Normal approximation for statistics of Gibbsian input in geometric probability

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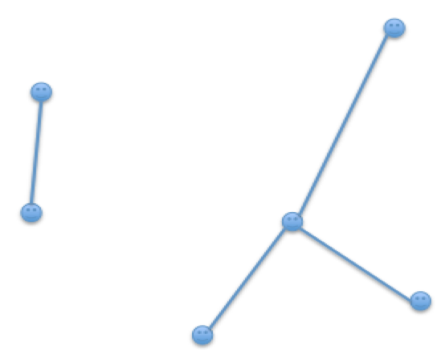
(joint work with J E Yukich)

Movitating examples

1. **k-nearest neighbours graph.**
   - $\mathcal{X}$ is a point configuration on $\mathbb{R}^d$.
   - $NG(\mathcal{X})$: the k-nearest neighbours (undirected) graph on the vertex set $\mathcal{X}$, i.e., the graph obtained by including $\{x, y\}$ as an edge whenever y is one of the k points nearest to x and/or x is one of the k points nearest to y.

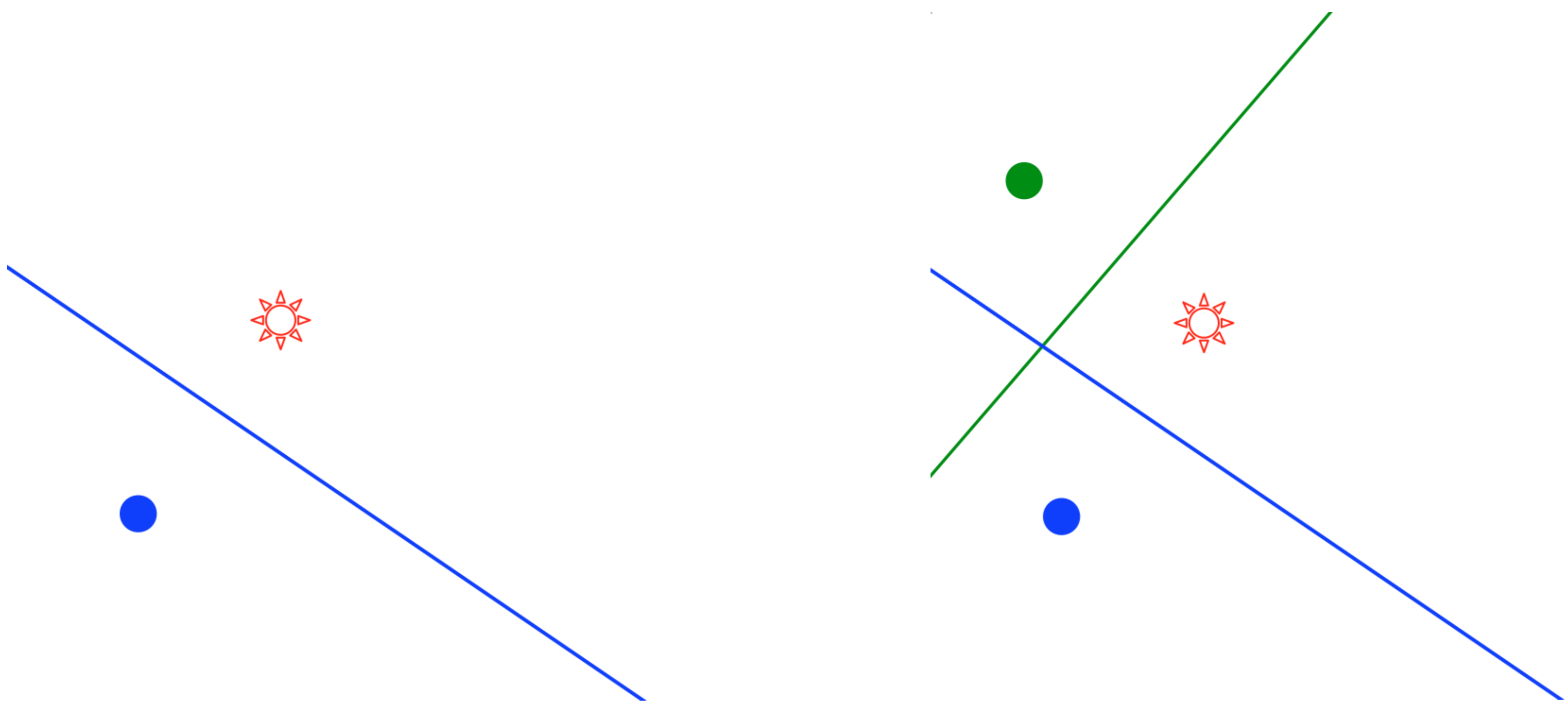


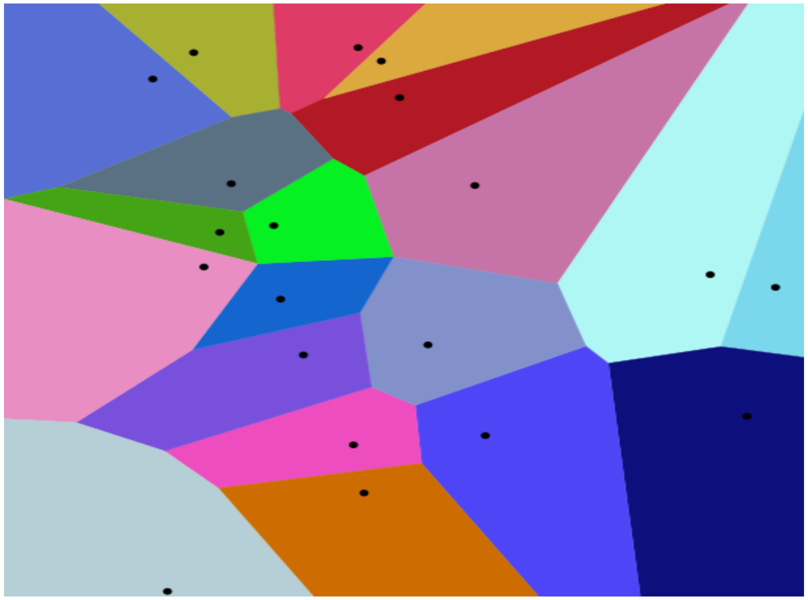
(a 1-nearest neighbours graph)

   - The total edge length of k-nearest neighbours graph?

