Approximation of Invariant Measures for path-dependent Regime-Switching Diffusions

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Outline

- Introduction
- Invariant Measure of the exact solutions
- Invariant Measure of the Numerical Solutions

Numerical solutions of SDEs

Consider an SDE on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \quad t > 0, \quad X_0 = x \in \mathbb{R}^n. \qquad (1)$$

Herein, $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \otimes \mathbb{R}^m \to \mathbb{R}^n$, and $(W(t))_{t\geq 0}$ is an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Discrete time Euler-Maruyama (EM) scheme:

$$\overline{Y}((k+1)\delta) = \overline{Y}(k\delta) + b(\overline{Y}(k\delta))\delta + \sigma(\overline{Y}(k\delta))\triangle W(k\delta), \ k \geq 0,$$

with $Y_0 = X_0 = x$, where $\triangle W(k\delta) := W((k+1)\delta) - W(k\delta)$.

Continuous time EM scheme:

$$Y(t) = Y_0 + \int_0^t b(\overline{Y}(\eta_s))\mathrm{d}s + \int_0^t \sigma(\overline{Y}(\eta_s))\mathrm{d}W(s),$$

where $\eta_t := \lfloor t/\delta \rfloor \delta$.

Regular Coefficients

If

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\|_{\mathrm{HS}} \leq K|x - y|, \quad x, y \in \mathbb{R}^n$$

for some $K > 0$, then

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \Big) \lesssim \delta^{p/2}.$$

- The convergence above is called strong convergence;
- The convergence rate is $1/2$.

Regular Coefficients

If the coefficients satisfy linear growth condition and

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\|_{\mathrm{HS}} \leq K_R|x - y|, \quad |x| \vee |y| \leq R$$

for some $K_R > 0$, then

$$\lim_{\delta \to 0} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \Big) = 0.$$

- The convergence above is called strong convergence;
- The convergence rate is not known

We also have

Theorem

Let $L_R^{(1)}$ and $L_R^{(2)}$ be the local growth constants of drift and diffusion, respectively. If $L_R^{(1)} \leq a_1 \log R, (L_R^{(2)})^2 \leq a_2 \log R$ for some postive constants a_1 and a_2, then the order of the convergence is half, that is

$$E\left[\sup_{0\leq t\leq T} |X(t)-Y(t)|^2\right] \lesssim \delta^{1/2}.$$

Convergence rate of EM scheme under various settings

- SDDEs with polynomial growth w.r.t. delay variables (B.-Yuan, 2013);
- SDEs with discontinuous coefficients (Ngo-Taguchi, arXiv:1604.01174v1);
- SDDEs under local Lipschitz and also under monotonicity condition (Gyöngy-Sabanis, 2013).

Irregular coefficients (Gyöngy, I., PA, 98)

Assume that

b satisfies a one-side Lipschitz condition in a domain D in $\mathbb{R}^n$ and σ is Lipschitzian. Then,

$$\sup_{t \le T} |X(t) - Y(t)| \le \xi \delta^\gamma, \quad \text{a.s.}, \quad \gamma \in (0, 1/4),$$

where ξ is a finite random variable.

Irregular coefficients (Yan, L.-Q., AOP, 2002)

Assume that there exist $c > 0$, $0 \le \alpha \le 1/2$, $0 \le \beta_1, \beta_2 \le 1$ such that

$$|b(t,x) - b(s,y)| \lesssim |x-y| + |t-s|^{\beta_1},$$

$$|\sigma(t,x) - \sigma(s,y)| \lesssim |x-y|^{1/2+\alpha} + |t-s|^{\beta_2}.$$

Then,

$$\mathbb{E}|X(t) - Y_{\eta_t}| \lesssim \delta^{\gamma},$$

where $\gamma := \beta_1 \wedge \alpha \wedge \frac{4\alpha\beta_2}{1+2\alpha}$.

Tools: Meyer-Tanaka formula & estimates for local time.

Irregular coefficients (Gyöngy & Rásonyi, SPA, 2011)

Let $b = f + g$, where g is monotone decreasing and assume further that there exist $\alpha \in [0, 1/2]$ and $\gamma \in (0, 1)$ such that

$$|f(t,x) - f(t,y)| \lesssim |x - y|, \qquad |g(t,x) - g(t,y)| \lesssim |x - y|^{\gamma},$$
$$|\sigma(t,x) - \sigma(t,y)| \lesssim |x - y|^{1/2+\alpha}.$$

Then,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X(t) - Y(t)|\Big) \lesssim \begin{cases} \frac{1}{(\log \delta^{-1})^{1/2}}, & \alpha = 0, \\ \delta^{2\alpha^2} + \delta^{\alpha\gamma}, & \alpha \in (0, 1/2]. \end{cases}$$

Approach: Yamada-Watanabe approximation approach.

Irregular coefficients(Ngo & Taguchi, Math. Comp., 2016)

Assume that

- $\langle x-y, b(t,x)-b(t,y)\rangle \lesssim |x-y|^2$;
- $\langle (\sigma\sigma^*)(t,x)\xi,\xi\rangle \asymp |\xi|^2$;
- $|\sigma(t,x)-\sigma(t,y)| \lesssim |x-y|^{1/2+\alpha}, \alpha\in[0,1/2]$;
- $|b(t,x)-b(s,x)|+|\sigma(t,x)-\sigma(s,x)| \lesssim |t-s|^{\beta}, \beta\geq 1/2$;
- $b^{(i)}\in\mathcal{A}$. Then,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|X(t)-Y(t)|\Big) \lesssim \begin{cases} \frac{1}{(\log\delta^{-1})^{1/2}}, & \alpha=0,\\ \delta^{2\alpha^2}, & \alpha\in(0,1/2]. \end{cases}$$

Key tools: Yamada-Watanabe approach and heat kernel estimate.

Irregular coeff. (Pamen & Taguchi, arXiv1508.07513v1)

Consider an SDE $\mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \mathrm{d}L_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^n$, where b is bounded and

$$|b(t,x) - b(t,y)| \lesssim |x-y|^\beta,\ \beta \in (0,1), \ \ |b(t,x) - b(s,x)| \lesssim |t-s|^\eta,\ \eta \in [1/2, 1].$$

- Then, $\mathbb{E}\Big(\sup_{0\le t\le T} |X(t) - Y(t)|^p\Big) \lesssim \delta^{\frac{p\beta}{2}}$ whenever $L =$ Wiener process.
- Moreover,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X(t) - Y(t)|^p\Big) \lesssim \begin{cases} \delta, & p \ge 2, p\beta \ge 2, \\ \delta^{\frac{p\beta}{2}}, & p \ge 2, 1 \le p\beta < 2 \text{ or } p \in [1,2) \end{cases}$$

whenever $L =$ truncated symmetric α-stable process with $\alpha \in (1,2)$ and $\alpha + \beta > 2$.

