Ergodicity of stochastic differential equations with jumps and singular coefficients

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Overview

Background

Consider the following ordinary differential equation (ODE):

$$x'(t) = b\big(x(t)\big), \quad x(0) = x_0 \in \mathbb{R}^d.$$

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For $d = 1$, $b(x) = 2sign(x)\sqrt{|x|}$ and $x_0 = 0$, the above equation has infinitely many solutions:

$$X(t) \equiv 0, \quad X(t) = t^2, \quad X(t) = -t^2, \quad \cdots$$

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Note that the function b is Hölder continuous.

It is interesting to find that noises may produce some regularization effects.

Background

Consider the following stochastic differential equation (SDE):

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + g(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t, \tag{1.1}$$

with $X_0 = x \in \mathbb{R}^d$.

$(W_t)_{t\geqslant 0}$ is an m-dimensional standard Brownian motion.

$(L_t)_{t\geqslant 0}$ is a k-dimensional pure jump Lévy process.

Background

In the case $g \equiv 0$:

- N. V. Krylov and M. Röckner (2005, PTRF):

$$\mathrm{d}X_t = \mathrm{d}W_t + b(X_t)\mathrm{d}t, \quad X_0 = x.$$

Condition: $b \in L^p(\mathbb{R}^d)$ with $p > d$.

- X. Zhang (2005, SPA):

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t, \quad X_0 = x.$$

Condition: σ is bounded and uniformly elliptic and $\nabla\sigma \in L^p(\mathbb{R}^d)$ with $p > d$.

Background

There are also many works devoted to study the properties of the unique strong solution:

- E. Fedrizzi and F. Flandoli (2013, JFA):
  The map $x \to X_t(x)$ is Sobolev differentiable.
- T. Zhang, etc. (2013, Math. Annalen):
  The map $\omega \to X_t(\omega)$ is Malliavin differentiable.
- L. Xie and X. Zhang (2016, AOP):
  The strong solution X_t is strong Feller and irreducible.

Background

In the case $\sigma \equiv 0$:

$$\mathrm{d}X_t = \mathrm{d}L_t + b(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

where L_t is a symmetric α-stable process.

- Tanaka, Tsuchiya and Watanabe (1974, JMKU):
  When $d = 1$, $\alpha < 1$, b is bounded and β-Hölder continuous with $\alpha + \beta < 1$, SDE may not has pathwise uniqueness strong solutions.

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- Priola (2012, OJM):
  Condition: $\alpha \geqslant 1$, b is bounded and β-Hölder continuous with $\beta > 1 - \alpha/2$.
- Zhang (2013, Poincare):
  Condition: $\alpha > 1$, $b \in L^\infty(\mathbb{R}^d) \cap W^{\beta,p}(\mathbb{R}^d)$ with $p > 2d/\alpha$ and $\beta \in (1 - \alpha/2, 1)$.

Main results

We shall consider two cases:

1. SDEs with multiplicative pure jump noise:

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t,$$

   where L_t is a symmetric α-stable process.

2. SDEs with general Lévy noise:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_{|z|>1} g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t,$$

   where N is a Poisson random measure.

Main results - Existence and uniqueness

Consider the following SDE in $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t, \quad X_0 = x, \tag{2.2}$$

where L_t is a symmetric α-stable process with $\alpha \in (1, 2)$.

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where L_t is a symmetric α-stable process with $\alpha \in (1,2)$.

Conditions:

◇ σ is bounded, uniformly elliptic and

$$\nabla\sigma \in L^p(\mathbb{R}^d) \text{ with } p > 2d/\alpha.$$

◇ $b \in W^{\beta,p}(\mathbb{R}^d)$ with $p > 2d/\alpha$ and $\beta \in (1-\alpha/2, 1)$.

Main results - Existence and uniqueness

Theorem 1

SDE (2.2) has a unique strong solution $X_t(x)$ which is strong Feller and irreducible. Moreover, $X_t(x)$ has a density $p(t,x,y)$ with the following estimates:

$$c_1 t(t^{1/\alpha} + |x-y|)^{-d-\alpha} \leqslant p(t,x,y) \leqslant c_2 t(t^{1/\alpha} + |x-y|)^{-d-\alpha}.$$

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Remark: We drop the boundness condition on the drift b, which is new even in the additive noise case.

