Directed Polymer in Random Environment with Correlation

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13th Workshop on Markov Processes and Related Topics —17-21, July, 2017

Contents

Basic notations

- $S = (S_k), k \geq 0$: a nearest-neighbor path starting from the origin in $\mathbb{Z}^d$
- $\omega = \{\omega(i,x), (i,x) \in \mathbb{N} \times \mathbb{Z}^d\}$: a family of real-valued random variables appearing as the environment
- $H_n^\omega(S) = \sum_{i=1}^n \omega(i, S_i)$: the n-step energy of a path S for a fixed environment ω
  $\mathrm{\mathbb{P}_n^\omega(S) = \frac{1}{Z_n^\omega(\beta)} e^{\beta H_n^\omega(S)} \mathbb{P}(S)}$: the random polymer measure
- partition function:

$$\mathrm{Z_n^\omega(\beta) = \sum_S e^{\beta H_n^\omega(S)} \mathbb{P}(S) = \mathbb{E}_{\mathbb{P}} e^{\beta H_n^\omega(S)}} \tag{1}$$

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Basic Results

- $W_n = Z_n^\omega(\beta)/\mathbb{E}(Z_n^\omega(\beta))$ is a supermartingale,
- $W_\infty = 0$: termed as strong disorder, $\beta > \beta_c$
- $W_\infty > 0$: termed as weak disorder, $\beta < \beta_c$
- $\beta_c = 0$, for $d = 1, 2$
- $0 < \beta_c < \infty$, for $d \geq 3$
- For more details see

  [1] Hubert Lacoin(2010). New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. *Comm. Math. Phys.*, 294(2):471-503.

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Recent results

An intermediate disorder regime: proposed in

[2]T. Alberts, K. Khanin and J. Quastel, THE INTERMEDIATE DISORDER REGIME FOR DIRECTED POLYMERS IN DIMENSION $1+1$, *The Annals of Probability* 2014, Vol. 42, No. 3, 1212-1256

[3]F. Caravenna, R. Sun, N. Zygouras, Polynomial chaos and scaling limits of disordered systems, *J. Eur. Math. Soc.* 19, 1-65,2017

Recent results

It says in [2]

Convergence of partition functions

- Under scaling $\beta_n = \beta n^{-\frac{1}{4}}$, the partition function

$$e^{-n\lambda(\beta n^{-\frac{1}{4}})} Z_n^{\omega}(\beta n^{-\frac{1}{4}}) \xrightarrow{D} Z_{\sqrt{2}\beta}.$$

$\lambda(\cdot)$: log Laplace of the environment variables.

$Z_{\sqrt{2}\beta}$: has explicit Wiener chaos decomposition.

Basic ideas

Using

$$e^x \approx 1 + x,$$

they have

modified function

$$\mathfrak{Z}_n^{\omega}(\beta n^{-\frac{1}{4}}) = \mathbb{E}_{\mathbb{P}}\left[\prod_{i=1}^{n}(1 + \beta n^{-\frac{1}{4}}\omega(i, S_i))\right]. \qquad (2)$$

Then expanding it as,

Basic ideas

Expanding modified function

$$\begin{aligned} &\mathfrak{Z}_n^{\omega}(\beta n^{-\frac{1}{4}}) \\ =&\mathbb{E}_{\mathbb{P}}\left[1+\sum_{k=1}^{n}\beta^k n^{-\frac{k}{4}}\sum_{\mathbf{i}\in D_k^n}\prod_{j=1}^{k}\omega(i_j,S_{i_j})\right] \\ =&1+\sum_{k=1}^{n}\beta^k n^{-\frac{k}{4}}\sum_{\mathbf{i}\in D_k^n}\sum_{\mathbf{x}\in\mathbb{Z}^k}\omega(\mathbf{i},\mathbf{x})p_k(\mathbf{i},\mathbf{x}). \end{aligned} \tag{3}$$

where

$$D_k^n=\{\mathbf{i}=(i_1,i_2,\ldots,i_k)\in[n]^k:1\le i_1<i_2<\cdots<i_k\le n\}.$$

Basic ideas

U-statistics

Let $\mathcal{S}_k^n(p_k^n) = \sum_{\mathbf{i}\in D_k^n} \sum_{\mathbf{x}\in\mathbb{Z}^k} \omega(\mathbf{i},\mathbf{x}) p_k(\mathbf{i},\mathbf{x})$. Then, in [2]

$$\begin{aligned} &\mathfrak{Z}_n^{\omega}(\beta n^{-\frac{1}{4}}) \\ =&1 + \sum_{k=1}^{n} \beta^k n^{-\frac{3k}{4}} \mathcal{S}_k^n(n^{-\frac{k}{2}} p_k^n) \\ \xrightarrow{D}& Z_{2\beta} = \sum_k I_k(p_k). \end{aligned} \tag{4}$$

p_k^n, p_k: k-order transition probability of random walk and density of Brownian motion, respectively.

