{\Delta}} \in U_{\Delta}),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$.

2. On the other hand, by the assumption (A1) and the Harnack principle,

$$h(x) \geq \mathbb{E}_x[h(X_{\tau_{\Delta}}); X_{\tau_{\Delta}} \in U_{\Delta}]$$

$$\geq c h(x_0) \mathbb{P}_x(X_{\tau_{\Delta}} \in U_{\Delta}).$$

3. Hence, $h(x) \asymp h(x_0) \mathbb{P}_x(X_{\tau_{\Delta}} \in U_{\Delta})$. 

Boundary Harnack Principle for Diffusion with
1. Let $h$ be a harmonic function in $\Delta(Q, 4r, 10r)$ and vanishes continuously on $\nabla(Q, 4r, 10r)$. Let $\Delta := \Delta(Q, 2r, 8r)$. Then for $x \in \Delta(Q, r, r)$,

$$h(x) = \mathbb{E}_x [h(X_{\tau_\Delta}); X_{\tau_\Delta} \in S_\Delta] + \mathbb{E}_x [h(X_{\tau_\Delta}); X_{\tau_\Delta} \in U_\Delta]$$

$$+ \mathbb{E}_x [h(X_{\tau_\Delta}); X_{\tau_\Delta} \in W_\Delta]$$

$$\leq c h(x_0) \mathbb{P}_x (X_{\tau_\Delta} \in U_\Delta),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$.

2. On the other hand, by the assumption $(A1)$ and the Harnack principle,

$$h(x) \geq \mathbb{E}_x [h(X_{\tau_\Delta}); X_{\tau_\Delta} \in U_\Delta]$$

$$\geq c h(x_0) \mathbb{P}_x (X_{\tau_\Delta} \in U_\Delta).$$

3. Hence, $h(x) \asymp h(x_0) \mathbb{P}_x (X_{\tau_\Delta} \in U_\Delta)$. 


Thank you!