Limit theorems for a supercritical branching process with immigration in a random environment

Yanqing WANG

(A joint work with Quansheng LIU)

Zhongnan University of Economics and Law, Wuhan, China

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Contents

Introduction

In recent years, there is a great progress in the study of branching processes in random environments.

- For the survival probability and conditional limit theorems in the subcritical and critical cases, see Afanasyev, Böinghoff, Kersting and Vatutin (2012,2014), Vatutin(2010).
- For large deviations of the process and asymptotic properties of the limit variable of the naturally normalized process in the supercritical case, see Bansaye and Berestycki(2009), Böinghoff and Kersting(2010), Bansaye and Böinghoff(2011,2013,2014), Huang and Liu(2012,2014),Liang and Liu(2013) and Grama, Liu and Miqueu(2016, 2017).

The model of branching process with immigration in a random environment can be used in various fields. For example:

- Kesten, Kozlov and Spitzer (1975) used a branching process in an i.i.d. random environment with one immigrant at each generation to give limit laws for a random walk in a random environment.
- Bansaye (2009) studied a model of cell contamination by investigating a branching process in a random environment with immigration.
- Vatutin (2011) considered a multitype branching process with immigration that evolves in a random environment to study polling systems with random regimes of service.

Motivation

However, too few results have been known for a branching process with immigration in a random environment. The lack of theoretical results makes the application to concrete models not so comfortable. This consideration motivates us to establish limit theorems on several basic problems for such a process in the supercritical case.

Description of BPIRE

Let $\xi = (\xi_0, \xi_1, \xi_2, \cdots)$ be a sequence of independent and identically distributed random variables taking values in some space Θ indexed by time $n \in \mathbb{N} = \{0, 1, 2 \cdots\}$. Each realization of ξ_n corresponds to two probability distributions on $\mathbb{N}$. One is the offspring distribution denoted by $p(\xi_n) = \{p_k(\xi_n); k \geq 0\}$ where

$$0 \leq p_k(\xi_n) \leq 1, \ \ and \ \sum_k p_k(\xi_n) = 1.$$

The other is the distribution of the number of immigration denoted by $\hat{p}(\xi_n) = \{\hat{p}_k(\xi_n); k \geq 0\}$ where

$$0 \leq \hat{p}_k(\xi_n) \leq 1, \ \ and \ \sum_k \hat{p}_k(\xi_n) = 1.$$

We call ξ a random environment.

Definition of a BPIRE

A branching process $(Z_n)_{n\geq 0}$ with immigration $(Y_n)_{n\geq 0}$ in the random environment ξ (BPIRE) can be defined as follows:

$$Z_0 = 1, \qquad Z_{n+1} = Y_n + \sum_{i=1}^{Z_n} X_{n,i}, \quad n = 0, 1, 2, \cdots$$

where given the environment ξ, $X_{n,i}(i = 1, 2, \cdots)$, Y_n and Z_n are independent of each other, $X_{n,i}(i = 1, 2, \cdots)$ have the same distribution $p(\xi_n)$ and Y_n has the distribution $\hat{p}(\xi_n)$.

Quenched law and annealed law

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is given. As usual, $\mathbb{P}_\xi$ is called quenched law.

The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$, τ is the law of the environment ξ. The total probability $\mathbb{P}$ is usually called annealed law.

The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of the annealed law $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}_\xi$ (resp. $\mathbb{P}$) will be denoted by $\mathbb{E}_\xi$ (resp. $\mathbb{E}$).

Notation

For $n \geq 0$ and $p > 0$, define

$$m_n(p) = m_n(p,\xi) = \sum_{i=0}^{\infty} i^p p_i(\xi_n), \quad m_n = m_n(1)$$

and

$$\hat{m}_n(p) = \hat{m}_n(p,\xi) = \sum_{i=0}^{\infty} i^p \hat{p}_i(\xi_n), \quad \hat{m}_n = \hat{m}_n(1).$$

Then $m_n(p) = \mathbb{E}_\xi X_{n,i}^p$ and $\hat{m}_n(p) = \mathbb{E}_\xi Y_n^p$.

