Jump Type Stochastic Differential Equations with Non-Lipschitz Coefficients and Feller and Strong Feller Properties

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Introduction

Non-Explosion

Pathwise Uniquness

Non Confluence Property

Feller and Strong Feller Properties

Multidimensional SDEs with Jumps

Consider the stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) + \int_U c(X(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u), \\ X(0) = x \in \mathbb{R}^d, \qquad d \geq 2 \end{cases} \tag{SDE}$$

where

- $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ and $c : \mathbb{R}^d \times U \mapsto \mathbb{R}^d$ are Borel measurable functions,
- W is a standard d-dimensional Brownian motion,
- ν is a σ-finite measure on a measurable space $(U, \mathfrak{U})$,
- N is a Poisson random measure on $[0, \infty) \times U$ with intensity $\mathrm{d}t\, \nu(\mathrm{d}u)$,
- $\widetilde{N}(\mathrm{d}t, \mathrm{d}u) := N(\mathrm{d}t, \mathrm{d}u) - \mathrm{d}t\, \nu(\mathrm{d}u)$.

The Classical Existence and Uniqueness Result

Theorem ([Ikeda and Watanabe, 1989])

Suppose there exists some $K > 0$ *such that*

$$\begin{aligned} &|\sigma(x)|^2 + |b(x)|^2 + \int_U |c(x,u)|^2 \nu(\mathrm{d}u) \leq K(1 + |x|^2), \\ &|\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)|^2 \\ &\qquad + \int_U |c(x,u) - c(y,u)|^2 \nu(\mathrm{d}u) \leq K|x - y|^2, \end{aligned}$$

for all $x, y \in \mathbb{R}^d$, *then for any initial condition* $x \in \mathbb{R}^d$, (SDE) *has a unique non-exploding strong solution* X^x.

Our Goals

1. Linear growth $\to$ super linear growth, but still no explosion in finite time?
2. Lipschitz $\to$ non-Lipschitz, but still pathwise unique?
3. The non confluence property:

$$x_0 \neq y_0 \Longrightarrow \mathbb{P}\{X^{x_0}(t) \neq X^{y_0}(t), \forall t \geq 0\} = 1.$$

4. Feller and strong Feller properties under non-Lipschitz conditions?
5. Well-posedness for SDEs driven by Lévy processes:

$$\mathrm{d}X(t) = \psi(X(t-))\mathrm{d}L(t),$$

   where $L \in \mathbb{R}^d$ is a Lévy process, and $\psi : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$ is measurable.
6. Feynman-Kac formula for a Cauchy problem associated with a Lévy type operator.

Relevant literature

- Yamada, T. and Watanabe, S. (1971). On the uniqueness of solutions of stochastic differential equations. I and II, *J. Math. Kyoto Univ.*

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  2. Fu, Z. and Li, Z. (2010). Stochastic equations of non-negative processes with jumps. *Stochastic Process. Appl.*
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- Multi-dimensional SDEs without jumps
  1. Fang, S. and Zhang, T. (2005). A study of a class of stochastic differential equations with non-Lipschitzian coefficients. *Probab. Theory Related Fields*.
  2. Lan, G. and Wu, J.-L. (2014). New sufficient conditions of existence, moment estimations and non confluence for SDEs with non-Lipschitzian coefficients. *Stochastic Process. Appl.*

A Multidimensional Lotka-Volterra Model

- Mao, X., Marion, G., and Renshaw, E. (2002). Environmental Brownian noise suppresses explosions in population dynamics. *Stochastic Process. Appl.*

$$\mathrm{d}x_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j(t)\mathrm{d}t\right) + \sum_{j=1}^{n} \sigma_{ij}x_j\mathrm{d}w_j(t), i = 1, \ldots, n.$$

- In this model, the coefficients are locally Lipschitz but do not satisfy the linear growth condition.
- It is shown that a unique non-exploding solution exists. The special structure of the model is crucial to obtain non-explosion.
- Many other related work in this area.

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- Many other related work in this area.
- Criteria for non-explosion for general SDEs?

