Asymptotic Properties of Regime-Switching Stochastic Damping Hamiltonian Systems

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Contents

Stochastic Damping Hamiltonian Systems

A **stochastic damping Hamiltonian system** is

$$\begin{aligned} dx(t) &= y(t)dt, \\ dy(t) &= -\big[c(x(t),y(t))y(t) + \nabla V(x(t))\big]dt \\ &\qquad\qquad +\sigma(x(t),y(t))dB(t), \end{aligned} \tag{1}$$

where B is a standard Brownian motion in $\mathbb{R}^d$. L.M. Wu(2001, SPA) studied this model.

The model (1) is very general, and it covers the generalized Duffing oscillator, the **van der Pol oscillator**, and the Liénard oscillator or Liénard equation.

This model certainly has a wide range of applications. For example, the **van der Pol equation** has a long history of being used in both the physical and biological sciences.

Stochastic Damping Hamiltonian System (cont'd)

Note also that the diffusion corresponding to SDE (1) is **degenerated**, and that the coefficients $\nabla V(x)$ and $c(x, y)$ usually are **only continuous but not C^∞**.

Hence, the hypoellipticity need not hold for (1), and the existence and uniqueness of solution and the strong Feller property of the corresponding Markov process are not obvious.

Letting $c(x, y) = 0$ and $V(x) = 0$, one gets

$$\begin{aligned} dx^0(t) &= y^0(t)dt, \\ dy^0(t) &= \sigma(x^0(t), y^0(t))dB(t). \end{aligned} \tag{2}$$

Note that SDE (2) has a unique non-explosive strong solution, and the corresponding diffusion is hypoelliptic.

Switching Models

One of the main motivations stems from many existing and emerging applications in which the usual continuous state variables alone do not provide satisfactory results.

In many real-world systems, due to random environments and other stochastic influences, the traditional differential equation setup is not adequate since the discrete events cannot be described by the usual differential equation alone.

We know that **the environments or regimes are always changing** from time to time.

Stochastic damping Hamiltonian systems with random switching considered below belong to the class of switching diffusions.

Switching Stochastic Damping Hamiltonian Systems

We consider a **stochastic damping Hamiltonian system with random switching** defined by (X, Y, Λ), where X and Y describe the position and velocity of a physical system moving in $\mathbb{R}^d$ respectively, whereas Λ delineates the randomly changing mechanical regimes (or environments).

$$\begin{aligned} dX(t) &= Y(t)dt, \\ dY(t) &= -\left[c(X(t), Y(t), \Lambda(t))Y(t) + \nabla V(X(t), \Lambda(t))\right]dt \\ &\qquad\qquad +\sigma(X(t), Y(t), \Lambda(t))dB(t). \end{aligned} \tag{3}$$

Λ is a right-continuous random process with a finite state space $\mathbb{S} = \{1, \ldots, n_0\}$ such that

$$\begin{aligned} &\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, (X(t), Y(t)) = (x, y)\} \\ &= \begin{cases} q_{kl}(x, y)\Delta + o(\Delta), & \text{if } k \neq l, \\ 1 + q_{kk}(x, y)\Delta + o(\Delta), & \text{if } k = l. \end{cases} \end{aligned} \tag{4}$$

Operators

For all functions $f \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{S})$, we define the following operator:

$$\mathcal{A}f(x,y,k) := \mathcal{L}_k f(x,y,k) + Q(x,y)f(x,y,k). \tag{5}$$

Here, for each $k \in \mathbb{S}$, $\mathcal{L}_k$ is a differential operator defined as follows:

$$\begin{aligned} \mathcal{L}_k f(x,y,k) \ :=& \ \frac{1}{2}\mathrm{tr}\big(a(x,y,k)\nabla_y^2 f(x,y,k)\big) + \langle y, \nabla_x f(x,y,k)\rangle \\ & -\langle c(x,y,k)y + \nabla_x V(x,k), \nabla_y f(x,y,k)\rangle, \end{aligned} \tag{6}$$

where $a(x,y,k) = \sigma(x,y,k)\sigma(x,y,k)^T$. In (5) the switching operator $Q(x,y)$ is defined as follows:

$$Q(x,y)f(x,y,k) := \sum_{l \in \mathbb{S}} q_{kl}(x,y)\big(f(x,y,l) - f(x,y,k)\big). \tag{7}$$

Spaces

Define a metric $\lambda(\cdot,\cdot)$ on $\mathbb{R}^{2d}\times\mathbb{S}$ as

$$\lambda\big((x,y,m),(\widetilde{x},\widetilde{y},\widetilde{m})\big)=|(x,y)-(\widetilde{x},\widetilde{y})|+d(m,\widetilde{m}),$$

where $d(\cdot,\cdot)$ is the discrete metric.

