Switching Diffusions: Past-Dependent Switching Having A Countable State Space

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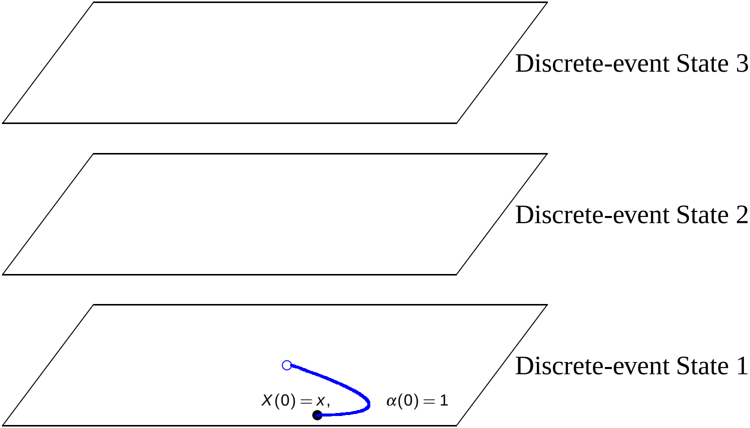
Outline

Switching Diffusion: An Illustration



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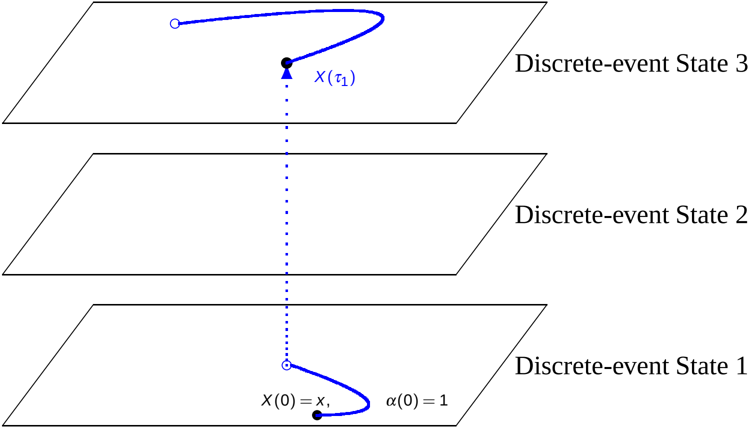

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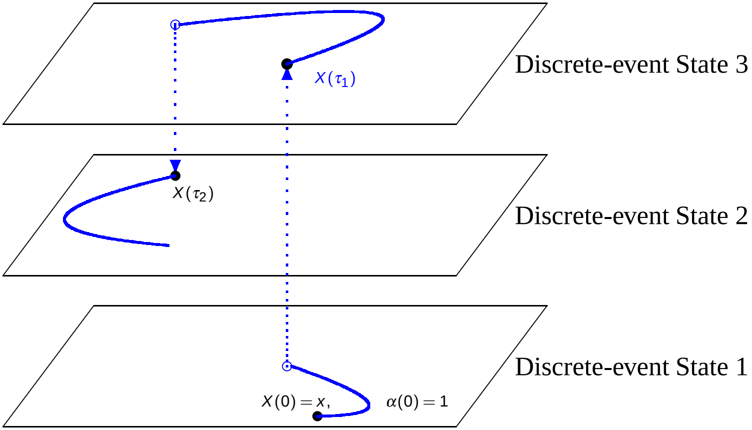

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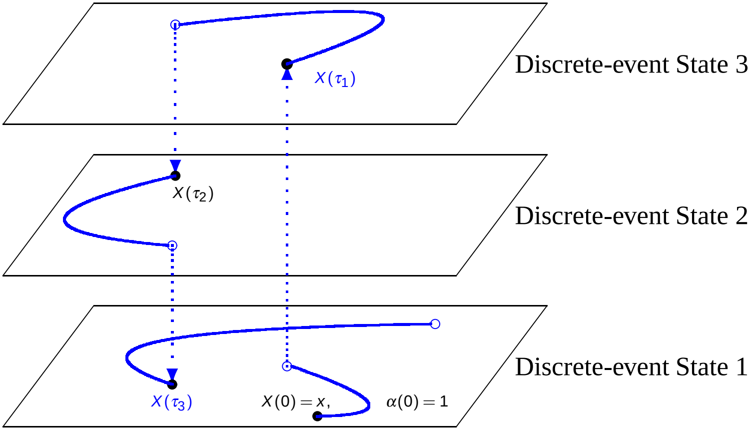

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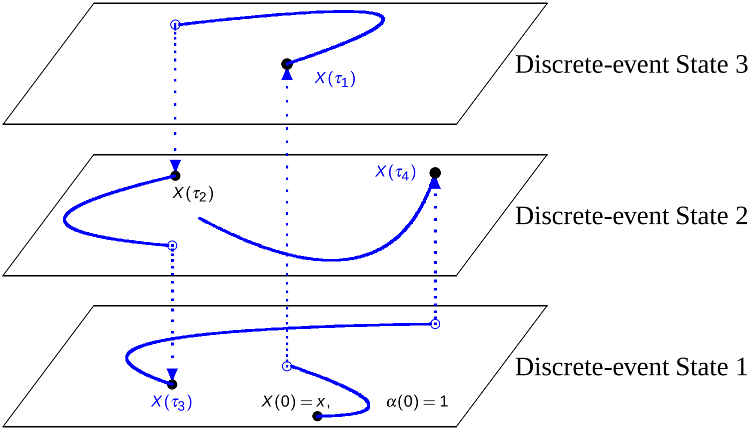


Figure: A “Sample Path” of Switching Diffusion $(X(t), \alpha(t))$.

Main Features

- continuous dynamics & discrete events coexist
- switching is used to model random environment or other random factors that cannot be formulated by the usual differential equations
- problems naturally arise in applications such as distributed, cooperative, and non-cooperative games, wireless communication, target tracking, reconfigurable sensor deployment, autonomous decision making, learning, etc.
- traditional ODE or SDE models are no longer adequate
- non-Gaussian distribution

An Example

Consider

$$\dot{x}(t) = A(\alpha(t))x(t) \tag{1.1}$$

where $\alpha(t)$ has two states $\{1,2\}$,

$$A(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

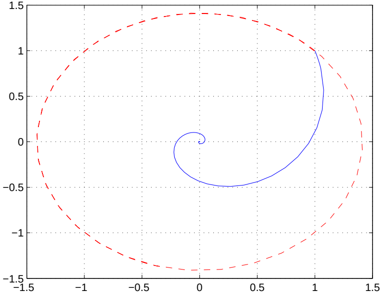
Associated with the hybrid system, there are two ODEs

$$\dot{x}(t) = A(1)x(t), \quad \text{and} \tag{1.2}$$

$$\dot{x}(t) = A(2)x(t) \tag{1.3}$$

switching back and forth according to $\alpha(t)$.