Normalized population

Set $\Pi_0 = 1$ and $\Pi_n = m_0 \cdots m_{n-1}$ for any $n \geq 1$.
Let

$$W_n := \frac{Z_n}{\Pi_n}.$$

We always assume $0 < m_0 < \infty$ a.s. and

$$\mathbb{E} \log m_0 > 0. \tag{1}$$

It is easy to check that $\{W_n\}_{n \geq 0}$ is a sub-martingale with respect to the natural filtration.

Main Results

Theorem 1 (Almost Sure Convergence)

Assume $\mathbb{E}\log^+(Y_0/m_0) < \infty$. Then there exits a finite random variable W such that

$$W_n \to W \ a.s..$$

Remark

Bansaye (2009) proved the a.s. convergence of $\{W_n\}$ under the assumptions $\mathbb{E}m_0^{-1} < 1$ and $\mathbb{E}\log^+ Y_0 < \infty$. Notice that $\mathbb{E}m_0^{-1} < 1$ implies $\mathbb{E}\log m_0 > 0$, and that $\mathbb{E}\log^+ Y_0 < \infty$ together with $\mathbb{E}m_0^{-1} < 1$ implies $\mathbb{E}\log^+(Y_0/m_0) < \infty$.

Let $X_0 = X_{0,1}$ which has the offspring distribution $p(\xi_0)$ given ξ.

Theorem 2 (Non-degeneracy of the Limit Variable)

Assume $\mathbb{E}\log^+(Y_0/m_0) < \infty$ and $\mathbb{E}(\log^- m_0)^2 < \infty$. Then

$$\mathbb{P}(W > 0) > 0$$

iff

$$\mathbb{E}(\frac{X_0}{m_0}\log^+ X_0) < \infty.$$

Convergence in $L^1(\mathbb{P})$

Theorem 3($L^1(\mathbb{P})$ Convergence)

Assume $\mathbb{P}(Y_0 = 0) < 1$, $\mathbb{E}\log^+(Y_0/m_0) < \infty$, and $\mathbb{E}(\log^- m_0)^2 < \infty$. Then, $W_n \xrightarrow{L^1(\mathbb{P})} W$ iff

$$\mathbb{E}m_0^{-1} < 1, \ \mathbb{E}\,(\hat{m}_0/m_0) < \infty \text{ and } \mathbb{E}(\frac{X_0}{m_0}\log^+ X_0) < \infty. \qquad (2)$$

Remark

If $\mathbb{P}(Y_0 = 0) = 1$ (the usual branching process in random environment), it is well-known (see Athreya and Karlin(1971) and Tanny(1988)) that $W_n \xrightarrow{L^1} W$ iff $\mathbb{E}(\frac{X_0}{m_0}\log^+ X_0) < \infty$.

Convergence in $L^p(\mathbb{P})$,$p > 1$

Theorem 4(Convergence in $L^p(\mathbb{P})$,$p > 1$)

Assume $\mathbb{P}(Y_0 = 0) < 1$, $\mathbb{E}\log^+(Y_0/m_0) < \infty$ and $\mathbb{E}(\log^- m_0)^2 < \infty$. Let $p > 1$ be fixed. Then, the sequence (W_n) converges in L^p under $\mathbb{P}$ iff

$$\mathbb{E}m_0^{-p} < 1, \ \mathbb{E}\left(\frac{Y_0^p}{m_0^p}\right) < \infty \ \text{ and } \ \mathbb{E}\left(\frac{X_0^p}{m_0^p}\right) < \infty. \tag{3}$$

Remark

If $\mathbb{P}(Y_0 = 0) < 1$ (the usual branching process in random environment), Guivarc'h and Liu(2001) proved that $W_n \xrightarrow{L^p} W$ iff $\mathbb{E}m_0^{1-p} < 1$ and $\mathbb{E}(\frac{X_0}{m_0})^p < \infty$.
Theorem 4 shows that a similar result holds in the immigration case, but the condition $\mathbb{E}m_0^{1-p} < 1$ should be replaced by $\mathbb{E}m_0^{-p} < 1$. This result, as well as the results obtained in Theorem 3 about the L^1 convergence, indicate that there are indeed different behaviors caused by the immigration, compared with a branching process without immigration (where $Y_0 = 0$ a.s.).

