Heat Kernel Estimates and Boundary Harnack Principle for Truncated Fractional Laplacian with Gradient Operator

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Background and Motivation

- A stochastic process $Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ is called a rotationally symmetric α-stable process with $\alpha \in (0,2)$ on $\mathbb{R}^d$ if it is a Lévy process such that its characteristic function is given by

$$\mathbb{E}_x\left[e^{i\xi\cdot(Z_t - Z_0)}\right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d.$$

- The infinitesimal generator for rotationally symmetric α-stable process is

$$\Delta^{\alpha/2} f(x) = \mathcal{A}(d,-\alpha)\int_{\mathbb{R}^d}\left(f(x+z) - f(x) - \nabla f(x)\cdot z 1_{\{|z|\le 1\}}\right)\frac{dz}{|z|^{d+\alpha}}$$

- Truncated symmetric α-stable process $\overline{Z}$ is the symmetric α-stable process Z with large jumps more than 1 removed.
- Denote by $\overline{\Delta}^{\alpha/2}$ the operator of $\overline{Z}_t$, then

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- Chen-Kim-Kumagai (2008) considered a wide class of symmetric finite range non-local operator

$$\mathcal{S}f(x) = \lim_{\varepsilon \downarrow 0} \int_{\{|y-x|>\varepsilon\}} (f(y) - f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}} 1_{\{|x-y|\le\kappa\}}\, dy,$$

  where $\alpha \in (0,2)$, κ is a positive constant and $c(x,y)$ is a measurable symmetric function and is bounded between two positive constants.

- The operator $\mathcal{S}$ is the operator $\overline{\Delta}^{\alpha/2}$ when $c(x,y) = 1$ and $\kappa = 1$.
- Associated with $\mathcal{S}$ is a symmetric jump process with jump density $\frac{c(x,y)}{|x-y|^{d+\alpha}} 1_{\{|x-y|\le\kappa\}}$. Sharp two-sided heat kernel estimate for all $t > 0$ and parabolic Harnack principle are established for them in this paper.

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Denote by $p(t,x,y)$ and $\overline{p}(t,x,y)$ the heat kernels of $\Delta^{\alpha/2}$ and $\overline{\Delta}^{\alpha/2}$ correspondingly.

- $\overline{p}(t,x,y)$ is jointly continuous and satisfies that

$$\overline{p}(t,x,y) \asymp p(t,x,y), \quad |x-y| \le 1, t \in (0,1],$$

and there are constants $c_k > 0$, $k = 1,2,3,4$ so that

$$c_1 \left(\frac{t}{|x-y|} \right)^{c_2|x-y|} \le \overline{p}(t,x,y) \le c_3 \left(\frac{t}{|x-y|} \right)^{c_4|x-y|}$$

for $|x-y| > 1$ and $t \in (0,1)$.

Purpose

- We are concerned with the operator

$$\mathcal{L}^b := \overline{\Delta}^{\alpha/2} + b \cdot \nabla,$$

where $\alpha \in (1,2)$ and $b \in K_d^{\alpha-1}$. Here a function $b \in K_d^{\alpha-1}$ with $\alpha \in (1,2)$, if

$$\lim_{r \to 0} \sup_x \int_{B(x,r)} |y-x|^{\alpha-1-d} |b|(y) = 0.$$

- The operator $\mathcal{L}^b$ can be viewed as the operator

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- Bogdan-Jakubowski (2007) constructed a continuous transition density function $p^b(t,x,y)$ associated with the operator

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where $\alpha \in (1,2)$ and $b \in K_d^{\alpha-1}$.

- $p^b(t,x,y)$ is comparable to the heat kernel of the operator $\Delta^{\alpha/2}$ in short time.
- Bogdan-Jakubowski (2012) established the Green function estimates and the boundary Harnack principle for the operator in bounded $C^{1,1}$ open set.
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Question:

- Does the transition density function exist of the operator $\mathcal{L}^b$? If so, is it comparable to that of $\overline{\Delta}^{\alpha/2}$ for $t \in (0,1)$?

- What about the boundary Harnack principle for $\mathcal{L}^b$?

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Main result

Theorem 1: Heat kernel estimates

(i)There exist a family of martingale solutions $\{\mathbb{P}_x, x \in \mathbb{R}^d\}$ for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value x. The solutions $\{\mathbb{P}_x, x \in \mathbb{R}^d\}$ form a strong Markov process X^b, which possesses a continuous transition density function $q^b(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$.