Phase Portrait of the Components



Phase portraits of the 'component' with a center (in dashed line) and the 'component' with a stable node (in solid line) with the same initial condition $x_0 = [1, 1]'$

Phase Portrait of Hybrid System

The phase portrait is given below.

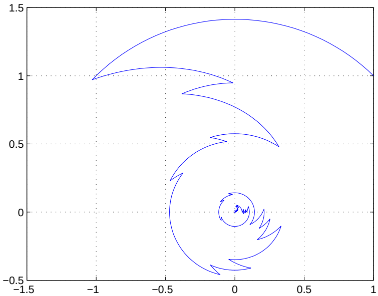


Figure: Switching linear system: Phase portrait of (1.1) with $x_0 = [1, 1]'$.

Seemingly Not Much Different from Diffusions without Switching?

Q: When we have a coupled system with $\mathscr{M} = \{1, 2\}$ and two stable linear systems, do we always get a stable system?

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Consider $\dot{x} = A(\alpha(t))x + B(\alpha(t))u(t)$, and a state feedback $u(t) = K(\alpha(t))x(t)$. Then one gets

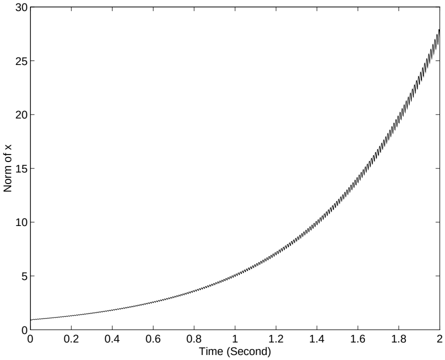
$$\dot{x} = [A(\alpha(t)) - B(\alpha(t))K(\alpha(t))]x.$$

Suppose that $\alpha(t) \in \{1,2\}$ such that

$$A(1) - B(1)K(1) = \begin{bmatrix} -100 & 20 \\ 200 & -100 \end{bmatrix}, \ A(2) - B(2)K(2) = \begin{bmatrix} -100 & 200 \\ 20 & -100 \end{bmatrix}.$$

The two feedback systems are stable individually. But if we choose $\alpha(t)$ so that it switches at $k\eta$, where $\eta = 0.01$. Then the resulting system is unstable.

The hybrid system is unstable



[L.Y. Wang, P.P. Khargonecker, and A. Beydoun, 1999, deterministic switching system]

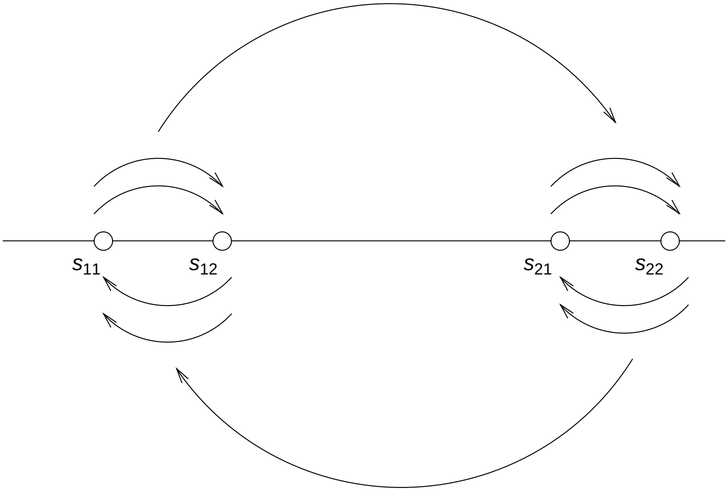
Why is the system unstable?

$$\frac{1}{2}[A(1) - B(1)K(1) + A(2) - B(2)K(2)] = \frac{1}{2}\begin{bmatrix} -200 & 220 \\ 220 & -200 \end{bmatrix}$$

is an unstable matrix.

The averaging effect dominates the dynamics.

Example: Two-time Scale (a demonstration)



Example: Schooling (Couzin et.al. Nature, 2005)

Example: High Way Traffic

Example: Honeybee Organization (Visscher, Nature, 2003)

What's new in this talk?

In our formulation

- The state space of $\alpha(t)$ is countably infinite.
- The switching rates of $\alpha(t)$ depends on $X(s), s \in [t-r, t]$.

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- Let r be a fixed positive number. Denote by $\mathscr{C}([a,b],\mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued continuous functions defined on $[a,b]$. We simply denote it by $\mathscr{C} := \mathscr{C}([-r,0],\mathbb{R}^n)$.

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- Let $W(t)$ be an $\mathscr{F}_t$-adapted and $\mathbb{R}^d$-valued Brownian motion.
- Suppose $b(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^{n\times d}$, where $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\} = \{1,2,\ldots\}$, the set of positive integers.

- Consider the two-component process $(X(t),\alpha(t))$, where $\alpha(t)$ is a pure jump process taking value in $\mathbb{Z}_+$, and $X(t)$ satisfies

$$dX(t) = b(X(t),\alpha(t))dt + \sigma(X(t),\alpha(t))dW(t). \tag{1.4}$$

- Assume that if $\alpha(t-) := \lim_{s\to t^-}\alpha(s) = i$, then it can switch to j at t with rate $q_{ij}(X_t)$ where $q_{ij}(\cdot) : \mathscr{C} \to \mathbb{R}$.

$$\begin{aligned}
&\mathbb{P}\{\alpha(t+\Delta) = j|\alpha(t) = i, X_s, \alpha(s), s \le t\} = q_{ij}(X_t)\Delta + o(\Delta) \text{ if } i \ne j \\
&\mathbb{P}\{\alpha(t+\Delta) = i|\alpha(t) = i, X_s, \alpha(s), s \le t\} = 1 - q_i(X_t)\Delta + o(\Delta).
\end{aligned} \tag{1.5}$$

- Assume the switching is conservative, i.e.,

$$q_i(\phi) = \sum_{j\ne i} q_{ij}(\phi) \text{ for any} (\phi, i) \in \mathscr{C} \times \mathbb{Z}_+. \tag{1.6}$$

Example 1

- Consider the evolution of two interacting species. One is micro: X, which is described by a logistic differential equation perturbed by a white noise. The other is macro: α, we assume that its number of individuals follows a birth-death process.