Long-term behavior

Applying the EM to the SDE

$$dx(t) = (x(t) - x^3(t))dt + 2x(t)dB(t).$$

gives

$$Y_{k+1} = Y_k(1 + \delta - Y_k^2\delta + 2\delta B_k).$$

Lemma

Given any initial value $Y_0 \neq 0$ and any $\delta > 0$,

$$\mathbb{P}\Big(\lim_{k\to\infty} |Y_k| = \infty\Big) > 0.$$

However, we have

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t)| \leq -1.$$

Actually, under some assumptions, we can show that Y_k^δ is a homogeneous Markov process, For any $x \in \mathbb{R}^n$ and Borel set A in $\mathbb{R}^n$, define

$$P(x, A) := P(Y_1 \in A | Y(0) = x) \text{ and } P^k(x, A) := P(Y_k \in A | Y(0) = x).$$

(H1) Both f and g are globally Lipschitz continuous, i.e. there exists a constant $L > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2 \leq L|u - v|^2, \ \forall u, v \in \mathbb{R}^n.$$

(H2) There exists a constant $\ell_1 > 0$ such that

$$|g(u) - g(v)|^2 + 2(u - v)^T (f(u) - f(v)) \leq -\ell_1 |u - v|^2, \ \forall u, v \in \mathbb{R}^n.$$

Under assumptions (H1) and (H2), we can show that $P_k(x, A) \to \pi^\delta(A)$ as $k \to \infty$ and δ sufficient small.

Regime-switching diffusion process

- A regime-switching diffusion process (RSDP), is a diffusion process in random environments characterized by a Markov chain.
- The state vector of a RSDP is a pair $(X(t), \Lambda(t))$, where $\{X(t)\}_{t\geq 0}$ satisfies a stochastic differential equation (SDE)

$$\mathrm{d}X(t) = b(X(t), \Lambda(t))\mathrm{d}t + \sigma(X(t), \Lambda(t))\mathrm{d}W_t, \quad t > 0, \tag{2}$$

with the initial data $X_0 = x \in \mathbb{R}^n, \Lambda_0 = i \in \mathbb{S}$, and $\{\Lambda(t)\}_{t\geq 0}$ denotes a continuous-time Markov chain with the state space $\mathbb{S} := \{1, 2 \cdots, N\}$, $1 \leq N \leq \infty$, and the transition rules specified by

$$\mathbb{P}(\Lambda(t+\triangle) = j | \Lambda(t) = i) = \begin{cases} q_{ij}\triangle + o(\triangle), & i \neq j, \\ 1 + q_{ii}\triangle + o(\triangle), & i = j. \end{cases} \tag{3}$$

- RSDPs have considerable applications in e.g. control problems, storage modeling, neutral activity, biology and mathematical finance (see e.g. the monographs by Mao-Yuan (2006), and Yin-Zhu (2010)). The dynamical behavior of RSDPs may be markedly different from diffusion processes without regime switchings, see e.g. Pinsky -Scheutzow (1992), Mao-Yuan (2006).

Mao, X. and Yuan, C., Stochastic Differential Equations with Markovian Switching, Imperial College Press, 2006

It is interesting to have a look of the following two equations

$$dx(t) = x(t)dt + 2x(t)dW(t) \tag{4}$$

and

$$dx(t) = 2x(t) + x(t)dW(t) \tag{5}$$

switching from one to the other according to the movement of the Markov chain $\Lambda(t)$. We observe that Eq. (4) is almost surely exponentially stable since the Lyapunov exponent is $\lambda_1 = -1$ while Eq. (5) is almost surely exponentially unstable since the Lyapunov exponent is $\lambda_2 = 1.5$.

Let $\Lambda(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ \gamma & -\gamma \end{pmatrix}.$$

Of course $W(t)$ and $\Lambda(t)$ are assumed to be independent. Consider a one-dimensional linear SDEwMS

$$dx(t) = a(\Lambda(t))x(t)dt + b(\Lambda(t))x(t)dW(t) \tag{6}$$

on $t \geq 0$, where

$$a(1) = 1, \quad a(2) = 2, \quad b(1) = 2, \quad b(2) = 1.$$

However, as the result of Markovian switching, the overall behaviour, i.e. Eq. (6) will be exponentially stable if $\gamma > 1.5$ but exponentially unstable if $\gamma < 1.5$ while the Lyapunov exponent of the solution is 0 when $\gamma = 1.5$.

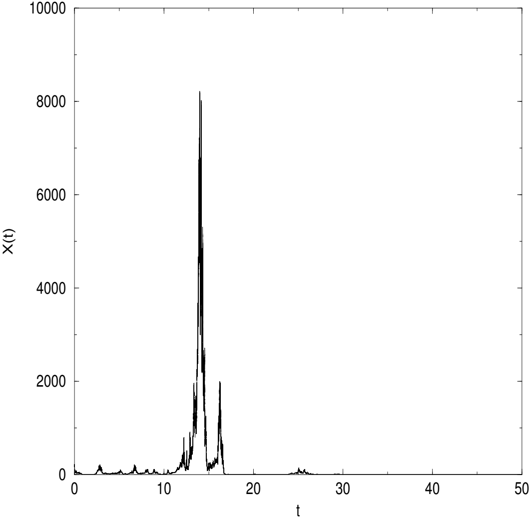

Figure: The graph of numerical solution when $\gamma = 2$.

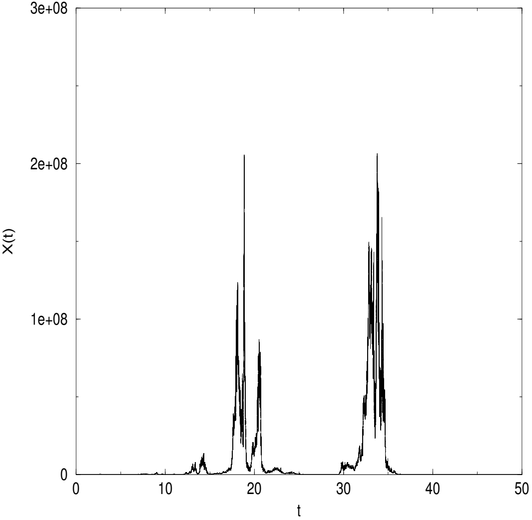

Figure: The graph of numerical solution when $\gamma = 1.5$.

