On purely discontinuous additive functionals of subordinate Brownian motions

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Outline

1. General result
2. Results for subordinate Brownian motion
3. Applications to absolute continuity/singularity

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$X = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, \mathbb{P}_x)$ a strong Markov process on $\mathbb{R}^d$,
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Note: If $\mathbb{P}_x(A_\infty < \infty) = 1$ for all $x \in \mathbb{R}^d$, then $\mathbb{P}_x(M_\infty > 0) = 1$ for all $x \in \mathbb{R}^d$, hence $u > 0$.

Properties of u

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Assume that $\mathbb{P}_x(A_\infty < \infty) = 1$ for all $x \in \mathbb{R}^d$. Then $\lim_{t\to\infty} u(X_t) = 1$ $\mathbb{P}_x$-a.s. for every $x \in \mathbb{R}^d$.

Properties of u, cont.

From now on we assume that $A_t = \sum_{s \le t} F(X_{s-}, X_S)$. Let $\widetilde{F}(x,y) = 1 - e^{-F(x,y)}$, and set $\widetilde{A}_t = \sum_{s \le t} \widetilde{F}(X_{s-}, X_s)$.

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X strongly Feller and $\lim_{t\to 0} \sup_x \mathbb{E}_x A_t = 0$ imply u is continuous.

Properties of u, cont.

Proposition 1: If $\mathbb{E}_x A_\infty \leq c < \infty$, then $u(x) \geq e^{-c} > 0$. Conversely, if $\mathbb{P}_x(A_\infty < \infty) = 1$ for all $x \in \mathbb{R}^d$ and $\inf_{x \in \mathbb{R}^d} u(x) = c > 0$, then $\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \widetilde{A}_\infty \leq 1/c - 1$.

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$$1 - u(x) = \mathbb{E}_x \int_0^\infty u(X_s)\, d\widetilde{A}_s \qquad .$$

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Note: If $\mathbb{P}_x(A_\infty < \infty) = 1$, then $u > 0$. Further, if X strongly Feller and $\lim_{t\to 0} \sup_x \mathbb{E}_x A_t = 0$, then u is continuous, hence bounded away from 0 on compacts. Finally, $\lim_{t\to\infty} u(X_t) = 1$.

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An application to perpetual integral functionals

Let $X = (X_t, \mathbb{P}_x)$ be a strong Markov process on $\mathbb{R}$, $\lim_{t\to\infty} X_t = +\infty$ and $\mathbb{P}_y(X_{T_x} = 1)$ for all $y < x$ where $T_x = \inf\{t > 0 : X_t \geq x\}$.

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Let $M_t = e^{-A_t}$, $M_t^x = e^{-A_t^x}$, $u(y) = \mathbb{E}_y M_\infty$ and $u^x(y) = \mathbb{E}_y M_\infty^x$.

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Then u and u^x are non-decreasing. Further, since $\mathbb{P}_y(M_{T_x}^x = 1) = 1$ for $y < x$, it follows that $\inf_{y\in\mathbb{R}} u^x(y) = u^x(x) > 0$.

An application to perpetual integral functionals, cont.

A simple application of Proposition 1 gives the following result originally proved in Khoshnevisan, Salminen, Yor (2006): The following are equivalent

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Note: Even when f is bounded, we cannot conclude that $\mathbb{E}_x A_\infty < \infty$ because of the lack of control of $u(x)$ as $x \to -\infty$.

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The Laplace exponent ϕ of S satisfies $\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}$. It is a Bernstein function, hence has a representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt)\,, \quad b \geq 0, \int_{(0,\infty)} (1 \wedge \lambda)\mu(dt) < \infty\,.$$

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Define $X = (X_t, \mathbb{P}_x)$ as $X_t := W_{S_t}$ – *subordinate BM*. It is a Lévy process with the characteristic exponent $\psi(\xi) = \phi(|\xi|^2)$, $\xi \in \mathbb{R}^d$:
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For $\phi(\lambda) = \lambda^{\alpha/2}$ we get $\psi(\xi) = |\xi|^\alpha$, so X is an isotropic α-stable process.