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- The life cycle of a micro species is usually very short, so it is reasonable to assume that the evolution of $X(t)$ can be described by

$$dX(t) = X(t)\big[a(\alpha(t)) - b(\alpha(t))X(t)\big]dt + \sigma(\alpha(t))X(t)dW(t), \quad (1.7)$$

where $a(i), b(i), \sigma(i)$ are positive constants for each $i \in \mathbb{Z}_+$.

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  where $a(i), b(i), \sigma(i)$ are positive constants for each $i \in \mathbb{Z}_+$.
- The reproduction process of $\alpha(t)$ is assumed to be non-instantaneous, depending on the period of time from egg formation to hatching, say r.

$$q_{i,j}(\phi) = \begin{cases} \delta_i(\phi) & \text{if } j = i-1 \\ \beta_i(\phi) & \text{if } j = i+1 \\ 0 & \text{if } j \neq i, i+1, i-1 \end{cases}$$

Existence and Uniqueness

- First in contrast to the case of switching process staying in a finite set, care needs to be exercised regarding uniformity with respect to the switching set.
- Second, the past dependence requires careful handling of the use of Lipschitz continuity etc. and the uniformity with respect to the element in the corresponding function spaces.
- Depending on the preference, Assumptions 2.1 allows certain bounds to be dependent of the switching state i, but uniform in the variable in the function space, whereas Assumption 2.2 requires uniformity in the bounds w.r.t. i, but requires the past dependent part be localized.

Assumption 2.1

(i) For each $i \in \mathbb{Z}_+$, there is a positive constant L_i such that

$$|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \le L_i |x - y| \forall x, y \in \mathbb{R}^n.$$

(ii) $q_{ij}(\phi)$ is measurable in $\phi \in \mathscr{C}$ for all i and $j \in \mathbb{Z}_+$. Moreover,

$$M := \sup_{\phi \in \mathscr{C}, i \in \mathbb{Z}_+} \{q_i(\phi)\} < \infty.$$

Assumption 2.2

(i) There is a positive constant L such that

$$|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \le L |x - y|, \ \forall x, y \in \mathbb{R}^n, i \in \mathbb{Z}_+.$$

(ii) $q_{ij}(\phi)$ is measurable in $\phi \in \mathscr{C}$ for each $(i,j) \in \mathbb{Z}_+^2$. Moreover, for any $H > 0$,

$$M_H := \sup_{\phi \in \mathscr{C}, \|\phi\| \le H, i \in \mathbb{Z}_+} \{q_i(\phi)\} < \infty.$$

Theorem 2.1

Under either Assumption 2.1 or Assumption 2.2, for each initial data (ξ, i_0)*, there exists a unique solution* $(X(t), \alpha(t))$ *to* (1.4) *and* (1.5).

Remark 1

To obtain the existence and uniqueness of solutions, Assumptions 2.1 and 2.2 can be relaxed by replacing the global Lipschitz conditions with local Lipschitz conditions together with Lyapunov-type functions.

Markov property

Theorem 2.2

Suppose that either Assumption 2.1 or Assumption 2.2 is satisfied. Let $(X(t), \alpha(t))$ *be a solution to* (1.4) *and* (1.5). *Then* $(X_t, \alpha(t))$ *is a homogeneous strong Markov process taking value in* $\mathscr{C} \times \mathbb{Z}_+$ *with transition probabilities*

$$P(\phi, i, t, A \times \{j\}) = \mathbb{P}\{X_t^{\phi,i} \in A, \alpha(t) = j\},$$

where $X^{\phi,i}(t)$ *is the solution to* (1.4) *and* (1.5) *with initial data* $(\phi, i) \in \mathscr{C} \times \mathbb{Z}_+$.

Feller property

- In addition to the sufficient conditions for the existence and uniqueness of solution, we prove the Feller property of the solution only with an additional condition that $q_{ij}(\phi)$ is continuous in ϕ for any $i, j \in \mathbb{Z}_+$.
- There are some difficulties because the process $\{X_t\}$ takes value in an infinite dimensional Banach space and the switching $\{\alpha(t)\}$ has an infinite state space. Moreover, although we suppose that $q_{ij}(\phi)$ is continuous, neither the uniform continuity in $\phi \in \mathscr{C}$ nor equi-continuity in $i, j \in \mathbb{Z}_+$ is assumed.

Auxiliary Process

Let $\gamma(t)$ be a Markov chain with generator $\tilde{Q} = (\rho_{ij})$ for $(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, where $\rho_{ii} = -1$ and $\rho_{ij} = 2^{-j}$ if $j < i$ and $\rho_{ij} = 2^{-j+1}$ if $j > i$, i.e.,

$$\tilde{Q} = \begin{pmatrix} -1 & 1/2 & 1/4 & \cdots \\ 1/2 & -1 & 1/4 & \cdots \\ 1/2 & 1/4 & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$dZ(t) = b(Z(t), \gamma(t))dt + \sigma(Z(t), \gamma(t))dW(t), t \geq 0$$

Similar to Girsanov’s theorem, which tells us how to convert an Itô process to a Brownian motion under a change of measure, we aim to establish a change of measure allowing us to “convert" a hybrid diffusion with past-dependent switching to a hybrid diffusion with Markov switching.

Let α_k, γ_k be values of $\alpha(t), \gamma(t)$ at τ_k, θ_k ((the k-th jump moments of $\alpha(t)$ and $\gamma(t)$), respectively. $X_{(k)}, Z_{(k)}$ be values of X_t, Z_t at τ_k, θ_k respectively.