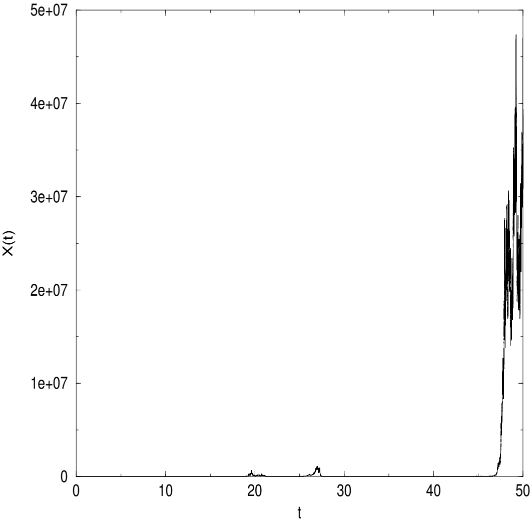

Figure: The graph of numerical solution when $\gamma = 0.5$.

- So far, the works on RSDPs have included ergodicity (Cloez-Hairer (2013), Shao (2014)) stability in distribution (Mao-Yuan (03), Xi-Yin (2010)), recurrence and transience (Pinsky-Scheutzow (1992),invariant densities (Bakhtin et al. (2014)) and so forth

- Since solving RSDPs is still a challenging task, numerical schemes and/or approximation techniques have become one of the viable alternatives (see e.g. Mao-Yuan (2006), Yin-Zhu (2010), Higham et al. (2007)).
- For more details on numerical analysis of diffusion processes without regime switching, please refer to the monograph by Kloeden and Platen (1992).
- Also, approximations of invariant measures for stochastic dynamical sysytems have attracted much attention, see e.g. Mattingly et al. (2010), Talay (1990), Bréhier (2014).

- For the counterpart associated with Euler-Maruyama (EM) algorithms, we refer to Yuan-Mao (2005), and Yin-Zhu (2010), where RSDPs therein enjoy finite state space.
- In this talk, we are concerned with the following questions: Consider

$$\mathrm{d}X(t) = b(X_t, \Lambda(t))\mathrm{d}t + \sigma(X_t, \Lambda(t))\mathrm{d}W(t), t > 0, X_0 = \xi \in \mathscr{C}, \Lambda(0) = i_0 \in \mathbb{S} \tag{7}$$

(i) Under what conditions, will the semigroup of the exact solution admit an invariant measure?

(ii) Under what conditions, will the discrete-time semigroup generated by EM scheme admit an invariant measure?

(iii) Will the numerical invariant measure, if it exists, converge in some metric to the underlying one?

We assume that $(\Lambda(t))$ is irreducible, which yields the positive recurrence together with the finiteness of $\mathbf{S}$. Let $\pi = (\pi_1, \cdots, \pi_N)$ denote its stationary distribution, which can be solved by $\pi Q = 0$ subject to $\sum_{i\in\mathbb{S}} \pi_i = 1$ with $\pi_i \geq 0$. Assume that $(\Lambda(t))$ is independent of $(W(t))$. Let $v(\cdot)$ be a probability measure on $[-\tau, 0]$ and $\|\cdot\|_{\mathrm{HS}}$ means the Hilbert-Schmidt norm. Let $\mathbf{E} = \mathbb{R}^n \times \mathbb{S}$. For any $\mathbf{x} = (x, i) \in \mathbf{E}$ and $\mathbf{y} = (y, j) \in \mathbf{E}$, define the metric ρ between $\mathbf{x}$ and $\mathbf{y}$ by

$$\rho(\mathbf{x}, \mathbf{y}) = |x - y| + \mathbf{1}_{\{i \neq j\}},$$

where, for a set A, $\mathbf{1}_A(x) = 1$ with $x \in A$; otherwise, $\mathbf{1}_A(x) = 0$. Let $\mathcal{P} = \mathcal{P}(\mathbf{E})$ be the space of all probability measures on $\mathbf{E}$. Define the Wasserstein distance W_ρ between two probability measures μ, $\nu \in \mathcal{P}$ as follows:

$$W_\rho(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left\{ \int_{\mathbf{E}} \int_{\mathbf{E}} \rho(\mathbf{x}, \mathbf{y}) \pi(\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{y}) \right\},$$

where $\mathcal{C}(\mu, \nu)$ denotes the collection of all probability measures on $\mathbf{E} \times \mathbf{E}$ with marginals μ and ν, respectively. In this work, $c > 0$ will stand for a generic constant which might change from occurrence to occurrence.

Invariant Measure: Additive Noises

We focus on a path-dependent random diffusion with additive noise

$$\mathrm{d}X(t) = b(X_t, \Lambda(t))\mathrm{d}t + \sigma(\Lambda(t))\mathrm{d}W(t), t > 0, X_0 = \xi \in \mathscr{C}, \Lambda(0) = i_0 \in \mathbb{S}, \quad (8)$$

where $b : \mathscr{C} \times \mathbb{S} \to \mathbb{R}^n$, $\sigma : \mathbb{S} \to \mathbb{R}^n \otimes \mathbb{R}^m$, and, for fixed $t \geq 0$, $X_t(\theta) = X(t+\theta)$, $\theta \in [-\tau, 0]$, used the standard notation.

We assume that, for each $i \in \mathbb{S}$ and arbitrary $\xi, \eta \in \mathscr{C}$,

(A) There exist $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}_+$ such that

$$2\langle \xi(0) - \eta(0), b(\xi, i) - b(\eta, i)\rangle \leq \alpha_i |\xi(0) - \eta(0)|^2 + \beta_i \|\xi - \eta\|_\infty^2.$$

Under **(A)**, (8) admits a unique strong solution $(X(t; \xi, i_0))$ with the initial datum $X_0 = \xi \in \mathscr{C}$ and $\Lambda(0) = i_0 \in \mathbb{S}$. The segment process (i.e., functional solution) associated with the solution process $(X(t; \xi, i_0))$ is denoted by $(X_t(\xi, i_0))$. The pair $(X_t(\xi, i_0), \Lambda(t))$ is a homogeneous Markov process.