(ii)There exist positive constants $C_k, k = 1, 2, 3$ such that

$$C_1^{-1}\overline{p}(t, C_2x, C_2y) \le q^b(t,x,y) \le C_1 p(t, C_3x, C_3y)$$

for all $t \in (0,1)$ and $x, y \in \mathbb{R}^d$. Especially, suppose b is a bounded function, there exist positive constants $C_k, k = 4, 5, 6$ such that

$$C_4^{-1}\overline{p}(t, C_5x, C_5y) \le q^b(t,x,y) \le C_4\overline{p}(t, C_6x, C_6y)$$

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Proposition

Suppose a strong Markov process Y associated with the following operator possesses a transition density function $q_Y(t,x,y)$

$$\mathcal{L}_Y f(x) = \int_{\{|z|\le 1\}} (f(x+z) - f(x) - \langle \nabla f(x), z\rangle) j(x,z)\, dz + b(x)\nabla f(x),$$

where $\int_{\mathbb{R}^d} (1 \wedge |z|^2) j(x,z)\, dz < +\infty$ for each $x \in \mathbb{R}^d$, b is a Borel measurable function in $\mathbb{R}^d$. Suppose $q_Y(t,x,y)$ satisfies

$$q_Y(t,x,y) \asymp p(t,x,y) \quad \text{for} \quad t \in (0,1),\ |x-y| \le 1, \qquad (1)$$

and there exists $c_1 > 0$ such that

$$q_Y(t,x,y) \le c_1 t \quad \text{for} \quad t \in (0,1),\ |x-y| > 1. \qquad (2)$$

Continued

Moreover, suppose there exists $c_2 > 0$ independent of r such that for any $x \in \mathbb{R}^d, t \in (0,1)$ and $r > 0$,

$$\mathbb{P}_x(\tau_{B(x,r)} \leq t) \leq c_2 t/\phi(r), \quad (3)$$

where ϕ is an increasing function in $\mathbb{R}^d$ such that $\phi(r) \to +\infty$ as $r \to +\infty$. Then there exist positive constants $C_k, k = 7, 8, 9$ such that for all $t \in (0,1)$ and $x, y \in \mathbb{R}^d$,

$$C_7^{-1}\overline{p}(t, C_8 x, C_8 y) \leq q_Y(t,x,y) \leq C_7 \overline{p}(t, C_9 x, C_9 y)$$

Boundary Harnack principle

- Suppose that X is a strong Markov process. Suppose U is an open set of $\mathbb{R}^d$. A Borel function h is said to be a harmonic function in U with respect to X, if
$$h(x) = \mathbb{E}_x[h(X_{\tau_B})], \quad x \in B$$
for every bounded open set B with $\overline{B} \subset U$.
- BHP for X in D: For a domain D in $\mathbb{R}^d$, there exist positive constants R_0 and C such that for any $Q \in \partial D, r \in (0, R_0]$ any positive harmonic functions u and v in $D \cap B(Q, r)$ that vanish continuously on $D^c \cap B(Q, r)$, we have
$$\frac{u(x)}{u(y)} \le C \frac{v(x)}{v(y)}, \quad x, y \in D \cap B(Q, r/2).$$
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Boundary Harnack principle

- Bogdan (1997) first obtained BHP for rotationally α-stable processes(or equivalently for $\Delta^{\alpha/2}$) with $\alpha \in (0, 2)$ in bounded Lipschitz domains.
- Later, Song-Wu (1999) extended it to κ-fat open sets.
- Bogdan-Kulczycki-Kwasnicki (2008) established it in any open set with the constant independent of the open set itself.
- The boundary Harnack principle has been extended to more purely discontinuous processes, including subordinate Brownian motion and a wide class of rotationally symmetric Lévy processes, as well as discontinuous processes with Brownian part.

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Boundary Harnack principle for truncated symmetric stable process

- Kim-Song (2007) established the boundary Harnack principle for truncated symmetric stable process on bounded and convex domain for harmonic functions vanishing continuously on D^c.
- The paper showed that unlike symmetric stable process, the boundary Harnack principle for truncated stable process does not always hold in any open set, especially not always in non-convex open sets.
- The main reason is that for each subset A in $\mathbb{R}^d$, some points can jump to it with positive jump density, while some others not.