Change of Measures

Proposition 2.3

For any $T > 0$, let $f(\cdot,\cdot) : \mathscr{C} \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be a bounded continuous function. For any $l = 0, 1, \ldots,$ any $i_k \in \mathbb{Z}_+$ with $i_k \neq i_{k+1}$ and $k = 1, \ldots, l+1$, and any $(\phi, i) \in \mathscr{C} \times \mathbb{Z}_+$,

$$\begin{aligned}
&\mathbb{E}_{\phi,i}\left[f(X_T, \alpha(T)) \mathbf{1}_{\{\tau_l \le T < \tau_{l+1}\}} \prod_{k=1}^{l} \mathbf{1}_{\{\alpha(\tau_k) = i_k\}} \right] \\
&\quad = e^T \mathbb{E}_{\phi,i}\Bigg[f(Z_T, i_l) \mathbf{1}_{\{\theta_l \le T < \theta_{l+1}\}} \\
&\qquad\qquad \times \prod_{k=1}^{l} \left(\mathbf{1}_{\{\gamma(\theta_k) = i_k\}} \frac{q_{i_k i_{k+1}}(Z_{\theta_k})}{\rho_{i_k i_{k+1}}} \right) \exp\left\{ - \int_0^T q_{\gamma(s)}(Z_s) ds \right\} \Bigg].
\end{aligned} \tag{2.1}$$

Lemma 2.1

Let $(\phi_0, i_0) \in \mathscr{C} \times \mathbb{Z}_+$ *with* $\|\phi_0\| \leq R$ *and* $T > 0$. *For each* $\Delta > 0$, *there exist* $m = m(\Delta) \in \mathbb{Z}_+$, $n_m = n_m(\Delta) \in \mathbb{Z}_+$, *and* $d_m = d_m(\Delta) > 0$ *such that*

$$\mathbb{P}_{\phi,i_0}\Big(\{\tau_{m+1} > T\} \cap \{\alpha(t) \leq n_m, \forall t \in [0,T]\}\Big) \geq 1 - \Delta, \quad \forall \|\phi - \phi_0\| < d_m,$$

where $N_k = \{1, \ldots, k\}$.

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where $N_k = \{1, \ldots, k\}$*.*

Under suitable conditions, sample paths of a diffusion $[0, T]$ are Hölder continuous. Thus, it is easy to find a compact set in which sample paths of a diffusion process lie with a large probability. Our arguments rely on this fact. However, the initial data ϕ of our process $X(t)$ does not always satisfy the Hölder continuity.

Theorem 2.4

Let either Assumption 2.1 or Assumption 2.2 be satisfied. Assume further that $q_{ij}(\cdot)$ *is a continuous function for any* $i, j \in \mathbb{Z}_+$*. Then the solution to* (1.4) *and* (1.5) *has the Feller property.*

Recurrence and Ergodicity

Functional Itô formula

In a recent insightful work, Dupire (2009) proposed a method to extend the Itô formula to a functional setting using a pathwise functional derivative that quantifies the sensitivity of a functional variation in the endpoint of a path.

This work encouraged subsequent development (for example, [5, 18]).

Let $\mathbb{D}$ be the space of cadlag functions $f : [-r, 0] \mapsto \mathbb{R}^n$. For $\phi \in \mathbb{D}$, we define horizontal (time) and vertical (space) perturbations for $h \geq 0$ and $y \in \mathbb{R}^n$ as

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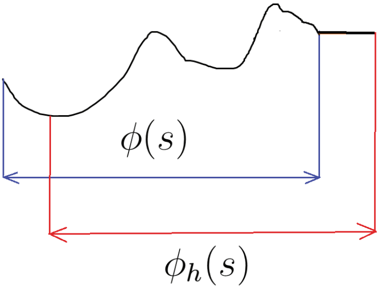
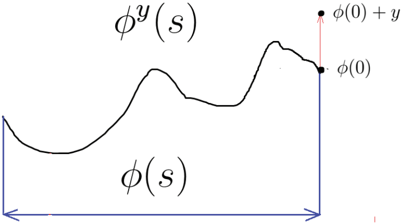
$$\phi_h(s) = \begin{cases} \phi(s+h) \text{ if } s \in [-r,-h], \\ \phi(0) \text{ if } s \in [-h,0], \end{cases}$$

and

$$\phi^y(s) = \begin{cases} \phi(s) \text{ if } s \in [-r,0), \\ \phi(0)+y, \end{cases}$$

respectively.





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$$\partial_\ell V(\phi, i) = \lim_{h \to 0} \frac{V(\phi^{he_\ell}, i) - V(\phi)}{h} \tag{3.2}$$

if these limits exist. In (3.2), e_ℓ is the standard unit vector in $\mathbb{R}^n$ whose ℓ-th component is 1 and other components are 0.

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- V, V_t, $V_x = (\partial_k V)$ and $V_{xx} = (\partial_{kl} V)$ are bounded in each $B_R := \{(\phi, i) : \|\phi\| \le R, i \le R\}$, $R > 0$.

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Let $V(\cdot,\cdot) \in \mathbb{F}$, we define the operator

$$\begin{aligned}
\mathscr{L}V(\phi,i) =& V_t(\phi,i) + V_x(\phi,i) b(\phi(0),i) + \frac{1}{2}\operatorname{tr}\Big(V_{xx}(\phi,i)A(\phi(0),i)\Big) \\
&+ \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)\big[V(\phi,j) - V(\phi,i\big] \\
=& V_t(\phi,i) + \sum_{k=1}^{n} b_k(\phi(0),i)V_k(\phi,i) + \frac{1}{2}\sum_{k,l=1}^{n} a_{kl}(\phi(0),i)V_{kl}(\phi,i) \\
&+ \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)\big[V(\phi,j) - V(\phi,i)\big],
\end{aligned} \tag{3.3}$$

for any bounded stopping time $\tau_1 \leq \tau_2$, we have the functional Itô formula:

$$\mathbb{E}V(X_{\tau_2}, \alpha(\tau_2)) = \mathbb{E}V(X_{\tau_1}, \alpha(\tau_1)) + \mathbb{E}\int_{\tau_1}^{\tau_2} \mathscr{L}V(X_s, \alpha(s))ds \qquad (3.4)$$

if the expectations involved exist.