Theorem 5(Boundedness of the moments of W_n of order $p \in (0,1)$)

Assume $\mathbb{P}(Y_0 = 0) < 1$. Let $p \in (0,1)$ be fixed. If $\mathbb{E}\log^+(Y_0/m_0) < \infty$, then $\sup_{n\geq 0} \mathbb{E} W_n^p < \infty$ iff

$$\mathbb{E} m_0^{-p} < 1, \ \mathbb{E}\left(\frac{Y_0}{m_0}\right)^p < \infty. \tag{4}$$

Theorem 6(Harmonic Moments of W)

Assume $p_0 = 0$ a.s., $\mathbb{E}\log^+(Y_0/m_0) < \infty$ and $\mathbb{E}m_0^{\lambda_0} < \infty$ for some $\lambda_0 > 0$. Then, for all $a \in (0, \lambda_0)$ satisfying $\mathbb{E}p_1 m_0^a < 1$, we have

$$\mathbb{E}W^{-a} < \infty \quad \text{and} \quad \limsup_{n\to\infty} \mathbb{E}W_n^{-a} < \infty. \tag{5}$$

Remark

Without additional condition on Y_0, the value of $a > 0$ in Theorem 6 cannot be improved. In fact, it was shown in Theorem 2.1 of Grama, Liu and Miqueu(2016) that when $Y_0 = 0\ a.s.$, for all $a \in (0, \lambda_0)$,

$$\mathbb{E}W^{-a} < \infty \quad \text{iff} \quad \mathbb{E}p_1 m_0^a < 1. \tag{6}$$

Central Limit Theorem

Theorem 7 (Central Limit Theorem)

Assume $\mathbb{E}\log^+(Y_0/m_0) < \infty$ and $\mathbb{E}\frac{X_0}{m_0}\log^+ X_0 < \infty$. Assume also $p_0 = 0$ *a.s.* and $\sigma^2 = var(\log m_0) \in (0, \infty)$. Then

$$\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \xrightarrow{d} N(0, 1). \qquad (7)$$

Large Deviation Principle

Let $\Lambda(t) = \log \mathbb{E} m_0^t$ and $\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{tx - \Lambda(t)\}$.

Theorem 8 (Large Deviation Principle)

Assume that $p_0 = p_1 = 0$ a.s. and that for all $s > 1$,

$$\mathbb{E} m_0^s < \infty, \quad \mathbb{E}\left(\frac{X_0}{m_0}\right)^s < \infty \quad \text{and} \quad \mathbb{E}\left(\frac{Y_0}{m_0}\right)^s < \infty. \tag{8}$$

then for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^o} \Lambda^*(x) &\le \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \le -\inf_{x \in \bar{B}} \Lambda^*(x). \end{aligned} \tag{9}$$

Large Deviation Principle

Remark

By a result of Bansaye and Berestycki(2009), when $p_0 = 0$ and $\mathbb{P}(p_1 = 0) < 1$, the result is no longer valid even for a branching process (without immigration) in an i.i.d. random environment, for which the proof is no nonger valid due to the fact that $\mathbb{E}W^{-\lambda} = \infty$ when $\lambda > 0$ is large enough (essentially when $\mathbb{E}(p_1 m_0^\lambda) \geq 1$).

Moderate Deviation Principle

Theorem 9 (Moderate Deviation Principle)

Assume that $p_0 = 0$ a.s. and that $\sigma^2 = var(\log m_0) \in (0, \infty)$. Assume also $\mathbb{E}\frac{X_0}{m_0} \log^+ X_0 < \infty$ and that

$$\mathbb{E} m_0^\delta < \infty \quad \text{and} \quad \mathbb{E}\left(\frac{Y_0}{m_0}\right)^\delta < \infty \tag{10}$$

for some $\delta > 0$. Let $\{a_n\}$ be a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \to \infty, \text{ as } n \to \infty.$$

Then for any measurable subset B of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in B^o} \frac{x^2}{2\sigma^2} &\leq \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B \right) \\ &\leq \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B \right) \qquad (11) \\ &\leq -\inf_{x \in \bar{B}} \frac{x^2}{2\sigma^2}. \end{aligned}$$

