Spine decomposition and $L \log L$ criterion for superprocesses with non-local branching mechanisms

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Outline

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The $L\log L$ criterion for Galton-Watson processes

$\{Z_n, n \geq 1\}$: a Galton-Watson process.
L: the number of offspring of an arbitrary individual.
$\{p_n, n \geq 1\}$: the distribution of L.

Set

$$m := \sum_{n=1}^{\infty} np_n.$$

m is the mean number of children given by one particle.
Suppose $m > 1$ (supercritical).

It is known that $EZ_n = m^n$ and $\left\{\frac{Z_n}{m^n}; n \geq 1\right\}$ is a martingale and thus

$$\lim_{n\to\infty} \frac{Z_n}{m^n} = W < \infty.$$

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Question: When is W nondegenerate? or equivalently, when does m^n gives the right growth rate of Z_n?

In 1966, Kesten and Stigum proved that W is nondegenerate if and only if

$$(L \log L) \qquad E(L \log^+ L)) = \sum_{n=1}^{\infty} p_n (n \log n) < \infty. \qquad (1)$$

Moreover, if (1) is satisfied,

$$W_t \to W \quad \text{a.s. and in } L^1$$

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In 1995, Lyons, Pemantle and Peres used a martingale change of measure method to give a probabilistic proof of the $L \log L$ criterion of Kesten and Stigum. The main technique is **a spine decomposition** under a martingale change of measure.

Later this method were extended to multitype branching processes (see Kurtz-Lyons-Pemantle- Peres(1997); Lyons(1997); Biggins-Kyprianou (2004)).

This technique was also used to study properties for branching random walks. See, for example, Hu-Shi(2009); Aidekon-Shi(2011, 2014); Faraud-Hu-Shi(2011, 2012); Gantert-Hu-Shi(2014).

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Recently, two papers discussed spine decomposition and $L \log L$ criterion for superprocesses with **nonlocal** branching mechanisms. :

1) Chen, R. and Song (2017+): for supercritical **multitype superdiffusions**
2) Kyprianou and Palau (2017+); Kyprianou, Palau and R. (2017+): for **super Markov chains**.

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Outline

Superprocesses

E: a locally compact separable metric space.
m: a σ-finite measure on $(E, \mathcal{B}(E))$ with full support.

The superprocess $X = \{X_t : t \geq 0\}$ is determined by two objects:

(i) a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E, which is an m-symmetric Hunt process on E.

(ii) a non-local branching mechanism ψ given by

$$\psi(x, f) = \phi^L(x, f(x)) + \phi^{NL}(x, f) \quad \text{for } x \in E,\ f \in \mathcal{B}_b^+(E). \tag{2}$$

Here ϕ^L is called the **local branching mechanism** given by

$$\phi^L(x, \lambda) = a(x)\lambda + b(x)\lambda^2 + \int_{(0,+\infty)} \left(e^{-\lambda\theta} - 1 + \lambda\theta\right) \Pi^L(x, d\theta)$$

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for $x \in E$, $\lambda \geq 0$, where $a(x) \in \mathcal{B}_b(E)$, $b(x) \in \mathcal{B}_b^+(E)$ and

$(\theta \wedge \theta^2)\Pi^L(x, d\theta)$ is a bounded kernel from E to $(0, +\infty)$.

ϕ^{NL} in (2) is called the **non-local branching mechanism** given by

$$\phi^{NL}(x, f) = -c(x)\pi(x, f) - \int_{(0,+\infty)} \left(1 - e^{-\theta\pi(x,f)}\right) \Pi^{NL}(x, d\theta),$$

where $c(x) \in \mathcal{B}^+(E)$, $\pi(x, dy)$ is a probability kernel on E with $\pi(x, \{x\}) = 0$ and $\theta\Pi^{NL}(x, d\theta)$ is a bounded kernel from E to $(0, +\infty)$.