I_k: k-multiple Wiener-Ito integral with respect to space-time white noise.

Basic ideas

Four-parameter fields

Let $\mathfrak{Z}^{\omega}(m, y; k, x; \beta) = \mathbb{P}[\Pi_{i=m+1}^{k}(1 + \beta\omega(i, S_i))1_{S_k=x}|S_m = y]$. Similarly, we have $Z^{\omega}(m, y; k, x; \beta), \mathfrak{Z}^{\omega}(k, x; \beta), Z^{\omega}(k, x; \beta)$.

Theorem (AKQ 2014)

Assuming that the ω have six moments with mean zero and variance one, the fields for $0 \leq s < t \leq 1, x, y \in R$

$$(s, y; t, x) \longrightarrow \frac{\sqrt{n}}{2}\mathfrak{Z}^{\omega}(ns, y\sqrt{n}; nt, x\sqrt{n}; \beta n^{-1/4})$$

converge weakly as $n \longrightarrow \infty$ to a random field $Z_{\sqrt{2}\beta}(s, y; t, x)$.

Basic ideas

Four-parameter fields

Let $\mathfrak{Z}^\omega(m, y; k, x; \beta) = \mathbb{P}[\Pi_{i=m+1}^k (1 + \beta\omega(i, S_i))1_{S_k=x}|S_m = y]$.

Theorem (continuued, AKQ 2014)

$Z_{\sqrt{2}\beta}(s, y; t, x)$ satisfies the following stochastic heat equation driven by white noise

$$\frac{\partial u(s, y; t, x)}{\partial t} = \frac{1}{2}\Delta u(s, y; t, x) + \sqrt{2}\beta u(s, y; t, x)\dot{W}(t, x) \qquad (5)$$

$$u(s, y; s, y) = \delta(t - s, x - y) \qquad (6)$$

Correlated environment

Environment

$\{\xi_{i,j} : i \in \mathbb{N}, j \in \mathbb{Z}\}$: i.i.d. with $\mathbb{E}_{\mathbb{Q}}\xi_{i,j} = 0$ and $\mathbb{E}_{\mathbb{Q}}\xi_{i,j}^2 = 1$ for any i, j.

$\omega = \{\omega(n, x) : n \geq 0, x \in \mathbb{Z}\}$: a stationary field by

$$\omega(n, x) = \sum_{y=-\infty}^{\infty} \psi_{y-x}\xi_{n,y}, \tag{7}$$

with $\psi_j \sim \delta|j|^{-\alpha}$ and $1/2 < \alpha < 1$. Then, one has

$$\mathbb{E}(\omega(i, x)\omega(j, y)) = \delta_{ij}\gamma(x - y),$$

where δ_{ij} is Kronecker and $\gamma(k) \sim \lambda|k|^{1-2\alpha}$ for large integer k and $\lambda = \delta^2 \frac{\Gamma(2\alpha-1)\Gamma(1-\alpha)}{\Gamma(\alpha)}$.

Correlated environment

Spectral measure of γ

Let $G(d\eta)$ be the spectral measure of the correlation function γ, i.e.,

$$\gamma(k) = \int_{-\pi}^{\pi} e^{\imath k\eta} G(d\eta), \quad \forall k \in \mathbb{Z}. \tag{8}$$

For every $N \in \mathbb{N}$, we define a new measure G_N by

$$G_N(A) = N^{\alpha-1/2} G(N^{-1/2}A), \qquad A \in \mathcal{B}(\mathbb{R}).$$

Then, $\lim_{N\to\infty} G_N = G_0$ (locally finite measure). Furthermore, G_0 has a spectral density $D^{-1}|\eta|^{1-2H}$ with $D = 2\Gamma(2-2H)\cos(1-H)\pi$, which is exactly the spectrum of fractional Brownian motion with Hurst parameter $H > 1/2$.

Correlated environment

A central limit theorem

Let ω be given by (7), S be the symmetrical nearest- random walk on $\mathbb{Z}$ started at the origin under probability measure $\mathbb{P}$, and let $\varrho = H/2$. Then

$$n^{-\varrho}\beta \sum_{i=1}^{n} \sum_{x\in\mathbb{Z}} \omega(i,x)\mathbb{P}(S_i = x) \xrightarrow{\mathcal{D}} N(0,\sigma^2) \tag{9}$$

with $\sigma^2 = \frac{4\beta^2\Gamma(1-H/2)}{DH}$.