Proof of Theorem 1 (a.s. convergence)

Let $Y = (Y_0, Y_1, \cdots)$ and $\mathbb{P}_{\xi,Y}$ be the conditional probability of $\mathbb{P}_\xi$ given Y. The corresponding expectation will be denoted by $\mathbb{E}_{\xi,Y}$. Then:

- $\{W_n\}$ is a sub-martingale under $\mathbb{P}_{\xi,Y}$ with respect to $\mathcal{F}_n = \sigma(X_{k,i}, \xi, Y : k < n, i = 1, 2, \cdots)$.
- $\mathbb{E}_{\xi,Y} W_n = \sum_{k=0}^{n-1} \frac{Y_k}{\Pi_k m_k} + 1$ and $\sum_{k=0}^{\infty} \frac{Y_k}{\Pi_k m_k} < \infty$ a.s. if $\mathbb{E} \log m_0 > 0$ and $\mathbb{E} \log^+(Y_0/m_0) < \infty$.

By the martingale convergence theorem, since the sub-martingale (W_n) is L^1 bounded under $\mathbb{P}_{\xi,Y}$, there is a finite random variable W such that $W_n \to W$ $\mathbb{P}_{\xi,Y}$ - a.s. As this holds for almost every realization of ξ and Y, it follows that $W_n \to W$ $\mathbb{P}$-a.s.

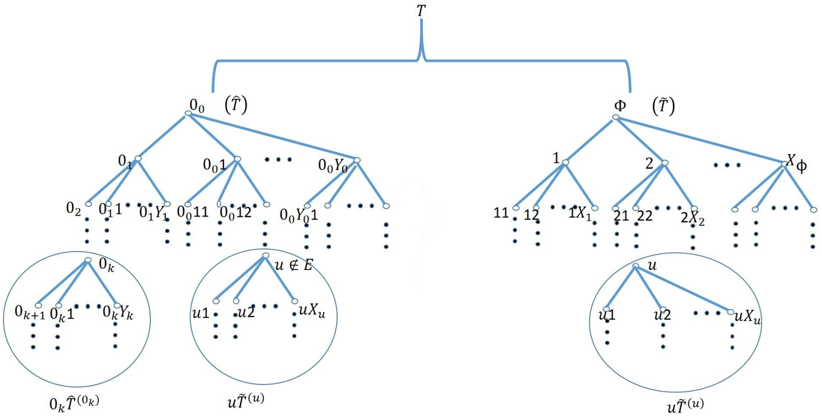
Decomposition of the BPIRE Model

We add one particle at each time n which we call eternal particle, denoted by $0_n := (\underbrace{0, \cdots, 0}_{n})$, giving birth to $Y_n + 1$ children including the Y_n immigrants and the eternal particle at time $n + 1$. We can easily find that the new branching tree can be seen as the union of two disjoint branching trees:

1. one starts from the initial particle $\emptyset$ denote by $\tilde{T}$;
2. the other starts from the eternal particle 0_0 denoted by $\hat{T}$.

Then, $\tilde{T}$ is an usual Branching tree without immigration; $\hat{T}$ is composed of all the immigrants and their descendants.

Figure: Geneological tree of a branching process with immigration



Decomposition of the BPIRE Model

Let $\tilde{Z}_n^{(u)}$ (resp. $\hat{Z}_n^{(u)}$) be the number of particles at time n of the shifted branching tree $\tilde{T}^{(u)}$ (resp. $\hat{T}^{(u)}$). Obviously, we have

$$Z_n = \tilde{Z}_n^{(\emptyset)} + \hat{Z}_n^{(0_0)} - 1.$$

Define $\tilde{W}_n^{(\emptyset)} = \tilde{Z}_n^{(\emptyset)}/\Pi_n$ and $\hat{W}_n^{(0_0)} = \hat{Z}_n^{(0_0)}/\Pi_n$, then we have