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Superprocesses

To be more specific, X is an $\mathcal{M}(E)$-valued Markov process such that for every $f \in \mathcal{B}_b^+(E)$ and every $\mu \in \mathcal{M}(E)$,

$$\mathrm{P}_\mu\left(e^{-\langle f, X_t\rangle}\right) = e^{-\langle u_f(\cdot,t),\mu\rangle} \quad \text{for } t \geq 0, \tag{3}$$

where $u_f(x,t) := -\log \mathrm{P}_{\delta_x}\left(e^{-\langle f, X_t\rangle}\right)$ is the unique non-negative locally bounded solution to the integral equation

$$\begin{aligned} u_f(x,t) &= P_t f(x) - \Pi_x\left[\int_0^t \psi(\xi_s, u_f^{t-s})ds\right] \\ &= P_t f(x) - \Pi_x\left[\int_0^t \phi^L(\xi_s, u_f(t-s,\xi_s))ds\right] \\ &\quad -\Pi_x\left[\int_0^t \phi^{NL}(\xi_s, u_f^{t-s})ds\right]. \end{aligned} \tag{4}$$

Superprocesses

The (P_t, ϕ^L, ϕ^{NL})-superprocess can be realized as right process in $\mathcal{M}(E)$.

$\mathcal{W}_0^+$: the space of right continuous paths from $[0, +\infty)$ to $\mathcal{M}(E)$ having the zero measure as a trap.

We assume X is the coordinate process in $\mathcal{W}_0^+$.
$(\mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0})$ is the natural filtration on $\mathcal{W}_0^+$ generated by the coordinate process.

We assume (1)

$$A := \{x \in E : \phi^{NL}(x, 1) > 0\} \neq \emptyset.$$

(2) ξ admits a transition density $p(t, x, y)$ with respect to the measure m,

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First moment

We define for $x \in E$,

$$\gamma(x) := c(x) + \int_{(0,+\infty)} \theta \Pi^{NL}(x, d\theta) \quad \text{and} \quad \gamma(x, dy) := \gamma(x)\pi(x, dy). \tag{5}$$

Proposition 1 For every $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_b(E)$,

$$\mathrm{P}_\mu \left(\langle f, X_t \rangle \right) = \langle \mathfrak{P}_t f, \mu \rangle,$$

where $\mathfrak{P}_t f(x)$ is the unique locally bounded solution to the following integral equation:

$$\mathfrak{P}_t f(x) = P_t f(x) - \Pi_x \left[\int_0^t a(\xi_s) \mathfrak{P}_{t-s} f(\xi_s) ds \right] + \Pi_x \left[\int_0^t \gamma(\xi_s, \mathfrak{P}_{t-s} f) ds \right]. \tag{6}$$

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Assumption 0. For any $x \in E$ such that $b(x) > 0$, there is a measure $\mathbb{N}_x$ such that for all $t > 0$ and $f \in \mathcal{B}_b^+(E)$,

$$\mathbb{N}_x \left(1 - e^{-\langle f, X_t \rangle}\right) = -\log \mathrm{P}_{\delta_x} \left(e^{-\langle f, X_t \rangle}\right). \tag{7}$$

This measure $\mathbb{N}_x$ is called the *Kuznetsov measure* or the *excursion law* for the (P_t, ϕ^L, ϕ^{NL})-superprocess.

Assumption 1. $\int_E \pi(x, \cdot) m(dx) \in \mathbf{K}(\xi)$.

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of ξ. Define a bilinear form $(\mathcal{Q}, \mathcal{F})$ by

$$\mathcal{Q}(u, v) := \mathcal{E}(u, v) + \int_E a(x) u(x) v(x) m(dx) - \int_E \int_E u(y) v(x) \gamma(x, dy) m(dx)$$

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for $\forall u, v \in \mathcal{F}$.

Under Assumption 1, for the closed form $(\mathcal{Q}, \mathcal{F})$ on $L^2(E, m)$, there are unique strongly continuous semigroups $\{T_t : t \geq 0\}$ and $\{\widehat{T}_t : t \geq 0\}$ on $L^2(E, m)$ such that $\|T_t\|_{L^2(E,m)} \leq e^{\beta_0 t}$, $\|\widehat{T}_t\|_{L^2(E,m)} \leq e^{\beta_0 t}$ ($\beta_0 > 0$ is a constant), and

$$(T_t f, g) = (f, \widehat{T}_t g) \quad \forall f, g \in L^2(E, m). \tag{8}$$

Assumption 2. There exist $\lambda_1 \in (-\infty, +\infty)$ and positive functions $h, \widehat{h} \in \mathcal{F}$ with h being bounded continuous, $\|h\|_{L^2(E,m)} = 1$ and $(h, \widehat{h}) = 1$ such that

$$\mathcal{Q}(h, v) = \lambda_1 (h, v), \quad \mathcal{Q}(v, \widehat{h}) = \lambda_1 (v, \widehat{h}) \quad \forall v \in \mathcal{F}. \tag{9}$$