Remark 3.1

Consider functions of the form

$$V(\phi,i) = f_1(\phi(0),i) + \int_{-r}^{0} g(t,i) f_2(\phi(t),i)dt.$$

where $f_2(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ is a continuous function and $f_1(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ is a function that is twice continuously differentiable in the first variable and $g(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be a continuously differentiable function in the first variable. Then at $(\phi,i) \in \mathscr{C} \times \mathbb{Z}_+$ we have (see [18] for the detailed computations)

$$V_t(\phi,i) = g(0,i)f_2(\phi(0),i) - g(-r,i)f_2(\phi(-r),i) - \int_{-r}^{0} f_2(\phi(t),i)dg(t,i),$$

$$\partial_k V(\phi,i) = \frac{\partial f_1}{\partial x_k}(\phi(0),i), \qquad \partial_{kl} V(\phi,i) = \frac{\partial^2 f_1}{\partial x_k \partial x_l}(\phi(0),i).$$

Irreducibility

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H1 (a) For any $i \in \mathbb{Z}_+$, $A(x,i)$ is elliptic uniformly on each compact set, where $A(x,i) = \sigma(x,i)\sigma^{\mathsf{T}}(x,i)$

(b) There is an i^* satisfying that for any $i \in \mathbb{Z}_+$, there exist $i_1, \ldots, i_k \in \mathbb{Z}_+$ and $\phi_1, \ldots, \phi_{k+1} \in \mathscr{C}$ such that $q_{ii_1}(\phi_1) > 0$, $q_{i_l,i_{l+1}}(\phi_{l+1}) > 0, l = 1, \ldots, k-1$, and $q_{i_k,i^*}(\phi(k+1)) > 0$.

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H2 (a) There exists an $i^* \in \mathbb{Z}_+$ such that $A(x,i^*)$ is elliptic uniformly on each compact set.

(b) For any $(\phi,i) \in \mathscr{C} \times \mathbb{Z}_+$, there exist positive integers $i = i_1,\ldots,i_k = i^*$ such that $q_{i_l,i_{l+1}}(\phi) > 0, l = 1,\ldots,k-1$.

Lemma 3.1 (Doeblin's condition)

Assume that either (H1) *or* (H2) *is satisfied. There is* $T_0 > 0$ *and a nontrivial measure* $\nu(\cdot)$ *on* $\mathfrak{B}(\mathscr{C})$ *such that* $\nu(D) > 0$ *if* D *is a nonempty open subset of* $\mathscr{C}$ *and that for any* $R > 0$, $T > T_0$, *there is a* $d_{R,T,i} > 0$ *satisfying*

$$\mathbb{P}_{\phi,i}\{X_T \in D \text{ and } \alpha(T) = i^*\} \geq d_{R,T,i}\nu(D), D \in \mathfrak{B}(\mathscr{C}) \text{ given that } \|\phi\| \leq R. \tag{3.5}$$

Definition 3.1

The process $\{(X_t,\alpha(t)) : t \geq 0\}$ is said to be recurrent (resp., positive recurrent) relative to a measurable set $E \in \mathscr{C} \times \mathbb{Z}_+$ if

$$\mathbb{P}_{\phi,i}\{(X_t,\alpha(t)) \in E \text{ for some } t \geq 0\} = 1$$

$$\left(\text{resp. } \mathbb{E}_{\phi,i}[\inf\{t > 0 : (X_t,\alpha(t)) \in E\}] < \infty\right)$$

for any $(\phi,i) \in \mathscr{C} \times \mathbb{Z}_+$.

Theorem 3.1

Suppose that either (H1) *or* (H2) *holds. Let D be a bounded open subset of $\mathscr{C}$ and N be a finite subset of $\mathbb{Z}_+$. If $(X_t, \alpha(t))$ is recurrent relative to $D \times N$ then $(X_t, \alpha(t))$ is recurrent relative to $D' \times N'$ for any open set $D' \subset \mathscr{C}$ and a finite set $N' \subset \mathbb{Z}_+$ containing i^* with i^* given in either* (H1) *or* (H2) *according to which hypothesis is satisfied.*

Theorem 3.2

Suppose that either (H1) *or* (H2) *holds. Let* $V(\cdot,\cdot) \in \mathbb{F}$ *satisfying*

$$\lim_{n\to\infty} \inf\{V(\phi,i) : |\phi(0)| \vee i \geq n\} = \infty. \tag{3.6}$$

Suppose further that there are $C > 0$ *&* $H > 0$ *satisfying*

$$\mathscr{L}V(\phi,i) \leq C\mathbf{1}_{\{V(\phi,i)\leq H\}}. \tag{3.7}$$

Then $(X_t, \alpha(t))$ *is recurrent relative to* $D \times N$*, where* D *is any open bounded subset of* $\mathscr{C}$ *and* $N \subset \mathbb{Z}_+$ *contains* i^*.

Theorem 3.3

Suppose either (H1) *or* (H2) *holds. Let* $V(\cdot,\cdot) \in \mathbb{F}$ *satisfying*

$$\lim_{n\to\infty} \inf\{V(\phi,i) : |\phi(0)| \vee i \geq n\} = \infty. \tag{3.8}$$

Suppose further there are $C_1, C_2 > 0$ *&* H *s.t.*

$$\mathscr{L}V(\phi,i) \leq -C_1 + C_2 \mathbf{1}_{\{V(\phi,i)\geq H\}}. \tag{3.9}$$

Then, $(X_t, \alpha(t))$ *is positive recurrent relative to any set of the form* $D \times N$ *where* D *is a nonempty open set of* $\mathscr{C}$ *and* $N \ni i^*$ *with* i^* *given in either* (H1) *and* (H2)*. Moreover, there is a unique invariant probability measure* μ^**, and for any* $(\phi,i) \in \mathscr{C} \times \mathbb{Z}_+$

$$\lim_{t\to\infty} \|P(t,(\phi,i),\cdot) - \mu^*\|_{TV} = 0.$$

Stability

- Sufficient conditions for stability can be given via Lyapunov functions.
- We focus on some practical conditions.
- To find sufficient conditions for stability, it is desirable to find some common threads that are shared by many specific systems. Our motivation is based on the following thoughts. First, although the dynamics of $X(t)$ depend on the residence of the state of $\alpha(t)$, the structures of equations for different states of $\alpha(t)$ are not drastically different but rather similar in certain sense. This observation suggests finding a Lyapunov function that has similar form in different states of $\alpha(t)$.

