Normal approximation for statistics of Gibbsian input in geometric probability

Aihua Xia

School of Mathematics and Statistics
The University of Melbourne, VIC 3010

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(joint work with J E Yukich)
Motivating examples

1. $k$-nearest neighbours graph.
   - $\mathcal{X}$ is a point configuration on $\mathbb{R}^d$.
   - $NG(\mathcal{X})$: the $k$-nearest neighbours (undirected) graph on the vertex set $\mathcal{X}$, i.e., the graph obtained by including $\{x, y\}$ as an edge whenever $y$ is one of the $k$ points nearest to $x$ and/or $x$ is one of the $k$ points nearest to $y$.

(a 1-nearest neighbours graph)

- The total edge length of $k$-nearest neighbours graph?
2. Gibbs-Voronoï tessellations.

- Voronoï tessellation (Georgy Voronoy 1908)
- For $x \in \mathcal{X}$, $C(x, \mathcal{X})$ is the set of points in $\mathbb{R}^d$ closer to $x$ than to any other point of $\mathcal{X}$.
– Each sub-divided region is a *Voronoi cell*.

• The Voronoi tessellation induced by $\mathcal{X}$ is the collection of cells $C(x, \mathcal{X}), x \in \mathcal{X}$.

• The total edge length of Gibbs-Voronoi tessellations?
3. **Maximal points**

- Consider the region as shown below.
- \( x \in \mathcal{X} \) is called *maximal* if no other points of \( \mathcal{X} \) in the top-right corner.
- The total number of maximal points?

(Red dots are maximal points)
4. Spatial birth-growth models

- When a seed is born, it has initial radius zero and then forms a cell within $\mathbb{R}^d$ by growing radially in all directions with a constant speed $v > 0$.
- Whenever one growing cell touches another, it stops growing in that direction.
- Seeds appear at locations $x_i \in \mathbb{R}^d$ at i.i.d. times $T_i, \ i = 1, 2, ...$
- If a seed appears at $x_i$ and if $x_i$ belongs to any of the cells existing at the time $T_i$, then the seed is discarded.
- The number of seeds accepted?
Poisson point process

- $Q_{\lambda} := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d \uparrow \mathbb{R}^d$.

- $\mathcal{P}$ is a Poisson point process on $Q_{\lambda}$ with unit intensity if
  1. for any Borel $B \subset Q_{\lambda}$, $\mathcal{P}(B) \sim \mathcal{P} \left( \text{Vol}(B) \right)$;
  2. for every $k \geq 1$, $\mathcal{P}(B_1), \ldots, \mathcal{P}(B_k)$ are independent for disjoint Borel sets $B_1, \ldots, B_k$.

- $\mathbf{P}(\mathcal{P} \text{ has } n \text{ points resp. sitting in } (x_i, x_i + dx_i)) = \frac{e^{-\lambda}}{n!} dx_1 \ldots dx_n =: j_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$
  - $j_n(x_1, \ldots, x_n)$ is called the Janossy density
From Poisson to Gibbs

- The points of Poisson process don’t interact
- Consider configurations of points $x_n := \{x_1, \ldots, x_n\}$ with interactions between particles taken in pairs, triples, etc.
- An $\mathbb{R} \cup \{+\infty\}$-valued measurable function $\Psi$ is called an energy function if it satisfies
  - $\Psi$ is non-degenerate: $\Psi(\emptyset) < +\infty$
  - $\Psi$ is hereditary: for any $x$ and $x \in x$, then $\Psi(x) < +\infty$ implies $\Psi(x \setminus \{x\}) < +\infty$.
  - $\Psi$ is stable: there exists a constant $c$ (usu. $< 0$) such that for any $x$, $\Psi(x) \geq c \#(x)$.
- A Gibbs point process with energy function $\Psi$ and inverse temperature $\beta \geq 0$ is a point process having Janossy density
  $$ j_n(x_n) = C(\beta)e^{-\beta \Psi(x_n)}. $$
Remarks

• $C(\beta)$ is a normalising constant.

• When $\beta = 0$, it reduces to Poisson point process.

• Stability ensures

$$\sum_{n \geq 0} \int_{Q_\lambda} e^{-\beta \Psi(x_n)} dx_n \leq \sum_{n \geq 0} e^{-\beta c n} \lambda^n = e^{e^{-\beta c} \lambda} < +\infty,$$

hence $C(\beta) > 0$.

• Non-degeneracy gives

$$1 = C(\beta) \sum_{n \geq 0} \int_{Q_\lambda} e^{-\beta \Psi(x_n)} dx_n \geq C(\beta) e^{-\beta \Psi(\emptyset)}$$

so $C(\beta) \leq e^{\beta \Psi(\emptyset)} < \infty$. 

[Slide 9]
The setup

- $\mathcal{P}^{\beta\Psi}$: a Gibbs point process on $\mathbb{R}^d$.
- $\mathcal{P}^{\beta\Psi}_\lambda$: the restriction of $\mathcal{P}^{\beta\Psi}$ to $Q_\lambda$.
- Our interest is on the asymptotic behaviour of the functionals

$$W_\lambda := \sum_{x \in \mathcal{P}^{\beta\Psi}_\lambda \setminus \{x\}} \xi(x, \mathcal{P}^{\beta\Psi}_\lambda \setminus \{x\})$$

as $\lambda \to \infty$. 
Examples (continued)

When \( \mathcal{X} \) is a realisation of \( \mathcal{P}_\lambda^{\beta \psi} \),

1. **\( k \)-nearest neighbours graph**: the asymptotic distribution of the total edge length? error estimates?