Non-explosion: General Criteria

Theorem ([Khasminskii, 2012, Meyn and Tweedie, 1993])

Suppose there exists a nonnegative C^2 function V satisfying

(i) $\mathcal{A}V(x) \leq cV(x)$, *for all* $x \in \mathbb{R}^d$

(ii) $\inf_{|x|\geq R} V(x) \to \infty$ *as* $R \to \infty$,

then the solution to the SDE is non-explosive with probability one.

1. Khasminskii, R. (2012). *Stochastic stability of differential equations*, volume 66 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2nd edition.
2. Meyn, S. P. and Tweedie, R. L. (1993). Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*

Non-Explosion: Multi-dimensional Diffusion Process

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) \tag{1}$$

Theorem ([Fang and Zhang, 2005])

Let ρ be a strictly positive, C^1-function defined on a neighborhood $[K, \infty)$ of ∞, satisfying

$$\lim_{s\to\infty} \rho(s) = \infty, \quad \lim_{s\to\infty} \frac{s\rho'(s)}{\rho(s)} = 0, \quad \int_K^\infty \frac{\mathrm{d}s}{s\rho(s)+1} = \infty.$$

Assume that for $|x| \geq K$,

$$|b(x)| \leq H(|x|\rho(|x|^2)+1), \quad |\sigma(x)|^2 \leq H(|x|^2\rho(|x|^2)+1),$$

then any solution to (1) will not explode in finite time a.s.

- Example: $\rho(s) = \log s$ for s large.

Non-Explosion: Multi-dimensional Jump Diffusion Process

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) + \int_U c(X(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u), \quad \text{(SDE)}$$

Theorem

Suppose there exists a nondecreasing function $\zeta : [0, \infty) \mapsto [1, \infty)$ *that is continuously differentiable and satisfies*

$$\int_0^\infty \frac{\mathrm{d}r}{r\zeta(r) + 1} = \infty,$$

such that for all $x \in \mathbb{R}^d$, *we have for some* $\kappa > 0$

$$2\langle x, b(x)\rangle + |\sigma(x)|^2 + \int_U |c(x, u)|^2 \nu(\mathrm{d}u) \leq \kappa[|x|^2\zeta(|x|^2) + 1],$$

then any solution to (SDE) *has no finite explosion time a.s.*

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then any solution to (SDE) *has no finite explosion time a.s.*

Examples: $\zeta(r) = 1$ and $\zeta(r) = \log r$ for r large.

Pathwise Uniquness under Lipschitz Condition

Suppose $X, \widetilde{X}$ are solutions with the same initial condition, then

$$\begin{aligned}\Delta_t &:= \widetilde{X}(t) - X(t) \\ &= \int_0^t [b(\widetilde{X}(t)) - b(X(t))]\mathrm{d}t + \int_0^t [\sigma(\widetilde{X}(t)) - \sigma(X(t))]\mathrm{d}W(t) \\ &\quad + \int_0^t \int_U [c(\widetilde{X}(t-), u) - c(X(t-), u)]\widetilde{N}(\mathrm{d}t, \mathrm{d}u).\end{aligned}$$

Lipschitz condition + the classical argument $\Longrightarrow$

$$\mathbb{E}[|\Delta_t|^2] \leq K \int_0^t \mathbb{E}[|\Delta_s|^2]\mathrm{d}s.$$

Then Gronwall's inequality implies that $\mathbb{E}[|\Delta_t|^2] = 0$, which, in turn, implies the pathwise uniqueness.

Yamada and Watanabe's Result

Consider the 1-d SDE:

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t).$$

Assume

(i) $\exists$ a positive increasing function $\rho(u), u \in (0, \infty)$ s.t.

$$|\sigma(x) - \sigma(y)| \leq \rho(|x - y|), \quad \int_{0+} \frac{\mathrm{d}u}{\rho^2(u)} = \infty,$$

(ii) $\exists$ a positive increasing concave function $\kappa(u), u \in (0, \infty)$ s.t.

$$|b(x) - b(y)| \leq \kappa(|x - y|), \quad \int_{0+} \frac{\mathrm{d}u}{\kappa(u)} = \infty.$$

Then pathwise uniqueness holds.