Weak scaling condition

We assume that the Laplace exponent ϕ is a complete Bernstein function and that it satisfies the following weak scaling condition: There exist $a_1, a_2 > 0$ and $0 < \delta_1 \le \delta_2 < 2 \wedge d$ such that

$$a_1 \left(\frac{R}{r}\right)^{\delta_1} \le \frac{\phi(R)}{\phi(r)} \le a_2 \left(\frac{R}{r}\right)^{\delta_2}, \quad 0 < r \le R < \infty\,.$$

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$$a_1 \left(\frac{R}{r}\right)^{\delta_1} \le \frac{\phi(R)}{\phi(r)} \le a_2 \left(\frac{R}{r}\right)^{\delta_2}, \quad 0 < r \le R < \infty .$$

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For $\phi(\lambda) = \lambda^{\alpha/2}$ we have exact scaling:

$$\frac{\phi(R)}{\phi(r)} = \left(\frac{R}{r}\right)^{\alpha} .$$

F-harmonic functions

Let $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be a symmetric function vanishing on the diagonal, and let $A_t^F := \sum_{s \le t} F(X_{s-}, X_s)$ be the corresponding AF.

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A non-negative function $h : \mathbb{R}^d \to [0, \infty)$ is F-harmonic in a bounded open set $D \subset \mathbb{R}^d$ with respect to X, if for every open $V \subset \overline{V} \subset D$,

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If $M_t^F = e^{-A_t^F}$ and $u(x) = \mathbb{E}_x[M_\infty^F]$, then it was shown that u is regular F-harmonic in every bounded open D.

Harnack inequality for F-harmonic functions

Theorem 2: Let $D \subset \mathbb{R}^d$ be a bounded open set and $K \subset D$ a compact subset of D. Fix $\beta > 1$ and $C > 0$. There exists a constant $c = c(d, a_1, a_2, \delta_1, \delta_2, \beta, C, D, K) > 1$ such that for every symmetric $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ vanishing on the diagonal and satisfying $F(x, y) \le C\left(\Phi(|x - y|)^\beta \wedge 1\right)$, and every $h : \mathbb{R}^d \to [0, \infty)$ which is F-harmonic with respect to X in D, it holds that

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The dependence of c on K and D is only through the ratio $(r_0 \wedge \mathrm{dist}(K, D^c))/\mathrm{diam}(K)$ where $r_0 = r_0(d, a_1, a_2, \delta_1, \delta_2, \beta)$ will be explained later.

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The HI will be used in case $D = V(0, 1, 2M + 1) := \{x \in \mathbb{R}^d : 1 < |x| < 2M + 1\}$ and $K = \overline{V}(0, 2, 2M) = \{x \in \mathbb{R}^d : 2 \leq |x| \leq 2M\}$, M some large number.

The key ingredient of the proof

The key ingredient in the proof of Theorem 2 is the following estimate: For every $\varepsilon > 0$, there exists $r_0 = r_0(d, a_1, a_2, \delta_1, \delta_2, \beta, \varepsilon)$ such that for every $r \in (0, r_0)$, all $x_0 \in \mathbb{R}^d$ and all $x, w \in B(x_0, r)$,

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By Jensen's inequality, this follows from $\mathbb{E}_x^w\left[A^F_{\tau_{B(x_0,r)}}\right] < \varepsilon$.
Set $B_r = B(x_0, r)$ and let G_{B_r} denote the Green function of the process X killed upon exiting the ball $B(x_0, r)$. It is well known that

$$\mathbb{E}_x^w\left[A^F_{\tau_{B_r}}\right] = \int_{B_r}\int_{B_r} \frac{G_{B_r}(x,y)G_{B_r}(z,w)}{G_{B_r}(x,w)} F(y,z)j(|y-z|)\,dz\,dy\,.$$

Here $j(|z|)$ is the density of the Lévy measure of X.

Key technical lemma

By using already existing two sided sharp estimates of G_{B_r} and j, we prove the following key technical lemma:

Lemma 3: Let $\beta > 1$ and $C > 0$. For every $\varepsilon > 0$ there exists a constant $r_0 = r_0(d, a_1, a_2, \delta_1, \delta_2, \beta, C, \varepsilon) \in (0, 1]$ such that for every $r \in (0, r_0]$ and symmetric $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ vanishing on the diagonal and satisfying $F(x, y) \le C\big(\Phi(|x-y|)^\beta \wedge 1\big)$, it holds that

$$\sup_{x,w \in B_r} \int_{B_r} \int_{B_r} \frac{G_{B_r}(x,y) G_{B_r}(z,w)}{G_{B_r}(x,w)} |F(y,z)| j(|y-z|)\, dz\, dy < \varepsilon.$$

Consequences of weak scaling

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Let $\Phi^R(s) = (\phi^R(s^{-2}))^{-1}$ and let X^R be the subordinate BM with the characteristic exponent $\psi^R(\xi) = \phi^R(|\xi|^2)$, $\xi \in \mathbb{R}^d$. Note that

$$(X_t^R)_{t\geq 0} \overset{\mathrm{D}}{=} (R^{-1}X_{t/\phi(R^{-2})})_{t\geq 0}.$$