Let $\Omega := C([0,\infty),\mathbb{R}^{2d})\times D([0,\infty),\mathbb{S})$ be endowed with the product topology of the sup norm topology on $C([0,\infty),\mathbb{R}^{2d})$ and the Skorohod topology on $D([0,\infty),\mathbb{S})$

For $\omega\in\Omega$, let $\omega(t)=(X(t),Y(t),\Lambda(t))$ be the coordinate process. Let $\mathcal{F}_t$ be the σ-field generated by the cylindrical sets on $C([0,\infty),\mathbb{R}^{2d})\times D([0,\infty),\mathbb{S})$ up to time t and set $\mathcal{F}=\bigvee_{t=0}^{\infty}\mathcal{F}_t$.

Martingale Approach

We know from Wu(2001, SPA) that the stochastic damping Hamiltonian system (1) has only **weak solution**.

Therefore, to establish the solutions to the system (3) and (4) in somewhat *weak sense*, we invoke the **martingale problem machinery**.

Indeed, one could see below that the martingale approach is appropriate to characterize the *weak solution* to the system (3) and (4).

Now we proceed to consider the martingale problem for the operator $\mathcal{A}$ defined in (5) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$.

Martingale Problem

For a given $(x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, we say a probability measure $\mathbb{P}^{(x,y,k)}$ on Ω is a solution to the martingale problem for the operator $\mathcal{A}$ starting from (x, y, k),

if $\mathbb{P}^{(x,y,k)}((X(0), Y(0), \Lambda(0)) = (x, y, k)) = 1$ and for each function $f \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{S})$,

$$M_t^{(f)} := f(X(t), Y(y), \Lambda(t)) - f(X(0), Y(0), \Lambda(0)) - \int_0^t \mathcal{A}f(X(s), Y(s), \Lambda(s))\mathrm{d}s \tag{8}$$

is an $\{\mathcal{F}_t\}$-martingale with respect to $\mathbb{P}^{(x,y,k)}$.

Sometimes, we simply say that the probability measure $\mathbb{P}^{(x,y,k)}$ is a martingale solution for the operator $\mathcal{A}$ starting from (x, y, k).

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Weak Solution and Notation

For a given $(x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, we call a martingale solution $\mathbb{P}^{(x,y,k)}$ for the operator $\mathcal{A}$ starting from (x, y, k) as a **weak solution to the system (3) and (4) with initial data (x, y, k)**.

Let $c^s(x, y, k)$ be the symmetrization of the matrix $c(x, y, k)$, given by $\left(\frac{1}{2}(c_{ij}(x, y, k) + c_{ji}(x, y, k))\right)$, $\|\cdot\|_{\text{H.S.}}$ be the Hilbert-Schmidt norm of matrix, the order relation on symmetric matrices be usual one defined by the definite non-negativeness; and let $\sigma > 0$ mean that σ is strictly positive definite.

For the existence and uniqueness of the weak solution to system (3) and (4), we make the following assumption.

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Standing Hypotheses

Assumption 1

For each $k \in \mathbb{S}$, we assmue that

(i) $V(\cdot, k)$ is lower bounded and continuously differentiable over $\mathbb{R}^d$;

(ii) $c(\cdot, \cdot, k)$ is continuous and for all $N > 0$:

$$\sup\{\|c(x, y, k)\|_{\text{H.S.}} : |x| \leq N, y \in \mathbb{R}^d\} < \infty,$$

and there exist c, $L > 0$ such that

$$c^s(x, y, k) \geq cI > 0 \text{ for all } |x| > L \text{ and } y \in \mathbb{R}^d;$$

(iii) $\sigma(\cdot, \cdot, k)$ is symmetric, infinitely differentiable and for some $\hat{\sigma} > 0$: $0 < \sigma(x, y, k) \leq \hat{\sigma} I$ over $\mathbb{R}^{2d}$;

(iv) $Q(x, y) := \big(q_{kl}(x, y)\big)$ is measurable such that for all $(x, y) \in \mathbb{R}^{2d}$, $q_{kl}(x, y) \geq 0$ for $k \neq l$, and $\sum_{l \in \mathbb{S}} q_{kl}(x, y) = 0$ and the function $q_{kk}(x) \leq 0$ is bounded from below for each $k \in \mathbb{S}$.