2. **Gibbs-Voronoi tessellations**.

- Voronoi tessellation (Georgy Voronoy 1908)
- For $x \in \mathcal{X}$, $C(x, \mathcal{X})$ is the set of points in $\mathbb{R}^d$ closer to x than to any other point of $\mathcal{X}$.





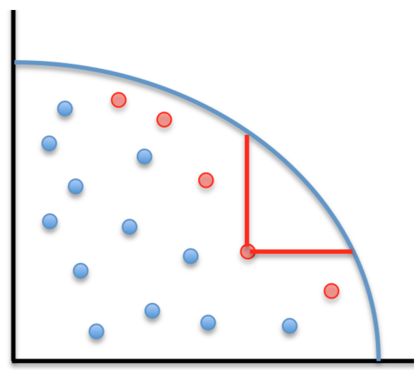
(Source: wiki)

– Each sub-divided region is a *Voronoi cell*.

- The Voronoi tessellation induced by $\mathcal{X}$ is the collection of cells $C(x, \mathcal{X}), x \in \mathcal{X}$.
- The total edge length of Gibbs-Voronoi tessellations?

3. **Maximal points**

- Consider the region as shown below.
- $x \in \mathcal{X}$ is called *maximal* if no other points of $\mathcal{X}$ in the top-right corner.
- The total number of maximal points?