- For instance, suppose there is a Lyapunov function $V(x)$ such that in each discrete state i, we have $\mathscr{L}_i V(x) \le c_i V(x)$, where $\mathscr{L}_i$ is the generator of the diffusion in regime i
- It is well known that the sign of c_i determines stability of the diffusion in each state i.
- For the switching diffusion, one can expect that the stability of the system depends not only on $\{c_i\}$ but also on the generator $Q(\cdot) = (q_{ij}(\cdot))_{\mathbb{Z}_+ \times \mathbb{Z}_+}$ of the switching part.

  A natural question is: under what relation between $\{c_i\}$ and $Q(\cdot)$, the switching diffusion is stable?

- Although the results hold for past-dependent cases, to elaborate on the main idea, we restrict ourselves to the case Q depends only on the current state of $X(t)$ rather than a history of $X(t)$.

Let $\widehat{\alpha}(t)$ be the Markov chain with generator $Q(0)$.
We suppose that $b(0,i) = 0, \sigma(0,i) = 0, i \in \mathbb{Z}_+$.
If $X(0) = 0$ then $X(t) \equiv 0$ and $\alpha(t) = \widehat{\alpha}(t)$.

Definition 4.1

The trivial solution $X(t) \equiv 0$ is said to be

- *stable in probability*, if for any $h > 0$,

$$\lim_{x \to 0} \inf_{i \in \mathbb{Z}_+} \mathbb{P}_{x,i}\{X(t) \le h \forall t \ge 0\} = 1.$$

- *asymptotically stable in probability*, if it is stable in probability and

$$\lim_{x \to 0} \inf_{i \in \mathbb{Z}_+} \mathbb{P}_{x,i}\left\{\lim_{t \to \infty} X(t) = 0\right\} = 1.$$

Definition 4.2

The Markov chain $\widehat{\alpha}(t)$ is said to be

- *ergodic*, if it has an invariant probability measure $\nu = (\nu_1, \nu_2, \ldots)$ and

$$\lim_{t\to\infty} \widehat{p}_{ij}(t) = \nu_j \text{ for any } i, j \in \mathbb{Z}_+ \text{ or equivalently,}$$

$$\lim_{t\to\infty} \sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \nu_j| = 0 \text{ for any } i \in \mathbb{Z}_+,$$

- *strongly ergodic*, if

$$\lim_{t\to\infty} \sup_{i\in\mathbb{Z}_+} \left\{ \sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \nu_j| \right\} = 0.$$

- *strongly exponentially ergodic*, if $\exists\, C > 0$ and $\lambda > 0$ such that

$$\sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \pi_j| \le Ce^{-\lambda t} \text{ for any } i \in \mathbb{Z}_+, t \ge 0. \tag{4.1}$$

Theorem 4.3

Suppose that the Markov chain $\widehat{\alpha}(t)$ is strongly exponentially ergodic with invariant probability measure $\nu = (\nu_1, \nu_2, \dots)$ and that

$$\sup_{i\in\mathbb{Z}_+} \sum_{j\neq i} |q_{ij}(x) - q_{ij}(0)| \to 0 \text{ as } x \to 0. \tag{4.2}$$

Let D be a neighborhood of 0 and $V : D \mapsto \mathbb{R}_+$ satisfying that $V(x) = 0$ if and only if $x = 0$ and that $V(x)$ is continuous on D, twice continuously differentiable in $D \setminus \{0\}$. Suppose that there is a bounded sequence of real numbers $\{c_i : i \in \mathbb{Z}_+\}$ such that

$$\mathscr{L}_i V(x) \leq c_i V(x) \forall x \in D \setminus \{0\}. \tag{4.3}$$

Then, if $\sum_{i\in\mathbb{Z}_+} c_i \nu_i < 0$, the trivial solution is asymptotic stable in probability.

Since $\widehat{\alpha}(t)$ is strongly exponentially ergodic, the Fredholm alternative works with $Q(0)$.

Since $\widehat{\alpha}(t)$ is strongly exponentially ergodic, the Fredholm alternative works with $Q(0)$.

Lemma 4.1 (Fredholm alternative)

If a Markov chain is strongly exponentially ergodic with generator $\tilde{Q}$ and invariant probability measure $\nu = (\nu_1, \nu_2, \dots)^\top$, then if $\mathbf{b} = (b_1, b_2, \dots)^\top$ is bounded satisfying $\nu^\top \mathbf{b} = 0$, then, there exists a bounded vector $\mathbf{c} = (c_1, c_2, \dots)^\top$ such that $\tilde{Q}\mathbf{c} = \mathbf{b}$.

Thus we can find (γ_i) such that the function $U(x,i) = \gamma_i V^p(x)$ for some sufficiently small p satisfies

$$\mathscr{L}U(x,i) \leq -\lambda U(x,i),\ \lambda > 0$$

Some questions

- Strongly exponential ergodicity is too strong. Can we relax it?
- What if we have $\mathscr{L}_i V(x) \leq c_i g(V(x))$ in lieu of $\mathscr{L}_i V(x) \leq c_i V(x)$
- Can we estimate pathwise convergence rate?

Let Γ be a family of increasing and continuously differentiable functions $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $g(y) = 0$ iff $y = 0$. The function

$$G(y) := -\int_y^h \frac{dz}{g(z)} \quad \text{on } [0,h] \tag{4.4}$$

is non-positive and strictly decreasing and $\lim_{y\to 0} G(y) = -\infty$. Its inverse $G^{-1} : (-\infty, 0] \mapsto (0,h]$ satisfies

$$\lim_{t\to-\infty} G^{-1}(t) = 0.$$

Assumption 4.1

Let D be a neighborhood of 0 and $V : D \mapsto \mathbb{R}_+$ satisfying that $V(x) = 0$ if and only if $x = 0$ and that $V(x)$ is continuous on D, twice continuously differentiable in $D \setminus \{0\}$. Suppose that there is a bounded sequence of real numbers $\{c_i : i \in \mathbb{Z}_+\}$ and a function $g(\cdot) \in \Gamma$ such that

$$\mathscr{L}_i V(x) \leq c_i g(V(x)) \forall x \in D \setminus \{0\}. \tag{4.5}$$

$$M_g := \sup_{0<|x|<h, i \in \mathbb{Z}_+} \left\{ \left| \frac{V_x(x)\sigma(x,i)}{g(V(x))} \right| \right\} < \infty, \tag{4.6}$$

$\mathscr{L}_i$ is the generator of the diffusion process at state i.