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BHP with $b \in K_d^{\alpha-1}$ in Lipschitz domain

- We shall first establish the boundary Harnack principle for X^b with $b \in K_d^{\alpha-1}$ in Lipshchitz open sets.
- Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a Lipschitz open set with characteristics (R, Λ) if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a Lipschitz function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $|\phi(x) - \phi(y)| \leq \Lambda|x - y|$, and an orthonormal coordinate system $CS_Q : y = (y_1, \cdots, y_{d-1}, y_d) =: (\tilde{y}, y_d)$ with its origin at Q such that

$$B(Q,R) \cap D = \{y = (\tilde{y}, y_d) \in B(0,R) \; in \; CS_Q : y_d > \phi(\tilde{y})\}.$$

Without loss of generality, we assume $R \in (0, 1/4)$ and $\Lambda \geq 1$.

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- Note that $\mathcal{L}^b = \overline{\Delta}^{\alpha/2}$ when $b \equiv 0$. Next we shall give an assumption. For the simplicity of notation, denote by $D_{Q,r}$ as the set $D \cap B(Q, r)$.
- Fix a constant $\kappa \in (0, 1/4)$. For each $r > 0$ and $Q \in \partial D$, define

$$\Omega_{\kappa,r,Q} := \{y \in D_{Q,r} : \delta_{D_{Q,r}}(y) > \kappa r\}.$$

For each point $u \in \mathbb{R}^d$ and each set $F \subseteq \mathbb{R}^d$, define

$$H_{u,F} := \{y \in F : |u - y| \le 1\} = \{y \in F : J(y, u) > 0\},$$

where $J(y, u) = |y - u|^{-(d+\alpha)} 1_{|u-y|\le 1}(y, u)$ is the jump density function of X^b.

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Define $m(\cdot)$ as the Lebesgue measure on $\mathbb{R}^d$.

- (A1) For each Lipschitz open set D with characteristics (R, Λ), suppose there exists a constant $c = c(d, R, \Lambda) > 0$ such that for any $r \in (0, R)$, $Q \in \partial D$ and $u \in \{u \in B(Q, 4r)^c : m(H_{u,D_{Q,r}}) > 0\}$,

$$\int_{\Omega_{\kappa,2r,Q}} 1_{|u-y|\leq 1}(y)\, dy \geq cr^d$$

$$\iff \int_{\Omega_{\kappa,2r,Q}} 1_{\{J(y,u)>0\}}(y)\, dy \geq cr^d.$$

Remarks:

- The assumption $(A1)$ is restricted to large jumps of the process X^b from the set $D \cap B(Q, r)$ near the boundary.
- If there exist some jumps from $D_{Q,r}$ to the point u, then there should be enough amount of jumps from $\Omega_{\kappa,2r,Q}$ to u.

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- (A1) For each Lipschitz open set D with characteristics (R, Λ), suppose there exists a constant $c = c(d, R, \Lambda) > 0$ such that for any $r \in (0, R)$, $Q \in \partial D$ and $u \in \left\{u \in B(Q, 4r)^c : m(H_{u, D_{Q,r}}) > 0\right\}$,

$$\int_{\Omega_{\kappa,2r,Q}} 1_{|u-y|\le 1}(y)\, dy \ge cr^d$$

$$\iff \int_{\Omega_{\kappa,2r,Q}} 1_{\{J(y,u)>0\}}(y)\, dy \ge cr^d.$$

Remarks:

- The assumption $(A1)$ is restricted to large jumps of the process X^b from the set $D \cap B(Q, r)$ near the boundary.
- If there exist some jumps from $D_{Q,r}$ to the point u, then there should be enough amount of jumps from $\Omega_{\kappa,2r,Q}$ to u.

Denote by G_D^b the Green function of the process X^b killed upon the open set D. Define

$$s_D(x) := \int_D G_D^b(x,y)\,dy = \mathbb{E}_x \tau_D.$$

Recall that $D_{Q,r}$ is the set $D \cap B(Q,r)$.