The proof consists of the computation of the variance and verification of Lindeberg's condition.

Remark

In the case of iid, the CLT holds with $\varrho = 1/4, \sigma^2 = 2\beta^2/\sqrt{\pi}$.

Fractional Gaussian fields

- A time-space fractional Brownian random field $W = \{W(t,x) : t \geq 0, x \in \mathbb{R}\}$ defined on some probability space $(\Omega_H, \mathcal{F}_H, \mathbb{P}_H)$ is a mean zero Gaussian field with covariance

$$\mathbb{E}_H(W(t,x)W(s,y)) = \frac{1}{2}(s \wedge t)(|x|^{2H} + |y|^{2H} - |x-y|^{2H}),$$

$H \in (0,1)$: Hurst parameter.

- Introduce the following Hilbert space:

$$\mathcal{L}_H = \{f : \|f\|_H^2 = \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} f(s,u)K(u,v)f(s,v)dsdudv < \infty\},$$

where $K(u,v) = H(2H-1)|u-v|^{2H-2}$.

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Stochastic integral

- For $f \in \mathcal{L}_H$, the stochastic integral $\int_0^1 \int_{\mathbb{R}} f(t,x) W(dtdx) := W(f)$ is defined as usual with

$$\mathbb{E}_H \left[\int_0^1 \int_{\mathbb{R}} f(t,x) W(dtdx) \right]^2 = \int_0^1 \int_{\mathbb{R}^2} f(s,u) K(u,v) f(s,v) dsdudv.$$

- Symmetric tensor product of $\mathcal{L}_H$.

$$\mathcal{L}_H^{\otimes k} = \{ f : ([0,1] \times \mathbb{R})^k \to \mathbb{R}; \int_{[0,1]^k} \int_{\mathbb{R}^{2k}} f(t_1, x_1, t_2, x_2, \dots, t_k, x_k) \prod_{i=1}^{k} K(x_i, y_i) f(t_1, y_1, t_2, y_2, \dots, t_k, y_k) dtdxdy < \infty \}$$

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Stochastic integral

- Multiple Ito integral $I_k(f^{\otimes k}) = H_k(W(f))$ for $f \in \mathcal{L}_H$, H_k, k-order Hermite polynomial. Then define $I_k(f) = \int_{([0,1]\times\mathbb{R})^k} f(\mathbf{t}, \mathbf{x}) W^{\otimes k}(d\mathbf{t}d\mathbf{x})$ for general $f \in \mathcal{L}_H^{\otimes k}$ by density argument with

$$\mathbb{E}(I_k(f)I_k(g)) = k! < f, g >_H .$$

- r-order contraction of two symmetry functions f and g by

$$\begin{aligned} & f \otimes_r g(t_1, x_1; \ldots; t_{m+n-2r}, x_{m+n-2r}) \\ = & \mathrm{Sym}\Big\{ \int_{[0,1]^r} \int_{\mathbb{R}^{2r}} f(t_1, x_1; \ldots; t_{n-r}, x_{n-r}; s_1, u_1; \ldots; s_r, u_r) \\ & \times g(t_1, x_1; \ldots; t_{m-r}, x_{m-r}; s_1, u_1; \ldots; s_r, u_r) \Pi_{i=1}^r K(u_i, v_i) \\ & f(t_1, x_1; \ldots; t_{n-r}, x_{n-r}; \tau_1, v_1; \ldots; \tau_r, v_r) \\ & \times g(t_1, x_1; \ldots; t_{m-r}, x_{m-r}; \tau_1, v_1; \ldots; \tau_r, v_r) ds d\tau du dv \Big\}, \end{aligned}$$

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& f(t_1, x_1; \ldots; t_{n-r}, x_{n-r}; \tau_1, v_1; \ldots; \tau_r, v_r) \\
& \times g(t_1, x_1; \ldots; t_{m-r}, x_{m-r}; \tau_1, v_1; \ldots; \tau_r, v_r) ds d\tau d\mathbf{u} d\mathbf{v} \Big\},
\end{aligned}$$