Theorem 4.4

Let Assumption 4.1 is satisfied. Suppose that the Markov chain $\widehat{\alpha}(t)$ is strongly ergodic with invariant probability measure $\nu = (\nu_1, \nu_2, \dots)$ and

$$\sup_{i \in \mathbb{Z}_+} \sum_{j \neq i} |q_{ij}(x) - q_{ij}(0)| \to 0 \text{ as } x \to 0. \quad (4.7)$$

is satisfied.

Then, if $\sum_{i \in \mathbb{Z}_+} c_i \nu_i < 0$, the trivial solution is asymptotically stable in probability. Moreover, there is a $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\lim_{t \to \infty} \frac{V(X(t))}{G^{-1}(-\lambda t)} \leq 1\right\} > 1 - \varepsilon \text{ for any } (x,i) \in B_\delta \times \mathbb{Z}_+. \quad (4.8)$$

Ideas of the proof

- We present the ideas for the case g is the identity function:

$$g(y) \equiv y.$$

- For general g, we can prove similarly by using a change of Lyapunov functions.
- The idea is as follows:
  When x is close to 0, then when T is sufficiently large, we have

$$\mathbb{E}_{x,i} \ln V(X(T)) \leq -\lambda T + \ln V(x). \tag{4.9}$$

  Make use of the log-Laplace transform to interchange the order of $\ln$ and $\mathbb{E}$ to obtain:

$$\mathbb{E}_{x,i} V^{\theta}(X(T)) \leq e^{-\theta\lambda T} V^{\theta}(x). \tag{4.10}$$

$$
\begin{aligned}
\ln\left(V(X(\tau_h\wedge t))\right) = & \ln(V(x)) + \int_0^{\tau_h\wedge t}\frac{\mathscr{L}_{\alpha(s)}V(X(s))}{V(X(s))}ds \\
& - \int_0^{\tau_h\wedge t}\frac{\left|V_x(X(s))\sigma(X(s),\alpha(s))\right|^2}{2V^2(X(s))}ds \\
& + \int_0^{\tau_h\wedge t}\frac{V_x(X(s))\sigma(X(s),\alpha(s))}{V(X(s))}dW(s).
\end{aligned} \tag{4.11}
$$

Thus

$$
\ln\left(V(X(\tau_h\wedge t))\right) \le \ln V(x) + H(t) \tag{4.12}
$$

where

$$
H(t) = \int_0^{\tau_h\wedge t} c(\alpha(s))ds + \int_0^{\tau_h\wedge t}\frac{V_x(X(s))\sigma(X(s),\alpha(s))}{V(X(s))}dW(s).
$$

and $\tau_h = \inf\{t \ge 0 : |X_t| > h\}$

Let

$$-\lambda := \sum c_i v_i < 0.$$

Because of the uniform ergodicity of $\widehat{\alpha}(t)$, there exists a $T > 0$ such that

$$\mathbb{E}_{0,i}\int_0^T c(\alpha(s))ds = \mathbb{E}_i\int_0^T c(\widehat{\alpha}(s))ds \le -\frac{3\lambda}{4}T \quad \forall i \in \mathbb{Z}_+. \tag{4.13}$$

By the assumption

$$\sup_{i\in\mathbb{Z}_+}\sum_{j\neq i}|q_{ij}(x)-q_{ij}(0)| \to 0 \text{ as } x \to 0. \tag{4.14}$$

there exists an $h_1 \in (0,h)$ such that

$$\mathbb{E}_{x,i}H(T) = \mathbb{E}_{x,i}\int_0^{\tau_h\wedge T} c(\alpha(s))ds \le -\frac{\lambda}{2}T \quad \forall |x| \le h_1, i \in \mathbb{Z}_+, \tag{4.15}$$

Lemma 4.2

Let Y be a random variable, $\theta_0 > 0$ a constant, and suppose

$$\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y) \leq K_1.$$

Then the log-Laplace transform $\phi(\theta) = \ln \mathbb{E}\exp(\theta Y)$ is twice differentiable on $\left[0, \frac{\theta_0}{2}\right)$ and

$$\frac{d\phi}{d\theta}(0) = \mathbb{E}Y, \quad \text{and } 0 \leq \frac{d^2\phi}{d\theta^2}(\theta) \leq K_2\,, \theta \in \left[0, \frac{\theta_0}{2}\right)$$

for some $K_2 > 0$. As a result of Taylor's expansion, we have

$$\phi(\theta) \leq \theta\mathbb{E}Y + \theta^2 K_2, \text{ for } \theta \in [0, 0.5\theta_0).$$

$$
\begin{aligned}
\ln \mathbb{E}_{x,i} e^{\theta H(T)} &\leq \theta \mathbb{E}_{x,i} H(T) + \theta^2 K \\
&\leq -\theta \frac{\lambda T}{2} + \theta^2 K
\end{aligned}
\tag{4.16}
$$

for some $K > 0$ depending on $T, \bar{c} = \sup\{|c_i|\}$ and M_g. If choosing θ such that

$$
\theta K < \frac{\lambda T}{4} \tag{4.17}
$$

we have

$$
\ln \mathbb{E}_{x,i} e^{\theta H(T)} \leq -\frac{\theta \lambda T}{4} \text{ for } 0 < |x| < h_1, i \in \mathbb{Z}_+
$$

or equivalently,

$$
\mathbb{E}_{x,i} e^{\theta H(T)} \leq \exp\left\{ -\frac{\theta \lambda T}{4} \right\} \text{ for } 0 < |x| < h_1, i \in \mathbb{Z}_+. \tag{4.18}
$$

This and the fact that $\ln\left(V(X(\tau_h \wedge t))\right) \leq \ln V(x) + H(t)$ imply

$$
\mathbb{E}_{x,i} V(X(\tau_h \wedge T)) \leq V(x) \mathbb{E}_{x,i} e^{\theta H(T)} \leq V(x) \exp\left\{ -\frac{\theta \lambda T}{4} \right\}. \tag{4.19}
$$

Theorem 4.5

Suppose that the Markov chain $\widehat{\alpha}(t)$ is ergodic with invariant probability measure $\nu = (\nu_1, \nu_2, \dots)$ and Assumption 4.1 is satisfied with additional conditions:

$$\limsup_{i\to\infty} c_i < 0, \tag{4.20}$$

Then, if $\sum_{i\in\mathbb{Z}_+} c_i \nu_i < 0$, the trivial solution is asymptotic stable in probability. Moreover, there is a $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\lim_{t\to\infty} \frac{V(X(t))}{G^{-1}(-\lambda t)} \leq 1\right\} > 1-\varepsilon \text{ for any } (x,i) \in B_\delta \times \mathbb{Z}_+. \tag{4.21}$$