Set

$$\widehat{\alpha} := \min_{i\in\mathbb{S}} \alpha_i, \quad \check{\alpha} := \max_{i\in\mathbb{S}} |\alpha_i| \quad \text{and} \quad \check{\beta} := \max_{i\in\mathbb{S}} \beta_i. \tag{9}$$

$$Q_1 := Q + \mathrm{diag}\Big(\alpha_1 + \mathrm{e}^{-\widehat{\alpha}\tau}\beta_1, \cdots, \alpha_N + \mathrm{e}^{-\widehat{\alpha}\tau}\beta_N\Big),$$

$$\eta_1 = -\max_{\gamma\in\mathrm{spec}(Q_1)} \mathrm{Re}(\gamma). \tag{10}$$

Let $(\Lambda^i(t), \Lambda^j(t))$ be the independent coupling of the Q-process $(\Lambda(t))$ with starting point $(\Lambda^i(0), \Lambda^j(0)) = (i, j)$. Let $T = \inf\{t \geq 0 : \Lambda^i(t) = \Lambda^j(t)\}$ be the coupling time of $(\Lambda^i(t), \Lambda^j(t))$. Since the cardinality of $\mathbb{S}$ is finite and (q_{ij}) is irreducible, there exists a constant $\theta > 0$ such that

$$\mathbb{P}(T > t) \leq \mathrm{e}^{-\theta t}, \qquad t > 0. \tag{11}$$

Let $P_t((\xi, i), \cdot)$ be the transition kernel of $(X_t(\xi, i), \Lambda^i(t))$. For $\nu \in \mathcal{P}$, νP_t denotes the law of $(X_t(\xi, i), \Lambda^i(t))$ when $(X_0(\xi, i), \Lambda^i(0))$ is distributed according to $\nu \in \mathcal{P}$.

Theorem 1 Suppose (**A**) holds and $\eta_1 > 0$. Then, it holds that

$$W_\rho(\nu_1 P_t, \nu_2 P_t) \le c\Big(1 + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\xi\|_\infty \nu_1(\mathrm{d}\xi, i) + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\xi\|_\infty \nu_2(\mathrm{d}\xi, i)\Big)\mathrm{e}^{-\frac{\theta\eta_1}{2(\theta+\eta_1)}t} \tag{12}$$

for any $\nu_1, \nu_2 \in \mathcal{P}$, where η_1 is defined in (10) and $\theta > 0$ is specified in (11). Furthermore, (12) implies that $(X_t(\xi, i), \Lambda^i(t))$, admits a unique invariant probability measure $\mu \in \mathcal{P}$ such that

$$W_\rho(\delta_{(\xi,i)} P_t, \mu P_t) \le c\Big(1 + \|\xi\|_\infty + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\eta\|_\infty \mu(\mathrm{d}\eta, i)\Big)\mathrm{e}^{-\frac{\theta\eta_1}{2(\theta+\eta_1)}t}, \tag{13}$$

where $\delta_{(\xi,i)}$ stands for the Dirac's measure at the point (ξ, i).

Remark: If the assumption $\eta_1 > 0$ is replaced by

$$\sum_{i\in\mathbb{S}}(\alpha_i + \mathrm{e}^{-\widehat{\alpha}\,\tau}\beta_i)\pi_i < 0$$

and

$$\min_{i\in\mathbb{S},\alpha_i+\mathrm{e}^{-\widehat{\alpha}\,\tau}\beta_i>0}\Big(-\frac{q_{ii}}{\alpha_i + \mathrm{e}^{-\widehat{\alpha}\,\tau}\beta_i}\Big) > 1,$$

Theorem 1 still holds.

Invariant Measures: Multiplicative Noises

Consider the following equation

$$\mathrm{d}X(t) = b(X_t, \Lambda(t))\mathrm{d}t + \sigma(X_t, \Lambda(t))\mathrm{d}W(t), \quad t > 0, \quad X_0 = \xi, \quad \Lambda(0) = i_0 \in \mathbb{S}, \tag{14}$$

where $b : \mathscr{C} \times \mathbb{S} \to \mathbb{R}^n$ and $\sigma : \mathscr{C} \times \mathbb{S} \to \mathbb{R}^n \otimes \mathbb{R}^m$. Let $v(\cdot)$ be a probability measure on $[-\tau, 0]$ and suppose that, for any $\xi, \eta \in \mathscr{C}$ and each $i \in \mathbb{S}$,

(H1) There exist $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}_+$ such that

$$\begin{aligned} &2\langle \xi(0) - \eta(0), b(\xi, i) - b(\eta, i)\rangle + \|\sigma(\xi, i) - \sigma(\eta, i)\|_{\mathrm{HS}}^2 \\ &\le \alpha_i |\xi(0) - \eta(0)|^2 + \beta_i \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2 v(\mathrm{d}\theta). \end{aligned}$$

(H2) There exists an $L > 0$ such that

$$\|\sigma(\xi, i) - \sigma(\eta, i)\|_{\mathrm{HS}}^2 \le L\Big(|\xi(0) - \eta(0)|^2 + \int_{-\tau}^0 |\xi(\theta) - \eta(\theta)|^2 v(\mathrm{d}\theta)\Big).$$

Set

$$Q_2 := Q + \mathsf{diag}\Big(\alpha_1 + \beta_1 \int_{-\tau}^0 \mathrm{e}^{\widehat{\alpha}\theta}\mu(\mathrm{d}\theta), \cdots, \alpha_N + \beta_N \int_{-\tau}^0 \mathrm{e}^{\widehat{\alpha}\theta}\mu(\mathrm{d}\theta)\Big),$$

$$\eta_2 = - \max_{\gamma \in \mathrm{spec}(Q_2)} \mathrm{Re}(\gamma) \tag{15}$$

Theorem 2 Let (**H1**)-(**H2**) hold and assume further $\eta_2 > 0$. Then,

$$W_\rho(\nu_1 P_t, \nu_2 P_t) \le c\left(1 + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\xi\|_\infty \nu_1(\mathrm{d}\xi, i) + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\eta\|_\infty \nu_2(\mathrm{d}\eta, i)\right)\mathrm{e}^{-\frac{\theta\eta_2}{2(\theta+\eta_2)}t} \tag{16}$$

for any $\nu_1, \nu_2 \in \mathcal{P}$, where $\theta > 0$ such that (11) holds and $\eta_2 > 0$ is defined in (15). Furthermore, (16) implies that $(X_t(\xi, i), \Lambda^i(t))$ admits a unique invariant probability measure $\mu \in \mathcal{P}$ such that

$$W_\rho(\delta_{(\xi,i)} P_t, \mu P_t) \le c\Big(1 + \|\xi\|_\infty + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\eta\|_\infty \mu(\mathrm{d}\eta, i)\Big)\mathrm{e}^{-\frac{\theta\eta_2}{2(\theta+\eta_2)}t}. \tag{17}$$