Main results - Existence and uniqueness

Consider the following SDE in $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_{|z|>1} g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t. \quad (2.3)$$

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Conditions:

- $\diamond$ σ is bounded, uniformly elliptic and $\nabla\sigma \in L^p(\mathbb{R}^d)$ with $p > d$.
- $\diamond$ $b \in L^p(\mathbb{R}^d)$ with $p > d$.
- $\diamond$ For any $0 < \varepsilon < 1$ and some $p > d/2$,

$$\sup_{x\in\mathbb{R}^d} \left(\int_{|z|\leqslant 1} |g(x,z)|^2 \nu(\mathrm{d}z) + \int_{\varepsilon<|z|\leqslant 1} |g(x,z)|\nu(\mathrm{d}z) \right) < +\infty,$$

$$\int_{|z|\leqslant 1} |\nabla g(x,z)|^2 \nu(\mathrm{d}z) \in L^p(\mathbb{R}^d).$$

Theorem 2

SDE (2.3) has a unique strong solution $X_t(x)$ which is strong Feller and irreducible. Moreover, for any bounded measurable φ,

$$\left|\mathbb{E}\varphi(X_t(x)) - \mathbb{E}\varphi(X_t(y))\right| \leqslant \frac{C}{\sqrt{t}}\|\varphi\|_\infty |x-y|.$$

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Theorem 2

SDE (2.3) has a unique strong solution $X_t(x)$ which is strong Feller and irreducible. Moreover, for any bounded measurable φ,

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Remark: Notice that we do not make any restrictions on the pure jump Lévy process. In particular, the large jump is allowed.

Main results - Long time behavior

Consider the following simplest SDE:

$$\mathrm{d}X_t = \mathrm{d}L_t + b(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d.$$

Classical results tell us that when b is locally Lipschitz continuous and dissipative in the sense that there exist $\kappa_1 > 0$ and $\kappa_2 \geqslant 0$ such that

$$\langle x, b(x)\rangle \leqslant -\kappa_1 |x|^{2+\ell} + \kappa_2, \quad \ell \geqslant 0,$$

then there exists a unique invariant measure μ for X_t.

Main results - Long time behavior

Recall that we consider the following two SDEs:

1. SDEs with multiplicative pure jump noise:

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t. \tag{2.4}$$

2. SDEs with general Lévy noise:

$$\begin{aligned}\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t &+ \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) \\ &+ \int_{|z|>1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t.\end{aligned} \tag{2.5}$$

For simplify, we shall focus on providing conditions in terms of the drift b.

Main results - Long time behavior

Assume that the drift coefficient b is divided into two parts:

$$b = b_1 + b_2.$$

Then, SDE (2.4) can be written as

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t + b_1(X_t)\mathrm{d}t + b_2(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d. \tag{2.6}$$

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Conditions:

◇ The first part b_1 is singular and satisfies

$$b_1 \in W^{\beta,p}(\mathbb{R}^d) \text{ with } p > 2d/\alpha \text{ and } \beta \in (1-\alpha/2, 1).$$

◇ The second part $b_2 \in W^{\beta,p}_{loc}(\mathbb{R}^d)$ is dissipative in the sense that

$$\langle x, b_2(x)\rangle \leqslant -\kappa_1|x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b_2(x)| \leqslant \kappa_3(1+|x|)^{1+\ell}.$$

Main results - Long time behavior

Theorem 3

There exists a unique invariant measure μ for the unique strong solution X_t of SDE (2.6). Moreover,

- If $\ell = 0$, then μ is V-uniformly exponential ergodic.
- If $\ell > 0$, then μ is uniformly exponential ergodic.

Remark: 1. We do not make any continuous assumptions on the drift b.
2. The whole drift $b = b_1 + b_2$ may not be dissipative, since b_1 can be unbounded.