Auxiliary Processes

For each $k \in \mathbb{S}$, let $Z^{(k)}(t) := (X^{(k)}(t), Y^{(k)}(t))$ satisfy

$$\begin{aligned} dX^{(k)}(t) &= Y^{(k)}(t)dt, \\ dY^{(k)}(t) &= -\big[c(X^{(k)}(t), Y^{(k)}(t), k)Y(t) + \nabla V(X^{(k)}(t), k)\big]dt \\ &\qquad +\sigma(X^{(k)}(t), Y^{(k)}(t), k)dB(t), \end{aligned} \tag{9}$$

and let $Z^{(k)0}(t) := (X^{(k)0}(t), Y^{(k)0}(t))$ satisfy

$$\begin{aligned} d\widehat{X}^{(k)0}(t) &= \widehat{Y}^{(k)0}(t)dt, \\ d\widehat{Y}^{(k)0}(t) &= \sigma(\widehat{X}^{(k)0}(t), \widehat{Y}^{(k)0}(t), k)dB(t). \end{aligned} \tag{10}$$

By virtue of Girsanov formula and by making using of Dunford-Pettis theorem and Egorov lemma, the following **two basic but important lemmas** were proved Wu (2001, SPA).

Two Lemmas of Wu

Lemma 1

For each $k \in \mathbb{S}$ *and for each initial state* $z = (x, y) \in \mathbb{R}^{2d}$, *SDE (9) admits* **a unique weak solution** $\mathbb{P}_k^{(z)}$, *a probability measure on the space* $C([0, \infty), \mathbb{R}^{2d})$, *and this solution is non-explosive.*

Lemma 2

For each $k \in \mathbb{S}$, *let* $\big(P_k(t, z, \cdot)\big)$ *be the transition probability family of Markov process* $\big((Z^{(k)}(t))_{t\geq 0}, (\mathbb{P}_k^{(z)})_{z\in\mathbb{R}^{2d}}\big)$ *(solution of (9)). For each* $k \in \mathbb{S}$, $t > 0$ *and* $z \in \mathbb{R}^{2d}$, $P_k(t, z, dz') = p_k(t, z, z')dz'$, $p_k(t, z, z') > 0$, dz'*-a.e. and*

$$z \to p_k(t, z, \cdot) \text{ is continuous from } \mathbb{R}^{2d} \text{ to } L^1(\mathbb{R}^{2d}, dz').$$

In particular, for each $k \in \mathbb{S}$, $P_k(t, z, \cdot)$ *is* **strong Feller** *for all* $t > 0$.

Weak Solution of Each Single System

For each $k \in \mathbb{S}$, it follows from Lemma 1 easily that for a given $z = (x, y) \in \mathbb{R}^{2d}$, **the probability measure $\mathbb{P}_k^{(z)}$ on $C([0,\infty), \mathbb{R}^{2d})$ is the unique solution to the martingale problem for the operator $\mathcal{L}_k$** starting from z;

that is, $\mathbb{P}_k^{(z)}(Z(0) = z) = 1$ and for each function $f \in C_c^\infty(\mathbb{R}^{2d})$,

$$M_t^{(k)(f)} := f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{L}_k f(Z(s))\mathrm{d}s \tag{11}$$

is a $\{\mathcal{G}_t\}$-martingale with respect to $\mathbb{P}_k^{(z)}$, where $\mathcal{G}_t$ is the σ-field generated by the cylindrical sets on $C([0,\infty), \mathbb{R}^{2d})$ up to time t.

Special Markovian Switching Case

We consider a simpler process $(\widehat{X}, \widehat{Y}, \psi)$. Let $(\widehat{X}, \widehat{Y})$ satisfy

$$\begin{aligned} d\widehat{X}(t) &= \widehat{Y}(t)dt, \\ d\widehat{Y}(t) &= -\big[c(\widehat{X}(t), \widehat{Y}(t), \psi(t))\widehat{Y}(t) + \nabla V(\widehat{X}(t), \psi(t))\big]dt \\ &\qquad + \sigma(\widehat{X}(t), \widehat{Y}(t), \psi(t))dB(t) \end{aligned} \tag{12}$$

and ψ that is independent of B, be a Markov chain with finite state space $\mathbb{S}$ satisfying

$$\mathbb{P}\{\psi(t+\Delta) = l | \psi(t) = k\} = \begin{cases} \Delta + o(\Delta), & \text{if } l \neq k, \\ 1 - (n_0 - 1)\Delta + o(\Delta), & \text{if } l = k. \end{cases} \tag{13}$$