(Red dots are maximal points)

4. **Spatial birth-growth models**

- When a seed is born, it has initial radius zero and then forms a cell within $\mathbb{R}^d$ by growing radially in all directions with a constant speed $v > 0$.
- Whenever one growing cell touches another, it stops growing in that direction.
- Seeds appear at locations $x_i \in \mathbb{R}^d$ at i.i.d. times $T_i,\ i = 1, 2, ...$
- If a seed appears at x_i and if x_i belongs to any of the cells existing at the time T_i, then the seed is discarded.
- The number of seeds accepted?

Poisson point process

- $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d \uparrow \mathbb{R}^d$.
- $\mathcal{P}$ is a Poisson point process on Q_λ with unit intensity if
  1. for any Borel $B \subset Q_\lambda$, $\mathcal{P}(B) \sim \mathcal{P}(\mathrm{Vol}(B))$;
  2. for every $k \geq 1$, $\mathcal{P}(B_1), \ldots, \mathcal{P}(B_k)$ are independent for disjoint Borel sets $B_1, \ldots, B_k$.
- $\mathbf{P}(\mathcal{P}$ has n points resp. sitting in $(x_i, x_i + dx_i)) =$ $\frac{e^{-\lambda}}{n!} dx_1 \ldots dx_n =: j_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$
  - $j_n(x_1, \ldots, x_n)$ is called the Janossy density

From Poisson to Gibbs

- The points of Poisson process don't interact
- Consider configurations of points $\mathbf{x}_n := \{x_1, \ldots, x_n\}$ with interactions between particles taken in pairs, triples, etc.
- An $\mathbb{R} \cup \{+\infty\}$-valued measurable function Ψ is called an *energy function* if it satisfies
  - Ψ is non-degenerate: $\Psi(\emptyset) < +\infty$
  - Ψ is hereditary: for any $\mathbf{x}$ and $x \in \mathbf{x}$, then $\Psi(\mathbf{x}) < +\infty$ implies $\Psi(\mathbf{x} \setminus \{x\}) < +\infty$.
  - Ψ is stable: there exists a constant c (usu. < 0) such that for any $\mathbf{x}$, $\Psi(\mathbf{x}) \geq c\#(\mathbf{x})$.
- A Gibbs point process with energy function Ψ and inverse temperature $\beta \geq 0$ is a point process having Janossy density

$$j_n(\mathbf{x}_n) = C(\beta) e^{-\beta \Psi(\mathbf{x}_n)}.$$

Remarks

- $C(\beta)$ is a normalising constant.
- When $\beta = 0$, it reduces to Poisson point process.
- Stability ensures

$$\sum_{n\geq 0}\int_{Q_\lambda} e^{-\beta\Psi(\mathbf{x}_n)}d\mathbf{x}_n \leq \sum_{n\geq 0} e^{-\beta cn}\lambda^n = e^{e^{-\beta c}\lambda} < +\infty,$$

  hence $C(\beta) > 0$.

- Non-degeneracy gives

$$1 = C(\beta)\sum_{n\geq 0}\int_{Q_\lambda} e^{-\beta\Psi(\mathbf{x}_n)}d\mathbf{x}_n \geq C(\beta)e^{-\beta\Psi(\emptyset)}$$

  so $C(\beta) \leq e^{\beta\Psi(\emptyset)} < \infty$.

The setup

- $\mathcal{P}^{\beta\Psi}$: a Gibbs point process on $\mathbb{R}^d$.
- $\mathcal{P}_\lambda^{\beta\Psi}$: the restriction of $\mathcal{P}^{\beta\Psi}$ to Q_λ.
- Our interest is on the asymptotic behaviour of the functionals

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\})$$

as $\lambda \to \infty$.

Examples (continued)

When $\mathcal{X}$ is a realisation of $\mathcal{P}_\lambda^{\beta\Psi}$,

1. k**-nearest neighbours graph:** the asymptotic distribution of the total edge length? error estimates?

2. **Gibbs-Voronoi tessellations:** the asymptotic distribution of the total edge length? error estimates?

3. **Maximal points:** the asymptotic distribution of the total number of maximal points? error estimates?