Theorem: BHP with $b \in K_d^{\alpha-1}$ in Lipschitz domain

Suppose the function $b \in K_d^{\alpha-1}$. For each Lipschitz open set D with characteristics (R,Λ) satisfying the assumption $(A1)$, there exists a positive constant $C = C(d,\alpha,R,\Lambda,b)$ such that for all $Q \in \partial D, r \in (0,R]$ and all function $h \geq 0$ on $\mathbb{R}^d$ that is harmonic with respect to the process X^b in $D \cap B(Q,2r)$ and vanishes continuously on $D^c \cap B(Q,2r)$, we have

$$\frac{h(x)}{h(y)} \leq C \frac{s_{D_{Q,r}}(x)}{s_{D_{Q,r}}(y)}, \quad x,y \in D \cap B(Q,r/2).$$

Examples:

- D is a bounded and convex domain. The boundary Harnack principle holds for harmonic functions vanishing continuously on D^c.
- $D = \mathbb{R}^d \setminus \overline{B(0, R_0)}$ with $R_0 > 12$. The boundary Harnack principle holds for harmonic functions vanishing in D^c.

Remark: The theorem gives the decay rate of harmonic function w.r.t. X^b. Especially, if D is a $C^{1,1}$ domain, then

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BHP with bounded function b in open sets

- (A2): For each open set D, suppose there exist constants $r_0 = r_0(D) > 0$ and $c = c(d, D) > 0$ such that for any $r \in (0, r_0)$, $Q \in \partial D$ and $u \in \{u \in B(Q, 4r)^c : m(H_{u, D_{Q,r}}) > 0\}$,

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Theorem: BHP with bounded function b in open sets

Suppose the function b is a bounded function. For each open set D satisfying the assumption $(A2)$, there exists a positive constant $C = C(d, \alpha, b, D)$ such that for all $Q \in \partial D, r \in (0, r_0)$ and all function $h \ge 0$ on $\mathbb{R}^d$ that is harmonic with respect to the process X^b in $D \cap B(Q, 2r)$ and vanishes continuously on $D^c \cap B(Q, 2r)$, we have

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- The proof of the boundary Harnack principle for X^b when b is bounded is adapted from the one in Bogdan-Kulczycki-Kwasnicki (2008) for the BHP for symmetric stable process in open sets.
- One key Lemma is as follows (see Kim-Song (2007) for symmetric truncated stable process and Bogdan-Kulczycki-Kwasnicki (2008) for symmetric stable process)

Lemma

Suppose D is an open set and ϕ is a $C_c^\infty(\mathbb{R}^d)$ function, then for $x \in D$,

$$\begin{aligned} \mathbb{E}_x\phi(X^b_{\tau_D}) &= \phi(x) + \int_D G^b_D(x,v)\mathcal{L}^b\phi(v)\,dv \\ &= \phi(x) + \int_D G^b_D(x,v)(\overline{\Delta}^{\alpha/2}\phi(v) + b(v)\nabla\phi(v))\,dv. \end{aligned}$$

- This method seems not work well when the function b belongs to Kato class $K_d^{\alpha-1}$.

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Sketch of the proof when $b \in K_d^{\alpha-1}$ in Lipschitz domain

- Suppose D is a Lipschitz domain, then

$$\mathbb{P}_x(X^b_{\tau_D} \in \partial D) = 0, \quad x \in D.$$

- By the Lévy system formula, for any Lipschitz open set D and nonnegative function h,

$$\mathbb{E}_x[h(X^b_{\tau_D})] = \int_{\overline{D}^c} \int_D G^b_D(x,y) J(y,u) h(u)\, dy\, du,$$

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Lemma 1

For each bounded Lipschitz open set U with characteristics (R, Λ), there exist constants $r_0 = r_0(d, \alpha, b)$ and $C = C(d, \alpha, R, \Lambda, b) > 1$ such that if $diam(U) \leq r_0$, then

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Lemma 2: BHP w.r.t. $\tilde{L}^b$ in Lipschitz domain

Suppose the function $b \in K_d^{\alpha-1}$. For each Lipschitz open set D with characteristics (R, Λ), there exists a positive constant $C = C(d, \alpha, R, \Lambda, b)$ such that for all $Q \in \partial D, r \in (0, R]$ and any two nonnegative harmonic functions u and v with respect to $\tilde{\mathcal{L}}^b$ in $D \cap B(Q, r)$ and vanishes continuously on $D^c \cap B(Q, r)$, we have

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Thank you!