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$$|b(x) - b(y)| \leq \kappa(|x - y|), \quad \int_{0+} \frac{\mathrm{d}u}{\kappa(u)} = \infty.$$

Then pathwise uniqueness holds.

Example: $\sigma(x) = x^\alpha$, $\alpha \geq \frac{1}{2}$.

The Idea

- Construct a sequence of even and C^2 functions $\psi_n : \mathbb{R} \mapsto [0, \infty)$ satisfying

$$\lim_{n\to\infty} \psi_n(x) = |x|, x \in \mathbb{R}$$

- Suppose $X, \widetilde{X}$ are solutions with the same initial condition and denote $\Delta_t := \widetilde{X}(t) - X(t)$. Show that

$$\mathbb{E}[\psi_n(\Delta_t)] \le K \int_0^t \kappa(\mathbb{E}[|\Delta_s|])\mathrm{d}s + \frac{t}{n}.$$

- Passing to the limit as $n \to \infty$ and use Bihari's inequality to obtain $\mathbb{E}[|\Delta_t|] = 0$.

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- Passing to the limit as $n \to \infty$ and use Bihari's inequality to obtain $\mathbb{E}[|\Delta_t|] = 0$.

This idea was further developed in [Fu and Li, 2010], [Li and Mytnik, 2011], [Li and Pu, 2012], [Fournier, 2013] and others to establish pathwise uniqueness for 1-d jump type SDEs.

Multi-Dimensional SDEs without Jumps

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t).$$

Theorem ([Fang and Zhang, 2005])

Let r be a strictly positive, C^1- function defined on a neighborhood $(0, c_0]$ of 0 satisfying

$$\lim_{s\downarrow 0} r(s) = +\infty, \quad \lim_{s\downarrow 0} \frac{sr'(s)}{r(s)} = 0, \quad \int_{0+} \frac{\mathrm{d}s}{sr(s)} = \infty.$$

Assume that for $|x - y| \leq c_0$

$$|\sigma(x) - \sigma(y)|^2 \leq K|x - y|^2 r(|x - y|^2),$$
$$|b(x) - b(y)| \leq K|x - y| r(|x - y|).$$

Then pathwise uniqueness holds.

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Assume that for $|x - y| \leq c_0$

$$|\sigma(x) - \sigma(y)|^2 \leq K|x-y|^2 r(|x-y|^2),$$
$$|b(x) - b(y)| \leq K|x-y| r(|x-y|).$$

Then pathwise uniqueness holds.

Examples: $r(s) = \log(1/s)$, $r(s) = \log(1/s)\log\log(1/s)$.

Idea of Proof

Note that modulus of continuity only required to hold on a small neighborhood of the diagonal line $x = y$.

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- For $\delta > 0$, define $\Phi_\delta(\xi) := e^{\int_0^\xi \frac{\mathrm{d}s}{sr(s)+\delta}}, \xi \in (0, c_0]$.
- Let $\widetilde{X}, X$ and Δ_t as before. Let $\tau := \inf\{t \geq 0 : |\Delta_t| \geq c_0\}$.
- Show that

$$\mathbb{E}[\Phi_\delta(|\Delta_{t\wedge\tau}|)] \leq \Phi_\delta(|\Delta_0|^2) + K \int_0^t \mathbb{E}[\Phi_\delta(|\Delta_{s\wedge\tau}|)]\mathrm{d}s$$

  and hence when $\Delta_0 = 0$, we have $\mathbb{E}[\Phi_\delta(|\Delta_{t\wedge\tau}|)] \leq e^{Kt}$.
- Now let $\delta \downarrow 0$ to obtain

$$\mathbb{E}[e^{\int_0^{|\Delta_{t\wedge\tau}|} \frac{\mathrm{d}s}{sr(s)}}] \leq e^{Kt} \implies \Delta_{t\wedge\tau} = 0 \text{ a.s.}$$

- Finally prove that $\tau = \infty$ a.s. and hence $\Delta_t = 0$ a.s.