Example I Let $\{\Lambda(t)\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S}=\{1,2\}$ with the generator

$$Q=\begin{pmatrix} -1 & 1 \\ \gamma & -\gamma \end{pmatrix} \quad (18)$$

for some constant $\gamma>0$. Consider a scalar path-dependent OU process

$$\mathrm{d}X(t)=\{a_{\Lambda(t)}X(t)+b_{\Lambda(t)}X(t-1)\}\mathrm{d}t+\sigma_{\Lambda_t}\mathrm{d}W(t),\ t>0,\ (X_0,\Lambda(0))=(\xi,1)\in\mathscr{C}\times \quad (19)$$

where $a_1,b_1,b_2>0, a_2<0$. Set $\alpha:=2a_1+(1+\mathrm{e}^{-a_2})b_1$, $\beta:=2a_2+(1+\mathrm{e}^{-a_2})b_2$.

$$\begin{cases} \alpha+\beta<1+\gamma \\ \beta-\frac{\beta}{\alpha}>\gamma. \end{cases} \quad (20)$$

then $(X_t(\xi,i),\Lambda^i(t))$, determined by (19) and (18), has a unique invariant probability measure, and converges exponentially to the equilibrium.

Let $\delta = \frac{\tau}{M} \in (0,1)$ for some integer $M > \tau$. Consider the following EM scheme

$$dY(t) = b(Y_{t_\delta}, \Lambda(t_\delta))dt + \sigma(\Lambda(t_\delta))dW(t), \qquad t > 0 \tag{21}$$

with the initial condition $Y(\theta) = \xi(\theta)$ for $\theta \in [-\tau, 0]$ and $\Lambda(0) = i_0 \in \mathbb{S}$, where, $t_\delta := \lfloor t/\delta \rfloor \delta$ with $\lfloor t/\delta \rfloor$ being the integer part of t/δ, and $Y_{k\delta} = \{Y_{k\delta}(\theta) : -\tau \le \theta \le 0\}$ is a $\mathscr{C}$-valued random variable defined as follows: for any $\theta \in [i\delta, (i+1)\delta]$, $i = -M, -(M-1), \cdots, -1$,

$$Y_{k\delta}(\theta) = Y((k+i)\delta) + \frac{\theta - i\delta}{\delta}\{Y((k+i+1)\delta) - Y((k+i)\delta)\}, \tag{22}$$

i.e., $Y_{k\delta}(\cdot)$ is the linear interpolation of $Y((k-M)\delta)$, $Y((k-(M-1))\delta), \cdots, Y((k-1)\delta), Y(k\delta)$.

We further assume that there exists an $L_0 > 0$ such that

$$|b(\xi, i) - b(\eta, i)| \leq L_0 \|\xi - \eta\|_\infty, \qquad \xi, \eta \in \mathscr{C}, \quad i \in \mathbb{S}. \tag{23}$$

Moreover, the pair $(Y_{t_\delta}(\xi, i), \Lambda(t_\delta))$ enjoys the Markov property. Let $P_{k\delta}^{(\delta)}((\xi, i)$ stand for the transition kernel of $(Y_{k\delta}(\xi, i), \Lambda^i(k\delta))$.

Theorem 3 Let the assumptions of Theorem 1 be satisfied and suppose further (23) holds. Then, there exist $\delta_0 \in (0,1)$ and $\alpha > 0$ such that for any $k \geq 0$ and $\delta \in (0, \delta_0)$,

$$W_\rho(\nu_1 P_{k\delta}^{(\delta)}, \nu_2 P_{k\delta}^{(\delta)}) \leq c\Big(1+\sum_{i\in\mathbb{S}}\int_{\mathscr{C}} \|\xi\|_\infty \nu_1(\mathrm{d}\xi, i)+\sum_{i\in\mathbb{S}}\int_{\mathscr{C}} \|\eta\|_\infty \nu_2(\mathrm{d}\eta, i)\Big)\mathrm{e}^{-\alpha k\delta}, \tag{24}$$

in which $\nu_1, \nu_2 \in \mathcal{P}$. Furthermore, (24) implies that $(Y_{k\delta}(\xi, i), \Lambda^i(k\delta))$ admits a unique invariant probability measure $\mu^{(\delta)} \in \mathcal{P}$ such that

$$W_\rho(\delta_{(\xi,i)} P_{k\delta}^{(\delta)}, \mu^{(\delta)} P_{k\delta}^{(\delta)}) \leq c\Big(1 + \|\xi\|_\infty + \sum_{i\in\mathbb{S}}\int_{\mathscr{C}} \|\eta\|_\infty \mu^{(\delta)}(\mathrm{d}\eta, i)\Big)\mathrm{e}^{-\alpha k\delta}.$$

Numerical Invariant Measures: Multiplicative Noises

Assume that there exists an $L_1 > 0$ such that

$$|b(\xi,i)-b(\eta,i)|^2 \leq L_1\Big(|\xi(0)-\eta(0)|^2+\int_{-\tau}^{0}|\xi(\theta)-\eta(\theta)|^2 v(\mathrm{d}\theta)\Big) \quad (25)$$

for any $\xi,\eta\in\mathscr{C}$ and $i\in\mathbb{S}$. Consider the EM scheme corresponding to (14)

$$\mathrm{d}Y(t)=b(Y_{t_\delta},\Lambda(t_\delta))\mathrm{d}t+\sigma(Y_{t_\delta},\Lambda(t_\delta))\mathrm{d}W(t), \qquad t>0 \quad (26)$$

with the initial condition $Y(\theta)=\xi(\theta)$ for $\theta\in[-\tau,0]$ and $\Lambda(0)=i_0\in\mathbb{S}$, where Y_{t_δ} is defined exactly as in (22). Set

$$Q_3:=Q+\mathrm{diag}\Big(\alpha_1+4\mathrm{e}^{-\widehat{\alpha}\tau}\beta_1,\dots,\alpha_N+4\mathrm{e}^{-\widehat{\alpha}\tau}\beta_N\Big),$$

and

$$\eta_3:=-\max_{\gamma\in\mathrm{spec}(Q_3)}\mathrm{Re}(\gamma). \quad (27)$$

Theorem 4 Let (**H1**), (**H2**), and (25) hold and assume further $\eta_3 > 0$. Then, there exist $\delta_0 \in (0,1)$ and $\alpha > 0$ such that, for any $k \geq 0$ and $\delta \in (0, \delta_0)$,

$$W_\rho(\nu_1 P_{k\delta}, \nu_2 P_{k\delta}) \leq c\Big(1 + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\xi\|_\infty \nu_1(\mathrm{d}\xi, i) + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\eta\|_\infty \nu_2(\mathrm{d}\eta, i)\Big)\mathrm{e}^{-\alpha k\delta}, \tag{28}$$

where $\nu_1, \nu_2 \in \mathcal{P}$. Furthermore, (28) implies that $(Y_{k\delta}(\xi, i), \Lambda^i(k\delta))$ admits a unique invariant probability measure $\mu^{(\delta)} \in \mathcal{P}$ such that

$$W_\rho(\delta_{(\xi,i)} P_{k\delta}, \mu^{(\delta)} P_{k\delta}) \leq c\Big(1 + \|\xi\|_\infty + \sum_{i\in\mathbb{S}} \int_{\mathscr{C}} \|\eta\|_\infty \mu^{(\delta)}(\mathrm{d}\eta, i)\Big)\mathrm{e}^{-\alpha k\delta}.$$