Main results - Long time behavior

For SDE

$$\mathrm{d}X_t = \sigma(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

we have the following result:

Corollary 1

Suppose that

$$b \in W^{\beta,p}_{loc}(\mathbb{R}^d) \text{ with } p > 2d/\alpha \text{ and } \beta \in (1-\alpha/2, 1),$$

and there exists a $R_0 > 0$ such that for $|x| \geqslant R_0$,

$$\langle x, b(x)\rangle \leqslant -\kappa_1|x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b(x)| \leqslant \kappa_3(1+|x|)^{1+\ell}.$$

Then, the conclusions in Theorem 3 still hold.

Main results - Long time behavior

Still, we first assume that the drift

$$b = b_1 + b_2.$$

Then, SDE (2.5) can be written as

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_{|z|>1} g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z) + b_1(X_t)\mathrm{d}t + b_2(X_t)\mathrm{d}t. \tag{2.7}$$

Still, we first assume that the drift

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Then, SDE (2.5) can be written as

$$\begin{aligned} \mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) \\ + \int_{|z|>1} g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z) + b_1(X_t)\mathrm{d}t + b_2(X_t)\mathrm{d}t. \end{aligned} \quad (2.7)$$

Conditions:

◇ The first part b_1 is singular and satisfies

$$b_1 \in L^p(\mathbb{R}^d) \quad \text{with} \quad p > d.$$

◇ The second part b_2 is dissipative in the sense that

$$\langle x, b_2(x)\rangle \leqslant -\kappa_1|x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b_2(x)| \leqslant \kappa_3(1 + |x|)^{1+\ell}.$$

Main results - Long time behavior

Theorem 4

There exists a unique invariant measure μ for the unique strong solution X_t of SDE (2.7). Moreover,

- If $\ell = 0$, then μ is V-uniformly exponential ergodic.
- If $\ell > 0$, then μ is uniformly exponential ergodic.

Main results - Long time behavior

For SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_{|z|>1} g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t,$$

we have the following result:

Corollary 2

Suppose that there exists a $R_0 > 0$ such that

$$b \in L^p(B_{R_0}) \text{ with } p > d,$$

and for $|x| \geqslant R_0$,

$$\langle x, b(x)\rangle \leqslant -\kappa_1|x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b(x)| \leqslant \kappa_3(1+|x|)^{1+\ell}.$$

Then, the conclusions in Theorem 4 still hold.

Idea of the proof - Invariant measure

- Important idea: use partial Zvonkin's transformation to kill only the first part b_1 of the drift coefficient.

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- Difficulties:

1. The non-explosion and Krylov estimate of the unique strong solution.
2. The drift b_2 will be involved together with the transformation function.
3. Verify that the dissipative property of the new system.

Examples

Example 1

Consider the following SDE of OU type:

$$\mathrm{d}X_t = -X_t\mathrm{d}t + b(X_t)\mathrm{d}t + \mathrm{d}L_t, \quad X_0 = x \in \mathbb{R}^d.$$

- L_t – Brownian motion: we assume $b \in L^p(\mathbb{R}^d)$, $p > d$;
- L_t – α-stable process with $\alpha \in (1,2)$: we assume
  $b \in W^{\theta,p}(\mathbb{R}^d)$, $\theta > 1 - \alpha/2$ and $p > 2d/\alpha$.

Then, the above SDE admits a unique strong solution and there exists a unique invariant measure.

Remark: In both cases, the classical Lyapunov condition can not be verified, our result is new even in the existence of invariant measures.

Examples

Example 2

Consider the following mixing SDE with jumps:

$$\mathrm{d}X_t = \mathrm{d}W_t + \lambda|X_{t-}|^\beta \mathrm{d}L_t - X_t|X_t|^{\gamma-1}\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

where $\lambda \in \mathbb{R}$, $\beta \in (0,1)$ and $\gamma \in (0,\infty)$, L_t is a d-dimensional pure jump Lévy process.

Then, the above SDE admits a unique strong solution and there exists a unique invariant measure which is V-ergodicity in the case $\gamma \in (0,1]$ and exponential ergodicity in the case $\gamma > 1$.

Remark: The main features of this SDE are that the jump coefficient $x \mapsto |x|^\beta$ is Hölder continuous and the drift term can be polynomial growth.

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Thank You !