Recursive identities

- $$I_n(f)I_m(g) = \sum_{r=0}^{m\wedge n} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f \otimes_r g) \qquad (10)$$

  for $f \in \mathcal{L}_H^{\otimes m}$, $g \in \mathcal{L}_H^{\otimes n}$. Especially, when $m = 1$, it is reduced to

  $$I_n(f)I_1(g) = I_{n+1}(f \otimes g) + nI_{n-1}(f \otimes_1 g). \qquad (11)$$

- $$\begin{aligned}&(f_1^{\otimes m} \otimes f_2^{\otimes(n-1)}) \otimes_1 f_2 \\ &= \frac{m}{m+n-1} f_1^{\otimes(m-1)} \otimes f_2^{\otimes(n-1)} < f_1, f_2 >_H \\ &\quad + \frac{n-1}{m+n-1} f_1^{\otimes m} \otimes f_2^{\otimes(n-2)} \|f_2\|_H^2.\end{aligned}$$

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Recursive identities

- Furthermore, according to (11), we have

$$
\begin{aligned}
& I_{m+n}(f_1^{\otimes m} \otimes f_2^{\otimes n}) \\
=& I_{m+n-1}(f_1^{\otimes m} \otimes f_2^{\otimes (n-1)}) I_1(f_2) \\
& - m < f_1, f_2 >_H I_{m+n-2}(f_1^{\otimes (m-1)} \otimes f_2^{\otimes (n-1)}) \qquad (12) \\
& - (n-1)\|f_2\|_H^2 I_{m+n-2}(f_1^{\otimes m} \otimes f_2^{\otimes (n-2)})
\end{aligned}
$$

Chaos expansion

Proposition

Let W *be the gaussian random field above with spatial parameter* $1/2 < H < 1$. *Let* $(\Omega_H, \mathcal{F}_H, P_H)$ *be the canonical probability space corresponding to* W. *Then for any* $F \in L^2(\Omega_H)$, *it admits the following chaos expansion:*

$$F = \sum_{k=0}^{\infty} I_k(f_k),$$

where $f_k \in \mathcal{L}_H^{\otimes k}, k = 0, 1, \ldots,$ *and the series converges in* $L^2(\Omega_H, \mathcal{F}_H, P_H)$. *Moreover,*

$$\mathbb{E}_H[F^2] = \sum_{k=0}^{\infty} k! \|f_k\|_H^2.$$

Stochastic heat equations

Mild solution

We turn to stochastic heat equations (5) with multiplicative noise with initial value $u(s,x) = u(x)$, $0 \le s \le t \le 1, x \in \mathbb{R}$. Its solution is formulated in the mild form, i.e.,

$$u(t,x;s) = P_{t-s}u(x) + \beta \int_s^t \int_{\mathbb{R}} P_{t-r}(x-z)u(r,z)W(dr,dz), \qquad (13)$$

where $P_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ and $P_t f(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} f(y)dy$. If, furthermore, let $u(x) = \delta(x-y)$, we get a four-parameter field $u(t,x;s,y)$ by

$$u(t,x;s,y) = P_{t-s}(x-y) + \beta \int_s^t \int_{\mathbb{R}} P_{t-r}(x-z)u(r,z;s,y)W(drdz).$$

Chaos expansion for solution

$$
\begin{aligned}
& u(t,x;s,y) \\
=& P_{t-s}(x-y) \\
& + \sum_{k=1}^{\infty} \beta^k \int_{\Delta(s,t]^k} \int_{\mathbb{R}^k} \Pi_{i=1}^k P_{t_i - t_{i-1}}(x_i - x_{i-1}) P_{t-t_k}(x - x_k) W(dt_i dx_i) \\
=& P_{t-s}(x-y) + \sum_{k=1}^{\infty} \beta^k I_k(P_k(\widetilde{t,x;s,y}))
\end{aligned}
$$

with $\Delta(s,t]^k = \{s < t_1 < \cdots < t_k < t\}$, $x_0 = y$, and

$$
P_k(t,x;s,y;t_1,\dots,t_k;x_1,\dots,x_k) = \Pi_{i=1}^k P_{t_i - t_{i-1}}(x_i - x_{i-1}) P_{t-t_k}(x - x_k)
$$

$$
\triangleq P_k(t,x;s,y;\tau;\mathbf{x})
$$

The convergence of partition functions

Theorem 1

Let $\{u(t,x),(t,x) \in [0,1] \times \mathbb{R}\}$ be the solution to (5) with parameter 2β, initial data $u(x) = \delta(x)$. And let Z_n^ω be the partition function (1) of random polymer in the random environment $\{\omega(n,x) : n \geq 0, x \in \mathbb{Z}\}$ with the representation (7). Then

$$e^{-n\lambda(\beta n^{-\varrho})} Z_n^\omega(\beta n^{-\varrho}, tn, x\sqrt{n}) \longrightarrow u(t,x) \qquad n \longrightarrow \infty,$$

in the sense of fdd.