Sketch of the Proof of Theorem 1

Let

$$\Omega_1 = \{\omega |\ \omega : [0,\infty) \to \mathbb{R}^m \text{ is continuous with } \omega(0) = 0\},$$

which is endowed with the locally uniform convergence topology and the Wiener measure $\mathbb{P}_1$ so that the coordinate process $W(t,\omega) := \omega(t)$, $t \geq 0$, is a standard m-dimensional Brownian motion.

Set

$$\Omega_2 := \Big\{\omega \Big|\ \omega : [0,\infty) \to \mathbf{S} \text{ is right continuous with left limit}\Big\},$$

endowed with Skorokhod topology and a probability measure $\mathbb{P}_2$ so that the coordinate process $\Lambda(t,\omega) = \omega(t)$, $t \geq 0$, is a continuous time Markov chain with Q-matrix (q_{ij}). Let

$$(\Omega, \mathscr{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathscr{B}(\Omega_1) \times \mathscr{B}(\Omega_2), \mathbb{P}_1 \times \mathbb{P}_2).$$

Lemma 1 Under the assumptions of Theorem 1,

$$\mathbb{E}\|X_t(\xi,i)-X_t(\eta,i)\|_\infty^2 \le c\,\|\xi-\eta\|_\infty^2 \mathrm{e}^{-\eta_1 t} \tag{29}$$

for any $\xi,\eta\in\mathscr{C}$ and $i\in\mathbb{S}$, where $\eta_1>0$ is defined in (10).

Proof. For fixed $\omega_2\in\Omega_2$, consider the following SDE

$$\mathrm{d}X^{\omega_2}(t)=b(X_t^{\omega_2},\Lambda^{\omega_2}(t))\mathrm{d}t+\sigma(\Lambda^{\omega_2}(t))\mathrm{d}\omega_1(t),\;\; t>0,\;\; X_0^{\omega_2}=\xi\in\mathscr{C},\;\; \Lambda^{\omega_2}(0)=i$$

Since $(\Lambda^{\omega_2}(s))_{s\in[0,t]}$ may own finite number of jumps, $t\mapsto\int_0^t\alpha_{\Lambda^{\omega_2}(s)}\mathrm{d}s$ need not to be differentiable. To overcome this drawback, let us introduce a smooth approximation of it. For any $\varepsilon\in(0,1)$, set

$$\alpha^{\varepsilon}_{\Lambda^{\omega_2}(t)}:=\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\alpha_{\Lambda^{\omega_2}(s)}\mathrm{d}s+\varepsilon t=\int_0^1\alpha_{\Lambda^{\omega_2}(\varepsilon s+t)}\mathrm{d}s+\varepsilon t.$$

Plainly, $t \mapsto \alpha^{\varepsilon}_{\Lambda^{\omega_2}(t)}$ is continuous and $\alpha^{\varepsilon}_{\Lambda^{\omega_2}(t)} \to \alpha_{\Lambda^{\omega_2}(t)}$ as $\varepsilon \downarrow 0$ due to the right continuity of the path of $\Lambda^{\omega_2}(\cdot)$. As a consequence, $t \mapsto \int_0^t \alpha^{\varepsilon}_{\Lambda^{\omega_2}(r)}\mathrm{d}r$ is differentiable by the first fundamental theorem of calculus and $\int_0^t \alpha^{\varepsilon}_{\Lambda^{\omega_2}(r)}\mathrm{d}r \to \int_0^t \alpha_{\Lambda^{\omega_2}(r)}\mathrm{d}r$ as $\varepsilon \downarrow 0$ according to Lebesgue's dominated convergence theorem. Let

$$\Gamma^{\omega_2}(t) = X^{\omega_2}(t;\xi,i) - X^{\omega_2}(t;\eta,i). \tag{30}$$

Applying Itô's formula and taking (**A**) into account ensures that

$$\begin{aligned} & \mathrm{e}^{-\int_0^t \alpha^{\varepsilon}_{\Lambda^{\omega_2}(s)}\mathrm{d}s}|\Gamma^{\omega_2}(t)|^2 \\ &= |\Gamma^{\omega_2}(0)|^2 + \int_0^t \mathrm{e}^{-\int_0^s \alpha^{\varepsilon}_{\Lambda^{\omega_2}(r)}\mathrm{d}r}\Big\{ -\alpha^{\varepsilon}_{\Lambda^{\omega_2}(s)}|\Gamma^{\omega_2}(s)|^2 \\ &\quad + 2\langle \Gamma^{\omega_2}(s), b(X_s^{\omega_2}(\xi,i),\Lambda^{\omega_2}(s)) - b(X_s^{\omega_2}(\eta,i),\Lambda^{\omega_2}(s))\rangle\Big\}\mathrm{d}s \\ &\le |\Gamma^{\omega_2}(0)|^2 + \Gamma_1^{\omega_2,\varepsilon}(t) + \int_0^t \beta_{\Lambda^{\omega_2}(s)}\mathrm{e}^{-\int_0^s \alpha^{\varepsilon}_{\Lambda^{\omega_2}(r)}\mathrm{d}r}\|\Gamma_s^{\omega_2}\|_\infty^2\mathrm{d}s, \end{aligned} \tag{31}$$

where

$$\Gamma_1^{\omega_2,\varepsilon}(t) := \int_0^t \mathrm{e}^{-\int_0^s \alpha^{\varepsilon}_{\Lambda^{\omega_2}(r)}\mathrm{d}r}|\alpha_{\Lambda^{\omega_2}(s)} - \alpha^{\varepsilon}_{\Lambda^{\omega_2}(s)}|\cdot|\Gamma^{\omega_2}(s)|^2\mathrm{d}s. \tag{32}$$