$$W_n = \tilde{W}_n^{(\emptyset)} + \hat{W}_n^{(0_0)} - \Pi_n^{-1},$$

Eternal Particles Model

$\tilde{W}_n$ is the famous martingale for a branching process in a random environment, and has been well studied in many works. So we only concentrate on the tree $\hat{T}$. For any expression on the tree $\hat{T}$, we use an upper symbol u for the expression evaluated on the shifted tree of $\hat{T}$ at u. If we take view backward by one step, we have

$$\hat{W}_n^{(0_0)} = \frac{1}{m_0} \hat{W}_{n-1}^{(0_1)} + \frac{1}{m_0} \sum_{i=1}^{Y_0} \tilde{W}_{n-1}^{(0_0 i)} \tag{12}$$

By iteration, we get

$$\hat{W}_n^{(0_0)} = \sum_{k=1}^{n} \Pi_k \sum_{i=1}^{Y_{k-1}} \tilde{W}_{n-k}^{(0_{k-1} i)} + \Pi_n^{-1}. \tag{13}$$

Proof of Theorem 2(Non-degeneracy of the Limit Variable)

Using the decomposition formula of $\hat{W}_n$, we have

$$W_n = \tilde{W}_n^{(\emptyset)} + \sum_{k=1}^{n} \Pi_k \sum_{i=1}^{Y_{k-1}} \tilde{W}_{n-k}^{(0_{k-1}i)}.$$

For any $u \in \hat{T}$, we denote $\tilde{W}^{(u)} = \lim_{n\to\infty} \tilde{W}_n^{(u)}$, *a.s.*.
Under the moment condition $\mathbb{E}(\log^- m_0)^2 < \infty$, we have

$$W := \tilde{W}^{(\emptyset)} + \sum_{k=1}^{\infty} \Pi_k \sum_{i=1}^{Y_{k-1}} \tilde{W}^{(0_{k-1}i)} \ a.s.$$

From this representation of W, we can easily get Theorem 2.

Proof of Theorem 3 ($L^1(\mathbb{P})$ convergence)

From the decomposition formulas of W_n and W , we have

$$\mathbb{E}W_n = \sum_{k=1}^{n} (\mathbb{E}m_0^{-1})^{k-1} \mathbb{E}\left(\frac{\hat{m}_0}{m_0}\right) + 1,$$

$$\mathbb{E}W = (\sum_{k=1}^{\infty} (\mathbb{E}m_0^{-1})^{k-1} \mathbb{E}\left(\frac{\hat{m}_0}{m_0}\right) + 1)\mathbb{E}\tilde{W}.$$

From Scheff$\acute{e}$'s Theorem, $W_n \xrightarrow{L^1} W$ if and only if $\mathbb{E}W_n \to \mathbb{E}W$, which holds if and only if $\mathbb{E}m_0^{-1} < 1$, $\mathbb{E}\left(\hat{m}_0/m_0\right) < \infty$ and $\mathbb{E}\tilde{W} = 1$.

From Theorem 2 in Tanny(1988), we know that

$$\mathbb{E}\tilde{W} = 1 \textit{ if and only if } \mathbb{E}(\frac{X_0}{m_0} \ln^+ X_0) < \infty.$$

Proof of Theorem 4 ($L^p(\mathbb{P})$ convergence)

We first prove the sufficiency.
Since W_n is a nonnegative sub-martingale, we need only to prove $\sup \mathbb{E} W_n^p < \infty$.
By the decomposition formula of W_n, together with the subadditivity of norm and the fact that the environment is i.i.d and $\sup_{n\geq 0} \mathbb{E}\left(\tilde{W}_n^{(\emptyset)}\right)^p = \mathbb{E}\left(\tilde{W}^{(\emptyset)}\right)^p$, we have

$$\sup_{n\geq 0} ||W_n||_p \leq ||\tilde{W}^{(\emptyset)}||_p + \sum_{k=1}^{\infty} \left(\mathbb{E}(m_0^{-p})\right)^{(k-1)/p} \left(\mathbb{E}\left(\frac{Y_0^p}{m_0^p}\right)\mathbb{E}(\tilde{W}^{(\emptyset)})^p\right)^{1/p}.$$