Convergence of modified partition functions

Theorem 2

Let $\{u(t,x),(t,x) \in [0,1]\times\mathbb{R}\}$ be the solution to (5) with parameter $\sqrt{2}\beta$, initial data $u(x)=\delta(x)$. Then

$$\mathfrak{Z}_n^{\omega}(\beta n^{-\varrho}, tn, x\sqrt{n}) \longrightarrow u(t,x) \quad \text{as} \quad n \longrightarrow \infty.$$

Tightness

For $0 \le t \le 1, x \in \mathbb{R}$, define two-parameter fields by

$$z_n(t,x) := \sqrt{n}\mathfrak{z}_n^{\omega}(nt, \sqrt{n}x; \beta n^{-\varrho}).$$

Then, we have

$$z_n(t,x) = p_n(t,x) + n^{-\frac{1}{2}}\beta \sum_{\substack{s\in[0,t]\cap n^{-1}\mathbb{Z} \\ y\in n^{-1/2}\mathbb{Z}}} p_n(t-s, x-y)\bar{z}_n(s,y)\omega_n(s,y),$$

where $\omega_n(s,y) = n^{-\varrho}\omega(ns, \sqrt{n}y)$. $\bar{z}_n(s,y)$ corresponding to $\bar{\mathfrak{z}}_n^{\omega}(k,x;\beta)$, and $\bar{\mathfrak{z}}_n^{\omega}(k,x;\beta) = \frac{1}{2}[\mathfrak{z}_n^{\omega}(k+1,x;\beta) + \mathfrak{z}_n^{\omega}(k-1,x;\beta)]$.

Tightness

Theorem 3

Let $\epsilon > 0$ be small enough. For any $n \in \mathbb{N}$, $t, s \in [\epsilon, 1]$ and $x, y \in \mathbb{R}$, for some $q > 1$, there exist constant $C_\epsilon > 0, 0 < \iota < H$, such that

$$\mathbb{E}|z_n(t,x) - z_n(s,y)|^{2q} \leq C_\epsilon(|t-s|^{Hq} + |x-y|^{\iota q}). \tag{14}$$

Moreover, if 2q-order moment of ω is finite for $q > \frac{2}{H}$, then the family of process $\{z_n\}_{n=1}^{\infty}$ is tight in $C([\epsilon, 1], \mathbb{R})$.

Convergence of *U*-statistics in correlated case

weighted U−statistics $\mathcal{S}_k^n$ by

$$\mathcal{S}_k^n(f) = 2^{k/2} \sum_{\mathbf{i} \in E_k^n} \sum_{\mathbf{x} \in \mathbb{Z}^k} \bar{f}_n\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}\right) \omega(\mathbf{i}, \mathbf{x}) 1_{\{\mathbf{i} \leftrightarrow \mathbf{x}\}}. \tag{15}$$

where $\bar{f}_n$ is the conditional expectation of $f \in L^2([0,1]^k \times \mathbb{R}^k)$ with respect to the sigma algebra generated by

$$\mathfrak{R}_k^n \triangleq \left\{ \left(\frac{\mathbf{i}-1}{n}, \frac{\mathbf{i}}{n} \right] \times \left(\frac{\mathbf{x}-1}{\sqrt{n}}, \frac{\mathbf{x}+1}{\sqrt{n}} \right] : \mathbf{i} \in D_k^n, \mathbf{i} \leftrightarrow \mathbf{x} \right\}.$$

$$\bar{f}_n(\mathbf{t}, \mathbf{x}) = \frac{1}{|R|} \int_R f d\mathbf{t} d\mathbf{x}, \quad (\mathbf{t}, \mathbf{x}) \in R \in \mathfrak{R}_k^n,$$

Convergence of U-statistics in correlated case

Theorem 4

Let $f \in \mathcal{H}^k$. Then, as $n \to \infty$,

$$n^{\frac{-(H+1)k}{2}} \mathcal{S}_k^n(f) \xrightarrow{D} \int_{[0,1]^k} \int_{\mathbb{R}^k} f(\mathbf{t}, \mathbf{x}) W^{\otimes k}(d\mathbf{t}d\mathbf{x}) = I_k(f), \qquad (16)$$

where $W(dtdx)$ is fractional Gaussian noise (**??**) and $I_k, k = 1, 2 \ldots,$ is k-multiple integral.