Since $\alpha^{\varepsilon}_{\Lambda^{\omega_2}(s)} \to \alpha_{\Lambda^{\omega_2}(s)}$ so that $\Gamma_1^{\omega_2,\varepsilon}(t) \to 0$ as $\varepsilon \to 0$, by taking $\varepsilon \downarrow 0$ one has

$$\mathrm{e}^{-\int_0^t \alpha_{\Lambda^{\omega_2}(s)}\mathrm{d}s}\|\Gamma_t^{\omega_2}\|_\infty^2 \le \mathrm{e}^{-\widehat{\alpha}\tau}\Big\{c\,\|\Gamma_0^{\omega_2}\|_\infty^2 + \int_0^t \beta_{\Lambda^{\omega_2}(s)}\mathrm{e}^{-\int_0^s \alpha_{\Lambda^{\omega_2}(r)}\mathrm{d}r}\|\Gamma_s^{\omega_2}\|_\infty^2\mathrm{d}s\Big\}.$$

Thus, employing Gronwall's inequality followed by taking expectation w.r.t. $\mathbb{P}$ yields that

$$\mathbb{E}\|X_t(\xi,i_0) - X(\eta,i_0)\|_\infty^2 \le c\,\|\xi-\eta\|_\infty^2\,\mathbb{E}\,\mathrm{e}^{\int_0^t(\alpha_{\Lambda(s)}+\mathrm{e}^{-\widehat{\alpha}\tau}\beta_{\Lambda(s)})\mathrm{d}s}.$$

The result follows by the following result

For a map $K : \mathbb{S} \to \mathbb{R}$ and $p > 0$, let $A_p = Q + p\text{diag}(K_1, \cdots, K_N)$ and $\eta_p = -\max_{\gamma \in \text{Spec}(A_p)} \text{Re}(\gamma)$, where Spec$(A_p)$ and Re(γ) denote the spectrum of Q_p (i.e., the multiset of its eigenvalues) and the real part of γ, respectively. Set $\kappa := \sup\{p \geq 0 : \eta_p > 0\}$.

Lemma

For any $p > 0$, there exist constants $C_1(p), C_2(p) > 0$ such that

$$C_1(p)\mathrm{e}^{-\eta_p t} \leq \mathbb{E} \exp\left(\int_0^t pK_{\Lambda_s} \mathrm{d}s \right) \leq C_2(p)\mathrm{e}^{-\eta_p t}.$$

Moreover, if $\max_{i \in \mathbb{S}} K_i \leq 0$, then $\eta_p > 0$ for any $p > 0$; if $\max_{i \in \mathbb{S}} K_i > 0$ and $\sum_{i \in \mathbb{S}} K_i \pi_i < 0$, then $\eta_p > 0$ for any $p \in (0, \kappa)$ and $\eta_p < 0$ for any $p \in (\kappa, \infty)$.

For the numerical solution, we need the following lemma

We further need to introduce some additional notation. For $p > 0$, let $K : \mathbb{S} \to \mathbb{R}$ and set

$$A_p := Q + p\,\mathrm{diag}(K_1, \cdots, K_N).$$

Furthermore, define

$$\eta_p = -\max_{\gamma \in \mathrm{spec}(A_p)} \mathrm{Re}(\gamma), \qquad p > 0,$$

and

$$\kappa = \sup\{p \geq 0 : \eta_p > 0\}.$$

The lemma below, which is concerned with the estimate on the exponential functional of the discrete observation for the Markov chain involved and may be interested by itself, plays a crucial role in the analyzing the long-time behavior of the discretization for $(X_t(\xi, i_0), \Lambda(t))$.

Lemma 2 Let $K : \mathbb{S} \to \mathbb{R}$, and $Q_K = Q + \mathrm{diag}(K_1, \cdots, K_N)$. Set

$$\eta_K = - \max_{\gamma \in \mathrm{spec}(Q_K)} \mathrm{Re}(\gamma).$$

Then there exist $\delta_0 \in (0,1)$ and $c > 0$ such that, for $\forall \, \delta \in (0, \delta_0)$,

$$\mathbb{E}\, \mathrm{e}^{\int_0^t K_{\Lambda(s_\delta)} \mathrm{d}s} \leq c\, \mathrm{e}^{-\eta_K t/2}, \qquad \forall \, t > 0. \tag{33}$$

Sketch of the Proof of Lemma 2

By Hölder's inequality, it follows that

$$\mathbb{E}\,\mathrm{e}^{\int_0^t K_{\Lambda(s_\delta)}\mathrm{d}s} = \mathbb{E}\,\mathrm{e}^{\int_0^t K_{\Lambda(s)}\mathrm{d}s+\int_0^t (K_{\Lambda(s_\delta)}-K_{\Lambda(s)})\mathrm{d}s}$$
$$\leq \left(\mathbb{E}\,\mathrm{e}^{(1+\varepsilon)\int_0^t K_{\Lambda(s)}\mathrm{d}s}\right)^{\frac{1}{1+\varepsilon}} \left(\mathbb{E}\,\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\int_0^t (K_{\Lambda(s_\delta)}-K_{\Lambda(s)})\mathrm{d}s}\right)^{\frac{\varepsilon}{1+\varepsilon}}, \ \varepsilon > 0. \tag{34}$$

Observe that there exists $\delta_1 \in (0,1)$ such that for any $\triangle \in (0,\delta_1)$,

$$\mathbb{P}(\Lambda(t+\delta) = i|\Lambda(t) = i) = 1 + q_{ii}\delta + o(\delta), \tag{35}$$

and that

$$\mathbb{P}(\Lambda(t+\delta) \neq i|\Lambda(t) = i) = \sum_{j\neq i}(q_{ij}\delta + o(\delta)) \leq \max_{i\in\mathbb{S}}(-q_{ii})\delta + o(\delta). \tag{36}$$