We now prove the necessity. Assume $\sup_{n\geq 0}\mathbb{E}W_n^p < \infty$.
By the decomposition formula for W_n, we know that $W_n \geq \tilde{W}_n^{(\emptyset)}$.
Therefore $\sup_{n\geq 0}\mathbb{E}(\tilde{W}_n^{(\emptyset)})^p < \infty$. Hence from Guivarc'h and Liu(2001), we conclude that

$$\mathbb{E}m_0^{1-p} < 1, \ \textit{and} \ \mathbb{E}(m_0^{-p}X_0^p) < \infty.$$

By the fact that the environment is i.i.d, we can get

$$\mathbb{E}W_n^p \geq 1 + \sum_{k=1}^{n}(\mathbb{E}m_0^{-p})^{k-1}\mathbb{E}(m_0^{-p}Y_0^p). \tag{14}$$

Therefore $\sup_{n\geq 0}\mathbb{E}W_n^p < \infty$ and $\mathbb{P}(Y_0 = 0) < 1$ imply that

$$\mathbb{E}m_0^{-p} < 1 \quad \text{and} \quad \mathbb{E}(m_0^{-p}Y_0^p) < \infty.$$

Proof of Theorem 6 (harmonic moments)

Under the condition $p_0 = 0\, a.s.$, we know $\tilde{W} > 0$, $W > 0$ a.s.. When $\mathbb{E}m_0^{\lambda_0} < \infty$ for some $\lambda_0 > 0$, it was shown in Theorem 2.1 of Grama, Liu and Miqueu(2016) that for all $a \in (0, \lambda_0)$,

$$\mathbb{E}\tilde{W}^{-a} < \infty \qquad \text{if and only if} \qquad \mathbb{E}p_1 m_0^a < 1. \tag{15}$$

Since $\tilde{W} \leq W$, $\mathbb{E}\tilde{W}^{-\alpha} < \infty$ implies $\mathbb{E}W^{-\alpha} < \infty$.

Proof of Theorem 7 (central limit theorem)

Notice that

$$\log Z_n = \log \Pi_n + \log W_n,$$

$$\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} = \frac{\log \Pi_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} + \frac{\log W_n}{\sqrt{n}\sigma}.$$

Since $W_n \to W < \infty$ *a.s.*, we have $\lim_{n\to} \frac{\log W_n}{\sqrt{n}\sigma} = 0$ *a.s.*.
By the classic central limit theorem for i.i.d random variables, we have $\frac{\log \Pi_n - n\mathbb{E}\log m_0}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1)$ which implies Theorem 7.

Proof of Theorems 8 and 9 (LDP and MDP)

From the classic large deviation theory for *i.i.d* random variables, we know that under the conditions of Theorems 8 and 9, $\log \Pi_n$ satisfies the same large deviation and moderate deviation principles,respectively. So we need only to prove the following two exponential equivalences:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{|\log Z_n - \log \Pi_n|}{n} > \delta\right) = -\infty, \tag{16}$$

and

$$\lim_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{|\log Z_n - \log \Pi_n|}{a_n} > \delta\right) = -\infty \tag{17}$$

for any $\delta > 0$. For the proof, we use the exponential Markov inequality, the elementary inequality $e^{\lambda|\log x|} \leq x^{\lambda} + x^{-\lambda}$, the L^p convergence and the existence of harmonic moments.

Remark

For a branching process (without immigration) in an i.i.d. random environment, the same LDP and MDP were proved in Huang and Liu (2012) under the following boundedness condition: for some constants $p, c_1, c_2 > 1$,

$$c_1 \leq m_0 \quad \text{and} \quad m_0(p) \leq c_2 \quad \text{a.s.} \tag{18}$$

(recall that $m_0(p) = \sum_{i=0}^{\infty} i^p p_i(\xi_0) = \mathbb{E}_\xi X_0^p, m_0 = m_0(1)$), instead of the much weaker moment condition $\mathbb{E} m_0^s < \infty$ that we use here. The proof that we present here is quite different and much simpler. Our argument is based on an exponential equivalence instead of the Gärtner - Ellis theorem used in Huang and Liu (2012).

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Thank you !

yanqingwang@znufe.edu.cn