The proof is more complicated than that of the case i.i.d. environment since the correlation exists between two disjoint spatial areas.

The way to Theorem 4

- $k = 1$: Show it is true for $f = 1_{\{t_0 \le t \le t_1, x_0 \le x \le x_1\}}$ for some $0 \le t_0 < t_1 \le 1, x_0 < x_1$. In this case,

$$\mathcal{S}_1^n(f) = 2^{1/2} \sum_{\{i \in E_k^n, nt_0 \le i \le nt_1\}} \sum_{\{x \in \mathbb{Z}, \sqrt{n}x_0 \le x \le \sqrt{n}x_1\}} \omega(i,x) 1_{\{i \leftrightarrow x\}}$$

We take the method used in the proof of CLT in this talk.

- $k > 1$: We show

$$n^{-\frac{(H+1)k}{2}} \mathcal{S}_k^n(g^{\otimes k}) \longrightarrow I_k(g^{\otimes k}) \quad \text{as} \quad n \longrightarrow \infty.$$

with

$$g(t,x) = 1_{\{t_0 \le t \le t_1, x_0 \le x \le x_1\}}(t,x) \in \mathcal{L}_H$$

for $\forall t_0 \le t_1, x_0 \le x_1$ by induction.

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The way to Theorem 4

- We show it holds for $f \in \mathcal{L}_H^k$ of the form $f = g_1^{\otimes k_1} \otimes \cdots \otimes g_s^{\otimes k_s}$ with $g_1, \ldots, g_s$ being indicators as $k = 1$ of some disjoint rectangles, $k_1 + \cdots + k_s = k, k_1 > 0, \ldots, k_s > 0, s = 2, \ldots$.
- We need to show for $s = 2$

$$
\begin{aligned}
& n^{-\frac{(H+1)(m+l)}{2}} S_{m+l}^n(f^{\otimes m} \otimes g^{\otimes l}) \\
\longrightarrow & I_{m+l-1}(f^{\otimes m} \otimes g^{\otimes (l-1)}) I_1(g) \\
& - m I_{m+l-2}(f^{\otimes (m-1)} \otimes g^{\otimes (l-1)}) < f, g >_H \\
& - (l-1) I_{m+l-2}(f^{\otimes m} \otimes g^{\otimes (l-2)}) \|g\|_H^2 .
\end{aligned}
$$

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The way to Theorem 4

Lemma

For all fixed k, n, $\mathcal{S}_k^n(f)$ is linear in h with probability one, and, by the definition of ω, for $k_1 \neq k_2$, $\mathbb{E}_{\mathbb{Q}}(\mathcal{S}_{k_1}^n(f_1)\mathcal{S}_{k_2}^n(f_2)) = 0$ for $f_i \in \mathcal{L}_H^{\otimes k_i}, i = 1, 2$. Furthermore, for $k_1 = k_2 = k$, we have

$$\mathbb{E}_{\mathbb{Q}}[(\mathcal{S}_k^n(f))^2] \leq C\lambda^k n^{(1+H)k} \|f\|_{\mathcal{H}^k}^2$$

for some generic positive constant C.

- Based on the lemma and previous disscusion for general $f \in \mathcal{L}_H^k$

$$n^{\frac{-(H+1)k}{2}} \mathcal{S}_k^n(f) \xrightarrow{D} \int_{[0,1]^k} \int_{\mathbb{R}^k} f(\mathbf{t}, \mathbf{x}) W^{\otimes k}(d\mathbf{t}d\mathbf{x}) = I_k(f),$$

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The way to Theorem 4

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is true.

Convergence of $\mathfrak{Z}_n^\omega(\beta n^{-\varrho})$

It is easy to get

$$\begin{aligned}\mathfrak{Z}_n^\omega(\beta n^{-\varrho}, tn, x\sqrt{n}) =& 1 + \sum_{k=1}^{n} \beta^k n^{-k\varrho} \sum_{\mathbf{i}\in D_k^n} \sum_{\mathbf{x}\in\mathbb{Z}^k} \left[\prod_{j=1}^{k} \omega(i_j, x_j) p_k(\mathbf{i}, \mathbf{x})\right] \\ =& 1 + \sum_{k=1}^{n} 2^k \beta^k n^{-\frac{k(H+1)}{2}} \mathcal{S}_k^n(n^{\frac{k}{2}} p_k^n) \\ \longrightarrow & 1 + \sum_{k=1}^{\infty} 2^k \beta^k I_k(p_k(t, x, \cdot)),\end{aligned}$$

where $p(t,x)$ is Brownian motion density and the last expression is just the chaos expansion of mild solution $u(t,x)$ to stochastic heat equation (5) driven by fractional Gaussian noise.