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$$(X^R_t)_{t\ge 0} \stackrel{\mathrm{D}}{=} (R^{-1}X_{t/\phi(R^{-2})})_{t\ge 0}.$$

The notions related to the process X^R will have the superscript R. E.g.,

$$A^{R,F}_t = \sum_{s\le t} F(X^R_{s-}, X^R_s)\,.$$

F-harmonic functions and scaling

For $h : \mathbb{R}^d \to [0, \infty)$, $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, $D \subset \mathbb{R}^d$, and any $R > 0$, set

$$h_R(x) := h(Rx)\,, \quad F_R(x, y) := F(Rx, Ry)\,, \quad D_R := \{Rx : \ x \in D\}\,.$$

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Lemma 4: Let D be a bounded open set in $\mathbb{R}^d$, $R>0$, $\zeta := \tau_{D_R}$ and $\eta := \tau_D^R$. Assume that h is regular F-harmonic in D_R for X, i.e.

$$h(x) = \mathbb{E}_x\left[e^{-A^F_\zeta} h(X_\zeta)\right] \quad \text{for all } \ x\in D_R.$$

Then h_R is regular F_R-harmonic in D for X^R, i.e.

$$h_R(x) = \mathbb{E}_x\left[e^{-A^{R,F_R}_\eta} h_R(X^R_\eta)\right] \quad \text{for all } \ x\in D.$$

Condition on the function F

Let $F:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ be symmetric and bounded. Assume that there exist constants $C>0$ and $\beta>1$ such that

$$F(x,y)\le C\frac{\Phi(|x-y|)^\beta}{1+\Phi(|x|)^\beta+\Phi(|y|)^\beta}, \qquad \text{for all } \ x,y\in\mathbb{R}^d.$$

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For $R\ge 1$ and a bounded open set $D\subset B(0,1)^c$ let

$$\widehat{F}_R(x,y)=\begin{cases}F_R(x,y) & \text{if}\ \ (x,y)\in(D\times\mathbb{R}^d)\cup(\mathbb{R}^d\times D)\\ 0 & \text{otherwise.}\end{cases}$$

Then $\widehat{F}_R$ is symmetric, bounded and satisfies $\widehat{F}_R(x,y)\le C\Phi^R(|x-y|)^\beta$ for all $x,y\in\mathbb{R}^d$.

Hitting infinitely many annuli

For a Borel set $C \subset \mathbb{R}^d$ let $T_C = \inf\{t > 0 : X_t \in C\}$ be its hitting time. If $0 < a < b$, let $V(0, a, b) := \{x \in \mathbb{R}^d : a < |x| < b\}$ be the open annulus, and denote by $\overline{V}(0, a, b)$ its closure.

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Lemma 5: There exists a positive integer $M = M(d, \delta_1, a_1) \geq 2$ such that for every strictly increasing sequence of positive numbers $(R_n)_{n\geq 1}$ satisfying $\lim_{n\to\infty} R_n = \infty$ it holds that

$$\mathbb{P}_x\left(\limsup_{n\to\infty}\{T_{\overline{V}(0,R_n,MR_n)} < \infty\}\right) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

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That is, with $\mathbb{P}_x$ probability 1, the process X visits infinitely many of the sets V_n.

Main theorem

Theorem 6: Assume that $F:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ is symmetric, bounded and that there exist constants $C>0$ and $\beta>1$ such that

$$F(x,y)\le C\frac{\Phi(|x-y|)^\beta}{1+\Phi(|x|)^\beta+\Phi(|y|)^\beta}, \qquad \text{for all } \ x,y\in\mathbb{R}^d.$$

Let $A_t^F=\sum_{0<s\le t}F(X_{s-},X_s)$. If $\mathbb{P}_x(A_\infty^F<\infty)=1$ for all $x\in\mathbb{R}^d$, then $\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[A_\infty^F]<\infty$.

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Let $A_t^F = \sum_{0<s\le t} F(X_{s-}, X_s)$. If $\mathbb{P}_x(A_\infty^F < \infty) = 1$ for all $x \in \mathbb{R}^d$, then $\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_\infty^F] < \infty$.

Proof: Let $u(x) := \mathbb{E}_x M_\infty^F$. Since X is strongly Feller and $\lim_{t\to 0} \sup_x \mathbb{E}_x A_t = 0$, u is continuous. It suffices to prove that $\liminf_{|x|\to\infty} u(x) > 0$.

Proof of the main theorem, cont.