4. **Spatial birth-growth models:**

- Seeds appear at random locations $X_i \in \mathbb{R}^d$ at i.i.d. times $T_i,\ i = 1, 2, ...$ according to a marked Gibbs point process $\mathcal{P} := \{(X_i, T_i) \in \mathbb{R}^d \times [0, \infty)\}$.
- If a seed appears at X_i and if X_i belongs to any of the cells existing at the time T_i, then the seed is discarded.
- $X_i, i \geq 1$, are independent of $T_i, i \geq 1$.
- The number of seeds accepted in Q_λ? error estimates?

A quick review of normal approximation

- If $\xi_1, \ldots, \xi_n$ are iid with mean 0, var 1 and finite 3rd moment, let $S_n = \sum_{i=1}^n \xi_i$, then $d_K\left(S_n, N(0, \mathrm{Var}(S_n))\right) = O(\mathrm{Var}(S_n)^{-1/2})$.
  - ChFs
- Stein's method: the above claim is still true if $\xi_1, \ldots, \xi_n$ have some short range dependence.
- Barbour and X. (2006): the above claim is also true if S_n is a result of an integral of a locally dependent process w. r. t. a locally dependent point process.

From Poisson to Gibbs by thinning

- We consider the energy functions which satisfy
  - nonnegative;
  - monotonic: $\Psi(\mathcal{X}) \leq \Psi(\mathcal{X}')$ if $\mathcal{X} \subset \mathcal{X}'$;
  - translation invariant: $\Psi(\mathcal{X} + y) = \Psi(\mathcal{X})$ for all $y \in \mathbb{R}^d$;
  - rotation invariant: $\Psi(\mathcal{X}) = \Psi(\mathcal{X}')$ if $\mathcal{X}'$ is a rotation of Ξ.
- Schreiber and Yukich (2013): One can start with a Poisson point process with very dense points, construct an ancestor clan for each point, and thin away some points in the clan.
  - The ancestor clan of each point x has a diameter $D(x, \mathcal{P}_\lambda^{\beta\Psi})$ which is exponentially decaying.

Thm (X. and Yukich 2015)

Recall $W_\lambda = \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\})$, under some mild conditions,

$$d_K\left(\frac{W_\lambda - \mathbb{E}\, W_\lambda}{\sqrt{\mathrm{Var} W_\lambda}}, N(0,1)\right) = O((\ln \lambda)^{2d} \lambda (\mathrm{Var} W_\lambda)^{-3/2}).$$

Why?

Since $W_\lambda = \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\})$, define

$$\hat{W}_\lambda = \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}) \mathbf{1}(D(x, \mathcal{P}_\lambda^{\beta\Psi}) \leq \rho)$$

$\uparrow$limits dependence range,

$$\tilde{W}_\lambda = \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\})$$

$\uparrow$removes boundary effects.

- $\hat{W}_\lambda$, $\tilde{W}_\lambda$ and W_λ are very "close".
- Using Barbour and X. (2006), $\hat{W}_\lambda$ can be approximated by a suitable normal with approximation error $O(\rho^{2d}\lambda(\mathrm{Var}\hat{W}_\lambda)^{-3/2})$.

- If ξ is translation invariant, then we can write down $\mathrm{Var}\tilde{W}_\lambda$ explicitly and derive that $\mathrm{Var}\tilde{W}_\lambda = \Omega(\lambda)$ so the normal approximation error is $O((\ln\lambda)^{2d}\lambda^{-1/2})$.
  - This includes all the motivating examples except maximal points
- For maximal points, ξ is not translation invariant, we can prove that $\mathrm{Var}\tilde{W}_\lambda \geq \Omega\left((\ln\lambda)^{-d}\lambda\right)$, giving the error estimate of $O((\ln\lambda)^{(7d-1)/2}\lambda^{-(d-1)/2d})$.

Remarks

- For inputs with marked Poisson and binomial point processes, Lachièze-Rey, Schultey and Yukichz (2017) can remove the log factor (by the Malliavin-Stein theory).
- For general Gibbsian input, it seems to be impossible to remove the log factor but its power may be reduced.

Thank you!