2. **Gibbs-Voronoi tessellations**: the asymptotic distribution of the total edge length? error estimates?

3. **Maximal points**: the asymptotic distribution of the total number of maximal points? error estimates?
4. **Spatial birth-growth models:**

- Seeds appear at random locations $X_i \in \mathbb{R}^d$ at i.i.d. times $T_i, \ i = 1, 2, \ldots$ according to a marked Gibbs point process $\mathcal{P} := \{(X_i, T_i) \in \mathbb{R}^d \times [0, \infty)\}$.

- If a seed appears at $X_i$ and if $X_i$ belongs to any of the cells existing at the time $T_i$, then the seed is discarded.

- $X_i, i \geq 1$, are independent of $T_i, i \geq 1$.

- The number of seeds accepted in $Q_\lambda$? error estimates?
A quick review of normal approximation

- If $\xi_1, \ldots, \xi_n$ are iid with mean 0, var 1 and finite 3rd moment, let $S_n = \sum_{i=1}^{n} \xi_i$, then
  
  $$d_K(S_n, N(0, \text{Var}(S_n))) = O(\text{Var}(S_n)^{-1/2}).$$

  - ChFs

- Stein’s method: the above claim is still true if $\xi_1, \ldots, \xi_n$ have some short range dependence.

- Barbour and X. (2006): the above claim is also true if $S_n$ is a result of an integral of a locally dependent process w. r. t. a locally dependent point process.
From Poisson to Gibbs by thinning

- We consider the energy functions which satisfy
  - nonnegative;
  - monotonic: $\Psi(\mathcal{X}) \leq \Psi(\mathcal{X}')$ if $\mathcal{X} \subset \mathcal{X}'$;
  - translation invariant: $\Psi(\mathcal{X} + y) = \Psi(\mathcal{X})$ for all $y \in \mathbb{R}^d$;
  - rotation invariant: $\Psi(\mathcal{X}') = \Psi(\mathcal{X}')$ if $\mathcal{X}'$ is a rotation of $\Xi$.

- Schreiber and Yukich (2013): One can start with a Poisson point process with very dense points, construct an ancestor clan for each point, and thin away some points in the clan.
  - The ancestor clan of each point $x$ has a diameter $D(x, P_\lambda^{\beta\Psi})$ which is exponentially decaying.
Thm (X. and Yukich 2015)

Recall $W_\lambda = \sum_{x \in \mathcal{P}_\lambda^\beta} \xi(x, \mathcal{P}_\lambda^\beta \setminus \{x\})$, under some mild conditions,

$$d_K \left( \frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var} W_\lambda}}, N(0, 1) \right) = O((\ln \lambda)^{2d} \lambda (\text{Var} W_\lambda)^{-3/2}).$$
Why?

Since \( W_{\lambda} = \sum_{x \in P_\lambda^{\beta \Psi}} \xi(x, P_\lambda^{\beta \Psi} \setminus \{x\}) \), define

\[
\hat{W}_{\lambda} = \sum_{x \in P_\lambda^{\beta \Psi}} \xi(x, P_\lambda^{\beta \Psi} \setminus \{x\}) \mathbf{1}(D(x, P_\lambda^{\beta \Psi}) \leq \rho)
\]

↑limits dependence range,

\[
\tilde{W}_{\lambda} = \sum_{x \in P_\lambda^{\beta \Psi}} \xi(x, P_\lambda^{\beta \Psi} \setminus \{x\})
\]

↑removes boundary effects.

• \( \hat{W}_{\lambda}, \tilde{W}_{\lambda} \) and \( W_{\lambda} \) are very “close”.

• Using Barbour and X. (2006), \( \hat{W}_{\lambda} \) can be approximated by a suitable normal with approximation error \( O(\rho^{2d} \lambda (\text{Var} \hat{W}_{\lambda})^{-3/2}) \).
• If $\xi$ is translation invariant, then we can write down $\text{Var} \tilde{W}_\lambda$ explicitly and derive that $\text{Var} \tilde{W}_\lambda = \Omega(\lambda)$ so the normal approximation error is $O((\ln \lambda)^{2d} \lambda^{-1/2})$.

  - This includes all the motivating examples except maximal points

• For maximal points, $\xi$ is not translation invariant, we can prove that $\text{Var} \tilde{W}_\lambda \geq \Omega ((\ln \lambda)^{-d} \lambda)$, giving the error estimate of $O((\ln \lambda)^{(7d-1)/2} \lambda^{-(d-1)/2d})$. 
Remarks

- For inputs with marked Poisson and binomial point processes, Lachièze-Rey, Schultey and Yukichz (2017) can remove the log factor (by the Malliavin-Stein theory).

- For general Gibbsian input, it seems to be impossible to remove the log factor but its power may be reduced.
Thank you!