This equation implies that

$$T_t h = e^{-\lambda_1 t} h \quad \text{and} \quad \widehat{T}_t \widehat{h} = e^{-\lambda_1 t} \widehat{h} \quad \text{in } L^2(E, m).$$

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Theorem 1 For every $\mu \in \mathcal{M}(E)$, $W_t^h(X) := e^{\lambda_1 t}\langle h, X_t\rangle$ is a non-negative P_μ-martingale with respect to $\{\mathcal{F}_t : t \geq 0\}$.

Question 1 Let $W_\infty^h(X)$ be the limit of $W_t^h(X)$. When $W_\infty^h(X)$ is non-degenerate?

We can define a new probability measure Q_μ for every $\mu \in \mathcal{M}(E)^0$ by the following formula:

$$d\mathrm{Q}_\mu|_{\mathcal{F}_t} := \frac{1}{\langle h, \mu\rangle} W_t^h(X) d\mathrm{P}_\mu\Big|_{\mathcal{F}_t} \qquad \text{for all } t \geq 0.$$

To answer Question 1, our first step is to establish a **spine decomposition** of X under Q_μ.

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Outline

The spine: Concatenation process

Put

$$q(x) := \gamma(x,h)/h(x) \quad \text{for } x \in E. \tag{10}$$

Define

$$H_t = \exp\left(\lambda_1 t - \int_0^t a(\xi_s)ds + \int_0^t q(\xi_s)ds\right)\frac{h(\xi_t)}{h(\xi_0)}.$$

$\{H_t, t \geq 0\}$ is a positive local martingale and thus a supermartingale.

Step 1 Define

$$d\Pi_x^h = H_t \, d\Pi_x \quad \text{on } \mathcal{H}_t \cap \{t < \zeta\} \quad \text{for } x \in E,$$

The process ξ under $\{\Pi_x^h, x \in E\}$ will be denoted as ξ^h, which is a conservative and recurrent $\widetilde{m}$-symmetric right Markov process on E with $\widetilde{m}(dy) := h(y)^2 m(dy)$.

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Concatenation process

Step 2 For every $x \in E$, there is a unique (up to equivalence in law) right process $((\widehat{\xi}_t)_{t\geq 0}; \widehat{\Pi}^h_x)$ on E with lifetime $\widehat{\zeta}$ such that

$$\widehat{\Pi}^h_x\left(\widehat{\xi}_t \in B\right) = \Pi^h_x\left[e_q(t); \xi^h_t \in B\right] \quad \forall B \in \mathcal{B}(E).$$

$\widehat{\xi}$ is called the $e_q(t)$-subprocess of ξ^h. Here

$$e_q(t) := \exp\left(-\int_0^t q(\xi_s)ds\right) \quad \forall t \geq 0,$$

Now we define

$$\pi^h(x, dy) := \frac{h(y)\pi(x, dy)}{\pi(x, h)} \quad \text{for } x \in E. \tag{11}$$

Obviously, $\pi^h(x, dy)$ is a probability kernel on E.

Step 3 Let $\widetilde{\xi} := (\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}_t, \widetilde{\theta}_t, \widetilde{\xi}_t, \widetilde{\Pi}_x, \widetilde{\zeta})$ be the right process constructed from $\widehat{\xi}$ and $\kappa(\omega, dy) := \pi^h(\widehat{\xi}_{\widehat{\zeta}(\omega)-}(\omega), dy)$:

The process $\widetilde{\xi}$ evolves as the process ξ^h until time $\widehat{\zeta}$, it is then revived by means of the kernel $\kappa(\omega, dy)$ and evolves again as ξ^h and so on...

We will call $\widetilde{\xi}$ a *concatenation* process (cf. Ikeda, Nagasawa and Watanabe (1966)).

$\widetilde{\xi}$ serves as the spine in the decomposition described below.

Now we define

$$\pi^h(x, dy) := \frac{h(y)\pi(x, dy)}{\pi(x, h)} \quad \text{for } x \in E. \tag{11}$$

Obviously, $\pi^h(x, dy)$ is a probability kernel on E.

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Spine decomposition

Let $\widetilde{P}_t$ be the transition semigroup of $\widetilde{\xi}$.