Let $D = V(0, 1, 2M+1)$. The function u is regular F-harmonic in D_R, $R \geq 1$. Then u_R is regular $\widehat{F}_R$-harmonic in D for X^R.

Proof of the main theorem, cont.

Let $D = V(0,1,2M+1)$. The function u is regular F-harmonic in D_R, $R \geq 1$. Then u_R is regular $\widehat{F}_R$-harmonic in D for X^R. Moreover, $\widehat{F}_R$ satisfies the upper bound from the Harnack inequality, hence

$$c^{-1} u_R(y) \leq u_R(x) \leq c u_R(y) \quad \text{for all} \quad x, y \in \overline{V}(0,2,2M)\,.$$

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Assume $\lim_{n\to\infty} u(x_n) = 0$ for a sequence $|x_n| \to \infty$. Then there exists an increasing sequence $(k_n)_{n\geq 1}$ such that $x_n \in V_n := \overline{V}(0, 2^{k_n}, 2^{k_n} M)$ for every $n \geq 1$.

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Proof of the main theorem, cont.

Therefore it follows from (1) that

$$c^{-1}u(X_{t_l}(\omega)) \le u(x_{n_l}) \le cu(X_{t_l}(\omega)),$$

which implies that $\lim_{l\to\infty} u(X_{t_l}(\omega)) = 0$. But this is a contradiction with $\lim\limits_{t\to\infty} u(X_t) = 1$ $\mathbb{P}_x$-a.s. Therefore, u is bounded away from zero. By Proposition 1, $\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A^F_\infty] < \infty$.

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The idea of the proof comes from the paper I. Ben-Ari and R. G. Pinsky, Absolute continuity/singularity and relative entropy properties for probability measures induced by diffusions on infinite time intervals, *Stochastic Process. Appl.* **115** (2005), 179–206, where it was used for a similar result for diffusions.

Two remarks

$$F(x,y) \leq C\frac{\Phi(|x-y|)^{\beta}}{1+\Phi(|x|)^{\beta}+\Phi(|y|)^{\beta}}\,, \qquad \text{for all} \quad x,y \in \mathbb{R}^d.$$

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There exists F satisfying the above condition such that $\mathbb{E}_x[A_\infty] = \infty$. Of course, in this case it cannot hold that $\mathbb{P}_x(A_\infty < \infty) = 1$.

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On the other hand, this condition is almost necessary for the validity of the main theorem.

Theorem 7: For all γ and β satisfying $0<\gamma<1<\beta$, there exists a symmetric F such that

$$0 \le F(x,y) \le \frac{\Phi(|x-y|)^\beta}{1+\Phi(|x|)^\gamma+\Phi(|y|)^\gamma}, \qquad \text{for all } x,y \in \mathbb{R}^d,$$

$\mathbb{P}_x(A^F_\infty<\infty)=1$ for all $x\in\mathbb{R}^d$, but $\mathbb{E}_x[A^F_\infty]=\infty$.

Lévy system

$X = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, \mathbb{P}_x)$ a symmetric (wrt Lebesgue measure) right Markov process on $\mathbb{R}^d$, $\Omega = D([0,\infty), \mathbb{R}^d)$, $X_t = \omega(t)$, $\mathcal{M} = \sigma(\cup_{t\geq 0}\mathcal{M}_t)$.

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$X = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, \mathbb{P}_x)$ a symmetric (wrt Lebesgue measure) right Markov process on $\mathbb{R}^d$, $\Omega = D([0,\infty), \mathbb{R}^d)$, $X_t = \omega(t)$, $\mathcal{M} = \sigma(\cup_{t\geq 0}\mathcal{M}_t)$.
(N, H) a Lévy system of X: For a Borel function F on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal

$$\mathbb{E}_x \sum_{s\leq t} F(X_{s-}, X_s) = \mathbb{E}_x \int_0^t \int_{\mathbb{R}^d} F(X_{s-}, y) N(X_{s-}, dy)\, dH_s$$

Two families of functions

Two families of functions (Chen, Song, PTRF 2003, Song JTP 2006)
$J(X)$: bounded, symmetric $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

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$J(X) \subset I_2(X)$; if $\inf_{x,y} F(x, y) > -1$ and $F \in I_2(X)$, then
$\log(1 + F) \in I_2(X)$.