Hermite expansion

Let

$$\tilde{\omega}_n(i,x) = \frac{e^{\beta n^{-\varrho}\omega(i,x) - \lambda(\beta n^{-\varrho})} - 1}{\beta n^{-\varrho}} \triangleq F^{(n)}(\omega(i,x)), \tag{17}$$

where $\lambda(\cdot)$ is the Log-Laplace of $\omega(i,x)$. Thus, we get a mean zero stationary field $\tilde{\omega}_n(i,x)$ (n-dependent), which is a non-linear functionals of $\omega(i,x)$. The covariance of $\tilde{\omega}_n(i,x)$ and $\tilde{\omega}_n(i,y)$ is given by

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\tilde{\omega}_n(i,x)\tilde{\omega}_n(i,y)) &= \frac{1}{\beta^2 n^{-2\varrho}}\mathbb{E}_{\mathbb{Q}}\{e^{\beta n^{-\varrho}(\omega_n(i,x)+\omega_n(i,y)) - 2\lambda(\beta n^{-\varrho})} - 1\} \\ &= \gamma(x-y)(1+o(1)) := \tilde{\gamma}_n(x-y).\end{aligned}$$

Hermite expansion

We can expand $F^{(n)}(z), z \in \mathbb{R}$, by

$$F^{(n)}(z) = \frac{1}{\beta n^{-\varrho}} \sum_{k=1}^{\infty} (\beta n^{-\varrho})^k A_k(z),$$

where $A_k(z), k \in \mathbb{N}$, is the system of Appell polynomials related to the distribution of ω with $A_0 = 1$. Let $c_k, k \in \mathbb{N}$, be the expansion coefficients of F^n with respect to Appell system $A_k, k \in \mathbb{N}$. We remark that the Appell rank, which is the least index k such that $c_k \neq 0$, of $F^{(n)}$ is 1. Now by (17), we have

$$e^{-n\lambda(\beta n^{-\varrho})} \mathbb{Z}_n^{\omega} = e^{-n\lambda(\beta n^{-\varrho})} \mathbb{E}_{\mathbb{Q}} e^{\beta n^{-\varrho} \sum_{i=1}^{n} \omega(i,S_i)} = \mathbb{E}_{\mathbb{Q}} \prod_{i=1}^{n} (1 + \beta n^{-\varrho} \tilde{\omega}_n(i, S_i)). \tag{18}$$

Modified partition functions for $\tilde\omega_n(i,x)$

$$
\begin{aligned}
\mathfrak{Z}_n^{\tilde\omega_n}(\beta n^{-\varrho}) =& \mathbb{E}_{\mathbb{P}}\left[\prod_{i=1}^{n}(1+\beta n^{-\varrho}\tilde\omega_n(i,S_i))\right] \\
=& \mathbb{E}_{\mathbb{P}}\left[1+\sum_{k=1}^{n}\beta^k n^{-k\varrho}\sum_{\mathbf{i}\in D_k^n}\prod_{j=1}^{k}\tilde\omega_n(i_j,S_{i_j})\right] \\
=& 1+\sum_{k=1}^{n}\beta^k n^{-k\varrho}\sum_{\mathbf{i}\in D_k^n}\sum_{\mathbf{x}\in\mathbb{Z}^k}\left[\prod_{j=1}^{k}\tilde\omega_n(i_j,x_j)p_k(\mathbf{i},\mathbf{x})\right],
\end{aligned}
$$

and the corresponding weighted $U-$statistics $\mathcal{S}_k^n$ by

$$
\mathcal{S}_k^n(f,\tilde\omega_n) = 2^{k/2}\sum_{\mathbf{i}\in E_k^n}\sum_{\mathbf{x}\in\mathbb{Z}^k}\bar f_n\left(\frac{\mathbf{i}}{n},\frac{\mathbf{x}}{\sqrt{n}}\right)\tilde\omega_n(\mathbf{i},\mathbf{x})1_{\{\mathbf{i}\leftrightarrow\mathbf{x}\}}.
$$

Modified partition functions for $\tilde{\omega}_n(i,x)$

Theorem 5

For $f \in \mathcal{L}_H^{\otimes k}$, $k \in \mathbb{N}$, $\tilde{\omega}_n$ is defined by (17). Then

$$n^{\frac{-k(H+1)}{2}} \mathcal{S}_k^n(f, \tilde{\omega}_n) \xrightarrow{D} I_k(f), \quad \text{as} \quad n \to \infty.$$