Pathwise Uniqueness: Multi-Dimensional SDEs with Jumps

Theorem

Suppose there exist a positive number δ_0 *and a nondecreasing and concave function* $\varrho : [0, \infty) \mapsto [0, \infty)$ *satisfying*

$$0 < \frac{\varrho(r)}{(1+r)^2} \le \varrho(r/(1+r)) \text{ for all } r > 0, \text{ and } \int_{0+} \frac{\mathrm{d}r}{\varrho(r)} = \infty,$$

such that for all $x, z \in \mathbb{R}^d$ *with* $|x - z| \le \delta_0$

$$2\langle x - z,\, b(x) - b(z)\rangle + |\sigma(x) - \sigma(z)|^2 + \int_U |c(x,u) - c(z,u)|^2 \nu(\mathrm{d}u) \le \kappa\varrho(|x - z|^2), \qquad (2)$$

where κ *is a positive constant. Then pathwise uniqueness holds for* (SDE) *up to a possibly finite explosion time.*

Remarks

- The modulus of continuity only required to hold on a small neighborhood of the diagonal line $x = y$.
- Examples: $\varrho(r) = r$, $\varrho(r) = r\log(1/r)$ and $\varrho(r) = r\log(\log(1/r))$ for $r \in (0, \delta)$ with $\delta > 0$ small.
- A new proof is developed to establish the pathwise uniqueness.
- Let $\widetilde{X}, X$ be two solutions to (SDE) with the same initial condition. Let $\Delta_t := \widetilde{X}(t) - X(t)$, $t \geq 0$.
- The key idea is to use a *single* smooth function to obtain

$$\mathbb{P}\{\Delta_t = 0\} = 1.$$

Idea of proof

- Let $H(r) := \frac{r^2}{1+r^2}$ for $r \geq 0$.
- Define
$$S_{\delta_0} := \inf\{t \geq 0 : |\Delta_t| \geq \delta_0\}.$$
- Show that
$$\mathbb{E}[H(|\Delta_{t\wedge S_{\delta_0}}|)] \leq \kappa \int_0^t \varrho\big(\mathbb{E}[H(|\Delta_{s\wedge S_{\delta_0}}|)]\big)\mathrm{d}s$$
- Bihari's inequality implies that $\mathbb{E}[H(|\Delta_{t\wedge S_{\delta_0}}|)] = 0$.
- Next show that $\mathbb{E}[H(|\Delta_t|)] = 0$.

Application: SDEs Driven by Lévy Processes

- Consider the stochastic differential equation

$$\mathrm{d}X(t) = \psi(X(t-))\mathrm{d}L(t), \quad X(0) = x \in \mathbb{R}^d, \tag{3}$$

where the function $\psi : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ is Borel measurable and $L \in \mathbb{R}^d$ is a Lévy process with triplet (b, Q, ν).

- $b \in \mathbb{R}^d, Q \in \mathcal{S}^{d\times d}$ is nonnegative definite, and ν is a Lévy measure on $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}_0^d} 1 \wedge |u|^2 \nu(\mathrm{d}u) < \infty$.
- Example: $b = 0$, $Q = 0$, and $\nu(\mathrm{d}x) = \frac{\mathrm{d}x}{|x|^{d+\alpha}}$ $(\alpha \in (0,2))$. Thus L is a d-dimensional symmetric stable process of order α.
- Question: When does (3) have a non-exploding unique strong solution?

- L admits the Lévy-Itô decomposition

$$L(t) = bt + \sigma W(t) + \int_{\mathbb{R}_0^d} u I_{\{|u|\leq 1\}} \widetilde{N}(t, \mathrm{d}u) + \int_{\mathbb{R}_0^d} u I_{\{|u|>1\}} N(t, \mathrm{d}u),$$

- So (3) can be rewritten as

$$\begin{aligned}
\mathrm{d}X(t) = \psi(X(t-))b\mathrm{d}t + \psi(X(t-))\sigma\mathrm{d}W(t) \\
+ \int_{\{|u|\leq 1\}} \psi(X(t-))u\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\
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- So (3) can be rewritten as

$$\begin{aligned} \mathrm{d}X(t) = \psi(X(t-))b\mathrm{d}t + \psi(X(t-))\sigma\mathrm{d}W(t) \\ + \int_{\{|u|\leq 1\}} \psi(X(t-))u\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\ + \int_{\{|u|>1\}} \psi(X(t-))uN(\mathrm{d}t, \mathrm{d}u). \end{aligned}$$