Special Q-matrix

Obviously, the **generator of ψ** is given by the special Q-matrix:

$$\widehat{Q} = (\widehat{q}_{kl}) = \begin{pmatrix} -(n_0-1) & 1 & \cdots & 1 \\ 1 & -(n_0-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(n_0-1) \end{pmatrix}.$$

Corresponding to this matrix $\widehat{Q}$, we introduce an operator $\widehat{Q}$ as follows:

$$\widehat{Q}f(k) = \sum_{l\in\mathbb{S}} \widehat{q}_{kl}\big(f(l) - f(k)\big), \quad k \in \mathbb{S}. \tag{14}$$

Martingale Solution to Special Q-matrix

For a given $k \in \mathbb{S}$, a probability measure $\mathbb{Q}^{(k)}$ on $D([0,\infty),\mathbb{S})$ is said to be **a solution to the martingale problem for the operator $\widehat{Q}$ starting from k**,

if $\mathbb{Q}^{(k)}(\Lambda(0)) = k) = 1$ and for each function $f \in \mathcal{B}(\mathbb{S})$,

$$N_t^{(f)} := f(\Lambda(t)) - f(\Lambda(0)) - \int_0^t \widehat{Q} f(\Lambda(s))\mathrm{d}s \tag{15}$$

is an $\{\mathcal{N}_t\}$-martingale with respect to $\mathbb{Q}^{(k)}$, where $\mathcal{N}_t$ is the σ-field generated by the cylindrical sets on $D([0,\infty),\mathbb{S})$ up to time t. **Here Λ is the coordinate process** $\Lambda(t,\omega) := \omega(t)$ with $\omega \in D([0,\infty),\mathbb{S})$ and $t \geq 0$.

Lemma 3

For any given $k \in \mathbb{S}$, there exists a unique martingale solution $\mathbb{Q}^{(k)}$ on $D([0,\infty),\mathbb{S})$ for the operator $\widehat{Q}$ starting from k.

Stopping Times

let $\{\tau_n\}$ be the sequence of stopping times defined by

$$\tau_0 \equiv 0, \quad \text{and for } n \geq 1, \quad \tau_n := \inf\{t > \tau_{n-1} : \Lambda(t) \neq \Lambda(\tau_{n-1})\}. \quad (16)$$

Then it is obvious that for any $k \in \mathbb{S}$, $\mathbb{Q}^{(k)} \{\lim_{n\to\infty} \tau_n = +\infty\} = 1$.

Moreover, we have $\mathbb{Q}^{(k)}\big(\tau_1 \geq t\big) = \exp(-(n_0 - 1)t)$ for all $t \geq 0$ and

$$\mathbb{Q}^{(k)}\big(\Lambda(\tau_1) = l\big) = 1/(n_0 - 1) \text{ for each } l \in \mathbb{S} \setminus \{k\}.$$

Clearly, the distributions of τ_1 and $\Lambda(\tau_1)$ under $\mathbb{Q}^{(k)}$ are very regular.

Special Operator

Now we introduce an operator $\widehat{\mathcal{A}}$ on $C_c^2(\mathbb{R}^{2d} \times \mathbb{S})$ as follows:

$$\widehat{\mathcal{A}} f(x, y, k) := \mathcal{L}_k f(x, y, k) + \widehat{Q} f(x, y, k), \tag{17}$$

where the operators $\mathcal{L}_k$ and $\widehat{Q}$ are defined in (6) and (14), respectively.

We will show that for each $(x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, **there exists a unique martingale solution $\widehat{\mathbb{P}}^{(x,y,k)}$ for the operator $\widehat{\mathcal{A}}$ starting from (x, y, k)**.