Example 2

Consider a real-valued switching diffusion

$$dX(t) = b(\alpha(t))X(t)[|X(t)|^{\gamma} \vee 1]dt + \sigma(\alpha(t))[|X(t)|^2 \vee 1]dW(t), \ 0 < \gamma < 0, \tag{4.22}$$

where $a \vee b = \max(a, b)$ for two real numbers a and b, and $Q(x) = \big(q_{ij}(x)\big)_{\mathbb{Z}_+ \times \mathbb{Z}_+}$ with

$$q_{ij}(x) = \begin{cases} -\check{p}_1(x) & \text{if } i = j = 1 \\ \check{p}_1(x) & \text{if } i = 1, j = 2 \\ -\widehat{p}_i(x) - \check{p}_i(x) & \text{if } i = j \geq 2 \\ \widehat{p}_i(x) & \text{if } i \geq 2, j = i - 1 \\ \check{p}_i(x) & \text{if } i \geq 2, j = i + 1. \end{cases}$$

$\widehat{\alpha}(t)$ is ergodic with the invariant measure ν given by

$$\nu_1 = \frac{1}{\nu^*}, \ \nu_k = \frac{1}{\nu^*}\prod_{\ell=2}^{k}\frac{\check{p}_{\ell-1}(0)}{\widehat{p}_{\ell}(0)}, k \geq 2.$$

Let $V(x) = x^2$, we have

$$\mathscr{L}_i V(x) = 2b(i)|x|^{2+2\gamma} + \sigma^2(i)|x|^4(x)$$

Since $\gamma < 1$ and $\sigma(i)$ is bounded,

$$\mathscr{L}_i V(x) \leq [2b(i)+\varepsilon]|x|^{2+2\gamma} = [2b(i)+\varepsilon]V^{1+\gamma}(x) \text{ in } [-\hbar,\hbar]\times\mathbb{Z}_+.$$

Suppose

$$\sum b_i \nu_i < 0, \text{ and } \limsup_{i\to\infty} b_i < 0$$

Thus, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $(x,i) \in [0,\delta]\times\mathbb{Z}_+$, then, there exists a $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\limsup_{t\to\infty} t^{1/\gamma}X^2(t) \leq \lambda\right\} > 1-\varepsilon.$$

Thank you

J. Bao, J. Shao, C. Yuan, Approximation of invariant measures for regime-switching diffusions, *Potential Anal.* **44** (2016), no. 4, 707-727.

Z.-Q. Chen, Z. Zhao, Harnack principle for weakly coupled elliptic systems, *J. Differential Equations*, **139** (1997), no. 2, 261ï¿½282.

Z.-Q. Chen, Z. Zhao, Potential theory for elliptic systems, *Ann. Probab.* **24** (1996), no. 1, 293-319.

R. Cont, D.-A. Fournié, Change of variable formulas for non-anticipative functionals on path space, *J. Funct. Anal.*, **259** (2010), no. 4, 1043-1072.

R. Cont, D.-A. Fournié, Functional Itô calculus and stochastic integral representation of martingales, *Ann. Probab.*, **41**, (2013) no. 1 109-133.

B. Dupire, Functional Itô's Calculus. Bloomberg Portfolio Research Paper No. 2009-04-FRONTIERS Available at SSRN: http://ssrn.com/abstract=1435551 or http://dx.doi.org/10.2139/ssrn.1435551.

A. Eizenberg, M. Freidlin. On the Dirichlet problem for a class of second order PDE systems with small parameter, *Stochastics Stochastic Rep.* **33** (1990) no.3-4, 111-148.

- L.C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.
- I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 2012.
- R.Z. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed., Springer, Berlin, 2012.
- R.Z. Khasminskii, C. Zhu, G. Yin, Stability of regime-switching diffusions, *Stochastic Process. Appl.* **117** (2007), no. 8, 1037-1051.
- N.V. Krylov, *Controlled Diffusion Processes*, Springer, 2008.
- X. Mao, *Stochastic Differential Equations and Applications*, 2nd Ed., Horwood, Chinester, 2008.
- X. Mao, C. Yuan. *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006.
- S.P. Meyn, R.L. Tweedie, Stability of Markovian processes. I. criteria for discrete-time chains, *Adv. in Appl. Probab.*, **24** (1992), no. 3, 542-574.

- D.H. Nguyen, G. Yin, Modeling and analysis of switching diffusion systems: Past-dependent switching with a countable state space, *SIAM J. Control Optim.* **54** (2016), no. 5, 2450-2477.
- E. Nummelin, *General Irreducible Markov Chains and Nonnegative Operators*, Cambridge University Press, Cambridge, 1984.
- T. Pang, A. Hussain, An application of functional Itô's formula to stochastic portfolio optimization with bounded memory, *Proceedings of 2015 SIAM Conference on Control and Its Applications* (CT15), 159-166.
- J. Shao, Criteria for transience and recurrence of regime-switching diffusion processes, *Electron. J. Probab.* **20** 2015, no. 63, 15 pp.
- J. Shao, F. Xi, Strong ergodicity of the regime-switching diffusion processes. *Stochastic Process. Appl.* **123** (2013), no. 11, 3903-3918.
- J. Shao, F. Xi, Stability and recurrence of regime-switching diffusion processes, *SIAM J. Control Optim.* **52** (2014), no. 6, 3496-3516.
- Stroock, D. W., Varadhan, S. R. On the support of diffusion processes with applications to the strong maximum principle. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, 1972, 333-359.

P. Tuominen, R. L. Tweedie. Subgeometric rates of convergence of f-ergodic Markov chains, *Adv. in Appl. Probab.* **26**(1994), 775-798.

Z. Wu, J. Yin, C. Wang, *Elliptic and Parabolic Equations.* Hackensack: World Scientific, 2006.

G. Yin, C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York, 2010.

C. Zhu, G. Yin, Asymptotic properties of hybrid diffusion systems, *SIAM J. Control Optim.* **46** (2007), no. 4, 1155-1179.