- If $|\psi(x)|^2 \leq K(|x|^2\zeta(|x|^2) + 1)$, for all $x \in \mathbb{R}^d$, then the solution to (3) has no finite explosion time a.s.
- If $|\psi(x) - \psi(z)|^2 \leq K\varrho(|x-z|^2)$, for all $x, z \in \mathbb{R}^d$ with $|x - z| \leq \delta_0$, where $\varrho(r) = r, r\log(1/r)$ or $r\log(\log(1/r))$ for $r > 0$ small, then pathwise uniqueness holds for (3).

Non Confluence Property

- We say that the solution X of (SDE) has *non confluence property*, if

$$\widetilde{x} \neq x \Longrightarrow \mathbb{P}\{X^{\widetilde{x}}(t) \neq X^{x}(t), \forall t \geq 0\} = 1.$$

- In the 1-d case, this leads to the comparison theorem: If $\widetilde{x} \leq x$, then $X^{\widetilde{x}}(t) \leq X^{x}(t), \forall t \geq 0$ a.s.

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Related work:

1. Emery, M. (1981). Non confluence des solutions d'une équation stochastique lipschitzienne.
2. Yamada, T. and Ogura, Y. (1981). On the strong comparison theorems for solutions of stochastic differential equations. *Z. Wahrsch. Verw. Gebiete*.
3. Fang, S. and Zhang, T. (2005). A study of a class of stochastic differential equations with non-Lipschitzian coefficients. *Probab. Theory Related Fields*.
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A Sufficient Condition for Non Confluence Property

Suppose that for all $x, z \in \mathbb{R}^d$ and $u \in U$,

$$|x - z + c(x,u) - c(z,u)| \geq |x - z|,$$

and that

$$K|x-z|^2 \geq 2|\langle x - z, b(x) - b(z)\rangle| + |\sigma(x) - \sigma(z)|^2 + \int_U \left[|c(x,u) - c(z,u)|^2\right]\nu(\mathrm{d}u)$$

where $K > 0$ is a constant, then the non confluence property for (SDE) holds up to its explosion time.

Feller and Strong Feller Properties

- Suppose that (SDE) has a non-exploding weak solution and that the solution is unique in the weak sense.
- Consider

$$P_t f(x) := \mathbb{E}[f(X^x(t))], \ f \in \mathfrak{B}_b(\mathbb{R}^d), t \geq 0.$$

- We say that (SDE) satisfies
  1. the Feller property if $P_t f$ is continuous for every $f \in C_b(\mathbb{R}^d)$ and $t \geq 0$;
  2. the strong Feller property if $P_t f$ is continuous for every $f \in \mathfrak{B}_b(\mathbb{R}^d)$ and $t > 0$.
- Goal: To establish non-Lipschitz sufficient condition for Feller and Strong Feller Properties.

Related References

1. Chen, M.-F. (2004). *From Markov chains to non-equilibrium particle systems*. World Scientific Publishing Co. Inc., River Edge, NJ, second edition.
2. Chen, M. F. and Li, S. F. (1989). Coupling methods for multidimensional diffusion processes. *Ann. Probab.*, 17(1):151–177.
3. Priola, E. and Wang, F.-Y. (2006). Gradient estimates for diffusion semigroups with singular coefficients. *J. Funct. Anal.*, 236(1):244–264.
4. Wang, J. (2010). Regularity of semigroups generated by Lévy type operators via coupling. *Stochastic Process. Appl.*, 120(9):1680–1700.

Feller Property

Theorem

Suppose there exist a positive constant δ_0 and a nondecreasing and concave function $\theta : [0, \infty) \mapsto [0, \infty)$ satisfying

$$0 < \theta(r) \leq (1+r)^2 \theta\left(\frac{r}{1+r}\right) \text{ for all } r > 0, \text{ and } \int_{0+} \frac{\mathrm{d}r}{\theta(r)} = \infty,$$

such that for all $x, z \in \mathbb{R}^d$ with $|x - z| \leq \delta_0$

$$\int_U \left[|c(x,u) - c(z,u)|^2 \wedge (4|x-z| \cdot |c(x,u) - c(z,u)|) \right] \nu(\mathrm{d}u) + 2\langle x - z, b(x) - b(z) \rangle + |\sigma(x) - \sigma(z)|^2 \leq 2\kappa_0 |x - z| \theta(|x - z|),$$

Then the process X is Feller continuous.