Weak Solution: Special Case

Theorem 4

For any given $(x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$*, there exists a unique martingale solution* $\widehat{\mathbb{P}}^{(x,y,k)}$ *on* $C([0, \infty), \mathbb{R}^{2d}) \times D([0, \infty), \mathbb{S})$ *for the operator* $\widehat{A}$ *starting from* (x, y, k)*.*

Sketch of Proof. For any given $(z, k) := (x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, we define a series of probability measures on $(\Omega, \mathcal{F})$ as follows:

$$\mathbb{P}^{(1)} = \mathbb{P}_k^{(z)} \times \mathbb{Q}^{(k)}, \quad \mathbb{P}^{(2)} = \left(\mathbb{P}_k^{(z)} \times \mathbb{Q}^{(k)}\right) \otimes_{\tau_1} \left(\mathbb{P}_{\Lambda(\tau_1)}^{(Z(\tau_1))} \times \mathbb{Q}^{(\Lambda(\tau_1))}\right),$$
$$\mathbb{P}^{(n+1)} = \mathbb{P}^{(n)} \otimes_{\tau_n} \left(\mathbb{P}_{\Lambda(\tau_n)}^{(Z(\tau_n))} \times \mathbb{Q}^{(\Lambda(\tau_n))}\right), \quad n \geq 2,$$

where $\Omega = C([0, \infty), \mathbb{R}^{2d}) \times D([0, \infty), \mathbb{S})$, $Z(\cdot) = (X(\cdot), Y(\cdot))$.

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where $\Omega = C([0,\infty),\mathbb{R}^{2d}) \times D([0,\infty),\mathbb{S})$, $Z(\cdot) = (X(\cdot), Y(\cdot))$.

A Martingale Function

To proceed, for any given $t \geq 0$, we define a function M_t on the sample path space as follows:

$$M_t(Z(\cdot), \Lambda(\cdot)) := \prod_{i=0}^{n(t)-1} q_{\Lambda(\tau_i)\Lambda(\tau_{i+1})}(Z(\tau_{i+1})) \times \exp\left(-\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}\wedge t} \left[q_{\Lambda(\tau_i)}(Z(s)) - n_0 + 1\right] ds\right),$$

where

$$q_k(z) = \sum_{l \in \mathbb{S}\setminus\{k\}} q_{kl}(z), \quad n(t) = \max\{i : \tau_i \leq t\},$$

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Martingale Solution: General Case

Lemma 5

$(M_t, \mathcal{F}_t, \widehat{\mathbb{P}})$ *is a non-negative martingale with mean one.*

By virtue of M_t **and** $\widehat{\mathbb{P}}$, we can **construct another probability measure** $\mathbb{P}$ on $\Omega = C([0,\infty), \mathbb{R}^{2d}) \times D([0,\infty), \mathbb{S})$ such that $\mathbb{P}$ is a solution to the martingale problem for the operator $\mathcal{A}$.

Theorem 6

For any given $(z,k) \in \mathbb{R}^{2d} \times \mathbb{S}$*, there exists a unique martingale solution* $\mathbb{P}^{(z,k)}$ *on* Ω *for the operator* $\mathcal{A}$ *starting from* (z,k)*. In other words,* ***there exists a weak solution solution*** $\mathbb{P}^{(z,k)}$ ***on*** Ω ***to system (3) and (4) with initial data*** (z,k)*.*

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Sketch of Proof

For each $t \geq 0$ and each $A \in \mathcal{F}_t$, define

$$\mathbb{P}_t^{(z,k)}(A) = \int_A M_t(Z(\cdot), \Lambda(\cdot))\, \mathrm{d}\widehat{\mathbb{P}}^{(z,k)}. \tag{18}$$

Clearly, the family of probability measures $\{\mathbb{P}_t^{(z,k)}\}_{t\geq 0}$ is consistent. Thus, there exists a unique probability measure $\mathbb{P}^{(z,k)}$ on $(\Omega, \mathcal{F})$ which coincides with $\mathbb{P}_t^{(z,k)}$ on $\mathcal{F}_t$ for all $t \geq 0$. Moreover, we can prove that the $\mathbb{P}$ is the desired martingale solution staring from (z, k).

Strong Feller Property: Special Case

Let $\left(\widehat{P}(t,(z,k),\cdot)\right)_{t\geq 0}$ be the transition probability family of Markov process $\left((\widehat{Z}(t),\psi(t)\right)$.

Let us fix a probability measure $\mu(\cdot)$ that is equivalent to the product measure on $\mathbb{R}^{2d}\times\mathbb{S}$ of the Lebesgue measure on $\mathbb{R}^{2d}$ and the counting measure on $\mathbb{S}$.

Theorem 7

The Markov process $(\widehat{Z}(t),\psi(t))$ has the strong Feller property. Moreover, the transition probability $\widehat{P}(t,(z,k),\cdot)$ of $(\widehat{Z}(t),\psi(t))$ has transition density $\widehat{p}(t,(z,k),\cdot)$ with respect to $\mu(\cdot)$.