Proposition 2 For every $f \in \mathcal{B}_b^+(E)$, $t \geq 0$ and $x \in E$,

$$\widetilde{P}_t f(x) = \frac{e^{\lambda_1 t}}{h(x)} \mathfrak{P}_t(fh)(x). \tag{12}$$

Moreover, for each $t > 0$ and $x \in E$, $\widetilde{\xi}$ has a transition density $\widetilde{p}(t,x,y)$ with respect to the probability measure

$$\rho(dy) := h(y)\widehat{h}(y)m(dy). \tag{13}$$

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Spine decomposition

For every $\mu \in \mathcal{M}(E)$ and $x \in E$, there is a probability space with probability measure $\mathbb{P}_{\mu,x}$ that carries the following processes.

(i) $((\widetilde{\xi}_t)_{t\geq 0}; \mathbb{P}_{\mu,x})$ is equal in law to $\widetilde{\xi}$, a copy of the concatenation process starting from x;

(ii) $(n; \mathbb{P}_{\mu,x})$ is a random measure such that, given $\widetilde{\xi}$ starting from x, n is a Poisson random measure which issues $\mathcal{M}(E)$-valued processes $X^{n,t} := (X^{n,t}_s)_{s\geq 0}$ at space-time points $(\widetilde{\xi}_t, t)$ with rate

$$d\mathbb{N}_{\widetilde{\xi}_t} \times 2b(\widetilde{\xi}_t)dt.$$

Let D^n denote the almost surely countable set of immigration times. Given $\widetilde{\xi}$, the processes $\{X^{n,t} : t \in D^n\}$ are mutually independent.

Spine decomposition

(iii) $(m; \mathbb{P}_{\mu,x})$ is a random measure such that, given $\widetilde{\xi}$ starting from x, m is a Poisson random measure which issues $\mathcal{M}(E)$-valued processes $X^{m,t} := (X^{m,t}_s)_{s\geq 0}$ at space-time points $(\widetilde{\xi}_t, t)$ with initial mass θ at rate

$$\theta \Pi^L(\widetilde{\xi}_t, d\theta) \times d\mathrm{P}_{\theta\delta_{\widetilde{\xi}_t}} \times dt.$$

Let D^m denote the almost surely countable set of immigration times. Given $\widetilde{\xi}$, the processes $\{X^{m,t} : t \in D^m\}$ are mutually independent, also independent of n and $\{X^{n,t} : t \in D^n\}$.

Spine decomposition

(iv) $\{((X^{r,i}_s)_{s\geq 0}; \mathbb{P}_{\mu,x}),\ i \geq 1\}$ is a family of $\mathcal{M}(E)$-valued processes such that, given $\widetilde{\xi}$ starting from x **(including its revival times** $\{\tau_i : i \geq 1\}$**)**, $X^{r,i} := (X^{r,i}_s)_{s\geq 0}$ is equal in law to $((X_s)_{s\geq 0}, \mathrm{P}_{\pi_i})$ where P_{π_i} denotes the law of the (P_t, ϕ^L, ϕ^{NL})-superprocess starting from $\Theta_i \pi(\widetilde{\xi}_{\tau_i-}, \cdot)$ and Θ_i is a $[0, +\infty)$-valued random variable with distribution $\eta(\widetilde{\xi}_{\tau_i-}, d\theta)$ given by

$$\eta(x, d\theta) := \left(\frac{c(x)}{\gamma(x)} 1_A(x) + 1_{E\setminus A}(x)\right)\delta_0(d\theta) + \frac{1}{\gamma(x)} 1_A(x) 1_{(0,+\infty)}(\theta)\theta\Pi^{NL}(x, d\theta). \tag{14}$$

Moreover, given $\widetilde{\xi}$ starting from x (including $\{\tau_i : i \geq 1\}$), $\{\Theta_i : i \geq 1\}$ are mutually independent, $\{X^{r,i} : i \geq 1\}$ are mutually independent, also independent of $\{X^{n,t} : t \in D^n\}$ and $\{X^{m,t} : t \in D^m\}$.