Utilizing Jensen's inequality

$$
\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\int_{i\delta}^{(i+1)\delta\wedge t}(K_{\Lambda(i\delta)}-K_{\Lambda(s)})\mathrm{d}s}\Big|\Lambda(i\delta)\right)\\
&\le \frac{1}{(i+1)\delta\wedge t-i\delta}\int_{i\delta}^{(i+1)\delta\wedge t}\mathbb{E}\left(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}((i+1)\delta\wedge t-i\delta)(K_{\Lambda(i\delta)}-K_{\Lambda(s)})}\Big|\Lambda(i\delta)\right)\mathrm{d}s\\
&=\frac{\sum_{j\in\mathbb{S}}\mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t-i\delta}\int_{i\delta}^{(i+1)\delta\wedge t}\mathbb{E}\left(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}((i+1)\delta\wedge t-i\delta)(K_j-K_{\Lambda(s)})}\Big|\Lambda(i\delta)=j\right)\mathrm{d}s\\
&=\frac{\sum_{j\in\mathbb{S}}\mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t-i\delta}\int_{i\delta}^{(i+1)\delta\wedge t}\mathbb{E}\left(\mathbf{1}_{\{\Lambda(s)=j\}}|\Lambda(i\delta)=j\right)\mathrm{d}s\\
&\quad+\frac{\sum_{j\in\mathbb{S}}\mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t-i\delta}\int_{i\delta}^{(i+1)\delta\wedge t}\mathbb{E}\left(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}((i+1)\delta\wedge t-i\delta)(K_j-K_{\Lambda(s)})}\mathbf{1}_{\{\Lambda(s)\neq j\}}\Big|\Lambda(i\delta)=j\right)
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&\le \frac{\sum_{j\in\mathbb{S}} \mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t - i\delta} \int_{i\delta}^{(i+1)\delta\wedge t} \mathbb{E}\,(\mathbf{1}_{\{\Lambda(s)=j\}}|\Lambda(i\delta)=j)\mathrm{d}s \\
&\quad + \mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}} \frac{\sum_{j\in\mathbb{S}} \mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t - i\delta} \int_{i\delta}^{(i+1)\delta\wedge t} \mathbb{P}\,(\mathbf{1}_{\{\Lambda(s)\neq j\}}|\Lambda(i\delta)=j)\mathrm{d}s \\
&\le \frac{\sum_{j\in\mathbb{S}} \mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t - i\delta} \int_{i\delta}^{(i+1)\delta\wedge t} (1+q_{jj}(s-i\delta)+o(s-i\delta))\mathrm{d}s \\
&\quad + \mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}} \frac{\sum_{j\in\mathbb{S}} \mathbf{1}_{\{\Lambda(i\delta)=j\}}}{(i+1)\delta\wedge t - i\delta} \int_{i\delta}^{(i+1)\delta\wedge t} \Big(\max_{i\in\mathbb{S}}(-q_{ii})(s-i\delta)+o(s-i\delta)\Big)\mathrm{d}s \\
&\le 1 + \frac{\max_{i\in\mathbb{S}}(-q_{ii})}{2}\,\delta\,\mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}} + o(\delta),
\end{aligned}
$$

where $\check{K} := \max_{i\in\mathbb{S}} |K_i|$.

By the property of conditional expectation, we deduce from (37) that

$$
\begin{aligned}
&\mathbb{E}\, \mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\int_0^t (K_{\Lambda(s_\delta)}-K_{\Lambda(s)})\mathrm{d}s} \\
&= \mathbb{E}\, \mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\sum_{i=0}^{\lfloor t/\delta\rfloor}\int_{i\delta}^{(i+1)\delta\wedge t}(K_{\Lambda(i\delta)}-K_{\Lambda(s)})\mathrm{d}s} \\
&= \mathbb{E}\Big(\mathbb{E}\Big(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\sum_{i=0}^{\lfloor t/\delta\rfloor}\int_{i\delta}^{(i+1)\delta\wedge t}(K_{\Lambda(i\delta)}-K_{\Lambda(s)})\mathrm{d}s}\Big|\Lambda(t_\delta)\Big)\Big) \\
&= \mathbb{E}\Big(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\sum_{i=0}^{\lfloor t/\delta\rfloor-1}\int_{i\delta}^{(i+1)\delta}(K_{\Lambda(i\delta)}-K_{\Lambda(s)})\mathrm{d}s}\mathbb{E}\Big(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\int_{t_\delta}^{(t_\delta+\delta)\wedge t}(K_{\Lambda(t_\delta)}-K_{\Lambda(s)})\mathrm{d}s}\Big|\Lambda(t_\delta)\Big)\Big) \\
&\le \Big(1+\frac{\max_{i\in\mathbb{S}}(-q_{ii})}{2}\,\delta\,\mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}}+o(\delta)\Big)\mathbb{E}\Big(\mathrm{e}^{\frac{1+\varepsilon}{\varepsilon}\sum_{i=0}^{\lfloor t/\delta\rfloor-1}\int_{i\delta}^{(i+1)\delta}(K_{\Lambda(i\delta)}-K_{\Lambda(s)})\mathrm{d}s}\Big) \\
&\le \Big(1+\frac{\max_{i\in\mathbb{S}}(-q_{ii})}{2}\,\delta\,\mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}}+o(\delta)\Big)^{\lfloor t/\delta\rfloor+1}, \qquad \delta\in(0,\delta_1),
\end{aligned}
\tag{38}
$$

Noting that

$$
\begin{aligned}
&\left(1+\frac{\max_{i\in\mathbb{S}}(-q_{ii})}{2}\,\delta\,\mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}}+o(\delta)\right)^{\frac{\varepsilon(\lfloor t/\delta\rfloor+1)}{1+\varepsilon}}\\
&=\exp\left(\frac{\varepsilon(\lfloor t/\delta\rfloor+1)}{1+\varepsilon}\ln\left(1+\frac{\max_{i\in\mathbb{S}}(-q_{ii})}{2}\,\delta\,\mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}}+o(\delta)\right)\right)\\
&\le\exp\left(\frac{\varepsilon(t+\delta)}{1+\varepsilon}\frac{1}{\delta}\ln\left(1+\frac{\max_{i\in\mathbb{S}}(-q_{ii})}{2}\,\delta\,\mathrm{e}^{\frac{2(1+\varepsilon)\check{K}\delta}{\varepsilon}}+o(\delta)\right)\right)\\
&\le\exp\left(\frac{\varepsilon}{1+\varepsilon}\max_{i\in\mathbb{S}}(-q_{ii})\right)\exp\left(\frac{\varepsilon}{1+\varepsilon}\max_{i\in\mathbb{S}}(-q_{ii})t\right),
\end{aligned}
\tag{39}
$$

and taking (38) into consideration, we deduce from (34) that

$$
\begin{aligned}
&\mathbb{E}\,\mathrm{e}^{\int_0^t K_{\Lambda(s_\delta)}\mathrm{d}s}\\
&\le\exp\left(\frac{\varepsilon}{1+\varepsilon}\max_{i\in\mathbb{S}}(-q_{ii})\right)\left(\mathbb{E}\,\mathrm{e}^{(1+\varepsilon)\int_0^t K_{\Lambda(s)}\mathrm{d}s}\right)^{\frac{1}{1+\varepsilon}}\exp\left(\frac{\varepsilon}{1+\varepsilon}\max_{i\in\mathbb{S}}(-q_{ii})t\right).
\end{aligned}
$$

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Thanks A Lot !