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Sketch of Proof

For any $t > 0$, $z \in \mathbb{R}^{2d}$, $k, l \in \mathbb{S}$ and $A \in \mathcal{B}(\mathbb{R}^{2d})$,

$$\begin{aligned}
&\widehat{P}(t,(z,k),A \times \{l\}) \\
&= \delta_{kl} P^{(k)}(t,z,A) e^{-(n_0-1)t} + e^{-(n_0-1)t} \sum_{m=1}^{+\infty} \int \cdots \int_{0<t_1<t_2<\cdots<t_m<t} \\
&\qquad \sum_{\substack{l_0,l_1,l_2,\cdots,l_m \in \mathbb{S} \\ l_i \neq l_{i+1}, l_0=k, l_m=l}} \int_{\mathbb{R}^{2d}} \cdots \int_{\mathbb{R}^{2d}} P^{(l_0)}(t_1,z,dz_1) \\
&\qquad\qquad P^{(l_1)}(t_2-t_1,z_1,dz_2) \cdots P^{(l_m)}(t-t_m,z_m,A) dt_1 dt_2 \cdots dt_m.
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Each $P_k(t,z,\cdot)$ is strong Feller and has positive transition density $p_k(t,z,\cdot)$ by Lemma 2.

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Strong Feller Property: General Case

By making using of M_t, we can **transfer the strong Feller property** for $(\widehat{X}, \widehat{Y}, \psi)$ to the one for (X, Y, Λ).

Assumption 2

Assume that for a constant $H > 0$,

$$|q_{kl}(z) - q_{kl}(z_0)| \leq H|z - z_0| \tag{19}$$

for all $z, z_0 \in \mathbb{R}^{2d}$ and $k, l \in \mathbb{S}$ with $k \neq l$.

Theorem 8

Suppose that Assumption 2 holds. The unique non-explosive solution $(Z(t), \Lambda(t))$ to system (3) and (4) also has the strong Feller property.

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Suppose that Assumption 2 holds. The unique non-explosive solution $(Z(t), \Lambda(t))$ *to system (3) and (4) also has the strong Feller property.*

Exponential Ergodicity

For the exponential ergodicity, here I do not mention the general results but only **a very concrete example**: the stochastic **van der Pol equation** with state-dependent switching.

Example 9

Let $d = 1$ and $\mathbb{S} = \{1, 2\}$. Take the functions $c(x, y, k)$ and $V(x, k)$ in (3) and $q_{kl}(x, y)$ in (4) as follows. For $(x, y, k) \in \mathbb{R}^2 \times \{1, 2\}$, define

$$c(x, y, k) = \alpha(k)(x^2 - 1), \quad V(x, k) = \frac{1}{2}\beta(k)x^2,$$

$$Q(x, y) := (q_{kl}(x, y)) = \begin{pmatrix} -\exp(-|x|^3) & \exp(-|x|^3) \\ \frac{\widetilde{H}}{|x|^2+|y|^2+1} & -\frac{\widetilde{H}}{|x|^2+|y|^2+1} \end{pmatrix},$$

where $\alpha(1) = 1$, $\alpha(2) = 2$, $\beta(1) = 2$ and $\beta(2) = 1$, and $\widetilde{H}$ is an arbitrary positive constant. Moreover, let $\sigma(x, y, k)$ in (3) be just as in Assumption 1.

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Exponential Ergodicity (cont'd)

Set a function $\widetilde{V}(x, y, k)$ on $\mathbb{R}^2 \times \{1, 2\}$ as

$$\widetilde{V}(x, y, k) = \exp\big(\beta(k)x^2 + y^2 + G(x)y + U(x, k)\big),$$

where $G(x)$ is infinitely differentiable such that

$$G(x) = \frac{x}{|x|} \text{ for } |x| > 1 \text{ and } |G(x)| \leq 1 \text{ for } x \in \mathbb{R};$$

and $U(x, k)$ is twice differentiable in x such that

$$U(x, k) = \alpha(k)\left(\frac{|x|^3}{3} - |x|\right) \text{ for } |x| > 1 \text{ and } k \in \{1, 2\}.$$

Then $\mathcal{A}\widetilde{V}(x, y, k) \leq -\widetilde{V}(x, y, k) + \beta$ for some positive constant β, so the system (X, Y, Λ) determined in the above example is **Ψ-exponentially ergodic**.

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Some References

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Thank you very much!