MAF and quadratic variations

If $F \in J(X)$, then

$$A_t^F := \sum_{s \le t} F(X_{s-}, X_s) - \int_0^t \int_{\mathbb{R}^d} F(X_{s-}, y) N(X_{s-}, dy)\, dH_s$$

is well-defined (pure jump) martingale additive functional with quadratic and predictable quadratic variation

$$\begin{aligned} [A^F]_t &= \sum_{s \le t} (\Delta A_s^F)^2 = \sum_{s \le t} F^2(X_{s-}, X_s) \\ \langle A^F \rangle_t &= \int_0^t \int_{\mathbb{R}^d} F^2(X_{s-}, y) N(X_{s-}, dy)\, dH_s \end{aligned}$$

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If $F \in I_2(X)$, one defines A_t^F by the limiting procedure in $L^2(\mathbb{P}_x)$ with the same formulae for $[A^F]_t$ and $\langle A^F \rangle_t$.

Purely discontinuous Girsanov transform

Assume $\inf_{x,y} F(x,y) > -1$ and let $L_t^F := \mathcal{E}(A^F)_t > 0$ be the Doleans-Dade exponential of A^F:

$$\begin{aligned}\mathcal{E}(A^F)_t &= \exp(A_t^F) \prod_{s \le t} (1 + F(X_{s-}, X_s)) \exp(-F(X_{s-}, X_s)) \\ &= \exp\left(A_t^F + \sum_{s \le t} (\log(1+F) - F)(X_{s-}, X_s)\right)\end{aligned}$$

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Under $\widetilde{\mathbb{P}}_x$ X is again a right Markov process with a Lévy system $((1+F)N, H)$. Notation: $\widetilde{X} = (\widetilde{X}_t, \mathcal{M}, \mathcal{M}_t, \widetilde{\mathbb{P}}_x)$ – purely discontinuous Girsanov transform of X.

Relative entropy of probability measures

Recall that the relative entropy of two probability measures μ and ν is defined by

$$\mathcal{H}(\nu;\mu) := \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu}\, d\mu = \int \log \frac{d\nu}{d\mu}\, d\nu \le \infty$$

if $\nu \ll \mu$ and $+\infty$ otherwise.

Theorem 8: X conservative symmetric right Markov process on $\mathbb{R}^d$, $F \in I_2(X)$ and $\inf_{x,y} F(x,y) > -1$.

(a) $\mathbb{P}_x \perp \widetilde{\mathbb{P}}_x$ iff $[A^F]_\infty = \sum_{t>0} F^2(X_{t-}, X_t) = \infty$ $\mathbb{P}_x$ a.s. or $\widetilde{\mathbb{P}}_x$ a.s.

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(c1) $\mathcal{H}(\widetilde{\mathbb{P}}_x; \mathbb{P}_x) = \widetilde{\mathbb{E}}_x \sum_{t>0} (\log(1+F) - \frac{F}{1+F})(X_{t-}, X_t)$ and is finite iff $\widetilde{\mathbb{E}}_x [A^F]_\infty < \infty$.

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Corollary 9: Assume X has strictly positive transition densities under $\mathbb{P}_x$ for all $x \in \mathbb{R}^d$. If $\mathbb{P}_x \ll \widetilde{\mathbb{P}}_x$ (respectively $\mathbb{P}_x \perp \widetilde{\mathbb{P}}_x$) for some $x \in \mathbb{R}^d$, then this is true for all $x \in \mathbb{R}^d$. Analogously if densities exist under $\widetilde{\mathbb{P}}_x$.

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Corollary 10: If the invariant σ-field $\mathcal{I} := \{\Lambda \in \mathcal{M} : \theta_t^{-1}\Lambda = \Lambda \ \ \forall t \geq 0\}$ is trivial under both $\mathbb{P}_x$ and $\widetilde{\mathbb{P}}_x$ then either $\mathbb{P}_x \perp \widetilde{\mathbb{P}}_x$ or $\mathbb{P}_x \sim \widetilde{\mathbb{P}}_x$.

Theorem 11: Suppose that X is the subordinate Brownian motion via the subordinator whose Laplace exponent is a complete Bernstein function and satisfies the weak scaling condition.

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$$0 \le F(x,y) \le C\frac{\Phi(|x-y|)^\beta}{1+\Phi(|x|)^\beta+\Phi(|y|)^\beta}\,, \quad \text{for all } x,y\in\mathbb{R}^d\,,$$

then $\mathcal{H}(\mathbb{P}_x;\widetilde{\mathbb{P}}_x) < \infty$.

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(b) For each γ and β satisfying $0 < \gamma < 1/2 < \beta$ there exists $F \in I_2(X)$ satisfying

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such that $\mathbb{P}_x \ll \widetilde{\mathbb{P}}_x$ and $\mathcal{H}(\mathbb{P}_x; \widetilde{\mathbb{P}}_x) = \infty$.