Remarks

- Example: $\theta(r) = r, r \log(1/r)$, or $r \log(\log(1/r))$ for $r \in (0, \delta_0]$.
- The continuity condition only required to hold in a small neighborhood of the diagonal line $x = z$.
- Proof by the coupling method.

The Basic Idea

- Denote by $\{P(t,x,A) : t \geq 0, x \in \mathbb{R}^d, A \in \mathfrak{B}(\mathbb{R}^d)\}$ the transition probability family of the process X.
- To obtain the Feller property, we need to show that $\forall t \geq 0$, $P(t,x,\cdot)$ converges weakly to $P(t,z,\cdot)$ as $x \to z$.
- It suffices to prove that

$$W_1(P(t,x,\cdot), P(t,z,\cdot)) \to 0 \text{ as } x \to z,$$

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- To obtain the Feller property, we need to show that $\forall t \geq 0$, $P(t,x,\cdot)$ converges weakly to $P(t,z,\cdot)$ as $x \to z$.
- It suffices to prove that

$$W_1(P(t,x,\cdot), P(t,z,\cdot)) \to 0 \text{ as } x \to z,$$

For two probability measures P_1 and P_2 on a complete separable metric space (E,d), define

$$W_1(P_1,P_2) := \inf_{\widetilde{P}} \int d(x,z)\widetilde{P}(\mathrm{d}x,\mathrm{d}z),$$

where $\widetilde{P}$ varies over all coupling probability measures with marginals P_1 and P_2, that is, for any $A \in \mathfrak{B}(E)$, we have

$$\widetilde{P}(A \times E) = P_1(A), \text{ and } \widetilde{P}(E \times A) = P_2(A).$$

The Basic Coupling Operator

- For $x, z \in \mathbb{R}^d$, we set $a(x) := \sigma(x)\sigma(x)^T$ and
$$a(x,z) = \begin{pmatrix} a(x) & \sigma(x)\sigma(z)^T \\ \sigma(z)\sigma(x)^T & a(z) \end{pmatrix}, \quad b(x,z) = \begin{pmatrix} b(x) \\ b(z) \end{pmatrix}.$$
- Next for $f(x,z) \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$, define
$$\widetilde{\mathcal{L}} f(x,z) := \left[\widetilde{\Omega}_{\mathrm{d}} + \widetilde{\Omega}_{\mathrm{j}}\right] f(x,z), \tag{4}$$
where
$$\widetilde{\Omega}_{\mathrm{d}} f(x,z) = \frac{1}{2}\mathrm{tr}\big(a(x,z)D^2 f(x,z)\big) + \langle b(x,z), Df(x,z)\rangle,$$
$$\widetilde{\Omega}_{\mathrm{j}} f(x,z) = \int_U [f(x + c(x,u), z + c(z,u)) - f(x,z) \\ - \langle D_x f(x,z), c(x,u)\rangle - \langle D_z f(x,z), c(z,u)\rangle]\nu(\mathrm{d}u).$$

The Sketch of Proof

- Let $(\widetilde{X}, \widetilde{Z})$ be the coupling process corresponding to the operator $\widetilde{\mathcal{L}}$ of (5) with $(\widetilde{X}(0), \widetilde{Z}(0)) = (x, z)$, where $\delta_0 > |x - z| > 0$.
- We have

$$W_1(P(t,x,\cdot), P(t,z,\cdot)) \le \mathbb{E}[|\widetilde{X}(t) - \widetilde{Z}(t)|].$$

- Show that $\mathbb{E}[|\widetilde{X}(t) - \widetilde{Z}(t)|] \to 0$ as $z - x \to 0$ This gives the Feller property.