Recall that

$$A := \{x \in E : \ \psi^{NL}(x, 1) > 0\}.$$

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(v) $((X_t)_{t\geq 0}; \mathbb{P}_{\mu,x})$ is equal in law to $((X_t)_{t\geq 0}; \mathrm{P}_{\mu})$, a copy of the (P_t, ϕ^L, ϕ^{NL})-superprocess starting from μ. Moreover $((X_t)_{t\geq 0}; \mathbb{P}_{\mu,x})$ is independent of $\widetilde{\xi}$, n, m and all the immigration processes.

We denote by

$$I_t^c := \sum_{s\in D_t^n} X_{t-s}^{n,s}, \quad I_t^d := \sum_{s\in D_t^m} X_{t-s}^{m,s} \quad \text{and} \quad I_t^r := \sum_{\tau_i\leq t} X_{t-\tau_i}^{r,i}$$

the continuous immigration, the discontinuous immigration and the revival-caused immigration, respectively. We define Γ_t by

$$\Gamma_t := X_t + I_t^c + I_t^d + I_t^r, \qquad \forall t\geq 0.$$

The process $\widetilde{\xi}$ is called the *spine* process, and the process $I_t := I_t^c + I_t^d + I_t^r$ is called the *immigration* process.

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(v) $((X_t)_{t\geq 0}; \mathbb{P}_{\mu,x})$ is equal in law to $((X_t)_{t\geq 0}; \mathrm{P}_\mu)$, a copy of the (P_t, ϕ^L, ϕ^{NL})-superprocess starting from μ. Moreover $((X_t)_{t\geq 0}; \mathbb{P}_{\mu,x})$ is independent of $\widetilde{\xi}$, n, m and all the immigration processes.

We denote by

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Spine decomposition

For any $\mu \in \mathcal{M}(E)$ and any measure ν on $(E, \mathcal{B}(E))$ with $0 < \langle h, \nu \rangle < +\infty$, we randomize the law $\mathbb{P}_{\mu,x}$ by replacing the deterministic choice of x with an E-valued random variable having distribution $h(x)\nu(dx)/\langle h, \nu \rangle$. We denote the resulting law by $\mathbb{P}_{\mu,\nu}$. That is to say,

$$\mathbb{P}_{\mu,\nu}(\cdot) := \frac{1}{\langle h, \nu \rangle} \int_E \mathbb{P}_{\mu,x}(\cdot) h(x) \nu(dx).$$

Clearly $\mathbb{P}_{\mu,\delta_x} = \mathbb{P}_{\mu,x}$.

For simplicity we also write $\mathbb{P}_\mu$ for $\mathbb{P}_{\mu,\mu}$.

Spine decomposition

Theorem 2 Suppose that Assumptions 0-2 hold. For every $\mu \in \mathcal{M}(E)^0$, the process $((\Gamma_t)_{t\geq 0}; \mathbb{P}_\mu)$ is Markovian and has the same law as $((X_t)_{t\geq 0}; \mathrm{Q}_\mu)$.

Remark (1) In the case of purely local branching mechanism, the revival-caused immigration does not occur. To be more specific, in that case the spine runs as a copy of the h-transformed process ξ^h while only continuous and discontinuous immigration occur along the spine. **The concatenating procedure and the revival-caused immigration are consequences of non-local branching.**

(2) Similar phenomenon has been observed in Kyprianou and Palau (2017+) for multitype continuous-state branching processes and in Chen, R. and Song (2017+) for multitype superdiffusions.

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Outline

$L \log L$ criterion

Assumption 3.

(i) $a(x), \gamma(x) \in L^2(E, m)$.

(ii) $x \mapsto \pi(x, h)/h$ is bounded from above on A.

Assumption 4.

$$\lim_{t \to +\infty} \sup_{x \in E} \mathrm{essup}_{y \in E} |\widetilde{p}(t, x, y) - 1| = 0.$$

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$L \log L$ criterion

Theorem 3 Suppose that Assumptions 0-4 hold and $\lambda_1 < 0$ (supercritical case). Then as $t \to \infty$,

$$W_t^h(X) \to W_\infty^h(X) \quad \text{in } L^1(\mathrm{P}_\mu)$$

if and only if

$$\left(\int_{(0,+\infty)} rh(\cdot) \log^+(rh(\cdot))\Pi^L(\cdot, dr), \widehat{h}\right) < \infty$$

and

$$\left(\int_{(0,+\infty)} r\pi(\cdot, h) \log^+(r\pi(\cdot, h))\Pi^{NL}(\cdot, dr), 1_A\widehat{h}\right) < \infty,$$

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Thank you!