Strong Feller Property

Theorem

Suppose

(i) $\exists$ *a* $\lambda_0 > 0$ *s.t.* $\langle \xi, a(x)\xi \rangle \geq \lambda_0 |\xi|^2$, $\forall x, \xi \in \mathbb{R}^d$, *where* $a(x) := \sigma(x)\sigma(x)^T$. *Let* $\sigma_{\lambda_0} \in \mathcal{S}^{d\times d}$ *be s.t.* $\sigma_{\lambda_0}^2 = a - \lambda_0 I$.

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(ii) $\exists$ $\delta_0 > 0, \kappa_0 > 0$ *and* $\vartheta : [0, \delta_0] \mapsto [0, \infty)$ *satisfying* $\lim_{r\to 0} \vartheta(r) = 0$ *s.t. for all* $x, z \in \mathbb{R}^d$ *with* $|x - z| \leq \delta_0$*:*

$$\int_U \big[|c(x,u) - c(z,u)|^2 \wedge (4|x-z| \cdot |c(x,u) - c(z,u)|)\big]\nu(\mathrm{d}u)$$

$$+ 2\big\langle x - z, b(x) - b(z)\big\rangle + |\sigma_{\lambda_0}(x) - \sigma_{\lambda_0}(z)|^2 \leq 2\kappa_0 |x - z| \vartheta(|x - z|),$$

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$$+ 2\Big\langle x - z, b(x) - b(z) \Big\rangle + |\sigma_{\lambda_0}(x) - \sigma_{\lambda_0}(z)|^2 \leq 2\kappa_0 |x-z| \vartheta(|x-z|),$$

Then $\forall t > 0$, $f \in \mathfrak{B}_b(\mathbb{R}^d)$, *there exists a* $K = K(t, \delta_0, \kappa_0) > 0$ *s.t.*

$$\sup_{x\neq z} \frac{|P_t f(x) - P_t f(z)|}{|x - z|} \leq K \|f\|_\infty.$$

In particular, X *of* (SDE) *is strong Feller continuous.*

- Again, the continuity condition only required to hold in a small neighborhood of the diagonal hyperplane $x = z$ of $\mathbb{R}^d \times \mathbb{R}^d$.
- In the case when $c \equiv 0$, strong Feller property holds as long as the functions b and σ_{λ_0} are Hölder continuous with exponents $\delta_b > 0$ and $\delta_{\sigma_{\lambda_0}} > \frac{1}{2}$, respectively.
- Proof by the coupling method.

Lévy Type Operator

- Let $b(x) \in \mathbb{R}^d$, $a(x) = (a_{jk}(x)) \in \mathcal{S}^{d\times d}$ be measurable. Suppose $a(x)$ is nonnegative definite for all $x \in \mathbb{R}^d$.
- $\nu(x, \mathrm{d}y)$ is a Lévy measure satisfying $\int_{\mathbb{R}_0^d} |y| \wedge |y|^2 \nu(x, \mathrm{d}y) < \infty$ for all $x \in \mathbb{R}^d$.
- Consider the Lévy type operator

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} f(x) + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} f(x) + \int_{\mathbb{R}_0^d} [f(x+y) - f(x) - y \cdot Df(x)]\nu(x, \mathrm{d}y),$$

in which $f \in C_c^2(\mathbb{R}^d)$.

A Feynman-Kac Formula

- Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) + \mathcal{L}u(t,x) - \rho(t,x)u(t,x) = g(t,x), \\ \qquad (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(T,x) = f(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{5}$$

  where $\rho(\cdot,\cdot) \geq 0$, $g(\cdot,\cdot)$, and $f(\cdot)$ are continuous.
- Under certain conditions, the solution to (6) admits a stochastic representation:

$$\begin{aligned} u(t,x) =& \mathbb{E}_{t,x}\Big[e^{-\int_t^T \rho(r,X(r))dr} f(X(T)) \\ & - \int_t^T e^{-\int_t^s \rho(r,X(r))dr} g(s,X(s))\mathrm{d}s \Big], \quad 0 \leq t \leq T. \end{aligned}$$

  where X is the unique weak solution to an SDE corresponding to the Lévy type operator $\mathcal{L}$.

Finally

Thank you!

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