Sketch of the proof: We only show it holding for $k = 1$ and f of the form $f(t,x) = 1_{\{t_0 \le t \le t_1, x_0 \le x \le x_1\}}$ for some $0 \le t_0 \le t_1, x_0 \le x_1 \in \mathbb{R}$ as before.

$$\mathcal{S}_1^n(f, \tilde{\omega}_n) = 2^{1/2} \sum_{i \in E_k^n, nt_0 \le i \le nt_1} \sum_{x \in \mathbb{Z}, \sqrt{n}x_0 \le x \le \sqrt{n}x_1} \tilde{\omega}_n(i,x) 1_{\{i \leftrightarrow x\}}.$$

It is easy to show that

$$n^{-(H+1)} \mathbb{E}_{\mathbb{Q}}[(\mathcal{S}_1^n(f, \tilde{\omega}_n))^2] \to \frac{\lambda(t_1 - t_0)(x_1 - x_0)^{2H}}{H(2H-1)} = \mathbb{E}_H \left(\int_0^1 \int_R f(t,x) W(dt$$

Modified partition functions for $\tilde{\omega}_n(i,x)$

Sketch of the proof: By following Dobrushin's lines (see[?]) to show the normal asymptotics.

$$c_1^{(n)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_1(x) \frac{e^{\beta n^{-\varrho} x - \beta^2 n^{-2\varrho}/2} - 1}{\beta n^{-\varrho}} e^{-\frac{x^2}{2}} dx = 1 \neq 0.$$

We split $n^{\frac{-(H+1)}{2}} \mathcal{S}_1^n(f, \tilde{\omega}_n)$ as the sum of $n^{\frac{-(H+1)}{2}} \mathcal{S}_1^n(f)$ and $\mathcal{R}_n^{\tilde{\omega}_n}$.

$$\mathcal{R}_n^{\tilde{\omega}_n} = 2^{1/2} \sum_{i \in E_k^n, nt_0 \le i \le nt_1} \sum_{x \in \mathbb{Z}, \sqrt{n}x_0 \le x \le \sqrt{n}x_1} \sum_{j=2}^{\infty} c_j^{(n)} H_j(\omega(i,x)) 1_{\{i \leftrightarrow x\}}.$$

Then, by gaussian property,

$$\mathbb{E}_{\mathbb{Q}}[\mathcal{R}_n^{\tilde{\omega}_n}]^2 = 2\sum_{i\in E_k^n, nt_0\le i\le nt_1}\mathbb{E}_{\mathbb{Q}}\left[\sum_{x\in\mathbb{Z},\sqrt{n}x_0\le x\le\sqrt{n}x_1}\sum_{j=2}^{\infty}c_j^{(n)}H_j(\omega(i,x))1_{\{i\leftrightarrow x\}}\right]^2$$

$$= 2\sum_{i\in E_k^n, nt_0\le i\le nt_1}\sum_{j=2}^{\infty}(c_j^{(n)})^2\mathbb{E}_{\mathbb{Q}}\left[\sum_{x\in\mathbb{Z},\sqrt{n}x_0\le x\le\sqrt{n}x_1}H_j(\omega(i,x))1_{\{i\leftrightarrow x\}}\right]^2$$

$$= \sum_{j=2}^{\infty}(c_j^{(n)})^2 j!M\left[N\gamma^j(0)+\sum_{l=1}^{N-1}(N-l)\gamma^j(l)\right]$$

with $M = \lfloor n(t_1 - t_0) \rfloor$. The rhs of the above identity is bounded by

$$MN \sum_{j=2}^{\infty} (c_j^{(n)})^2 j! \left[\gamma^j(0) + \sum_{l=1}^{N-1} \gamma^j(l) \right] \sim MN \sum_{j=2}^{\infty} (c_j^{(n)})^2 j! \left[\gamma^j(0) + \sum_{l=1}^{N-1} \gamma^j(l) \right]$$
$$\leq CMN \sum_{j=2}^{\infty} (c_j^{(n)})^2 j! N^{1+j(1-2\alpha)}.$$

Whence we have

$$n^{-(H+1)} \mathbb{E}_{\mathbb{Q}} [\mathcal{R}_n^{\tilde{\omega}_n}]^2 \sim C(t_1 - t_0)(x_1 - x_0)^2 \sum_{j=2}^{\infty} (c_j^{(n)})^2 j! n^{(j-1)(H-1)} (x_1 - x_0)^{2j(H-1)} -$$

and

$$n^{-\frac{H+1}{2}} \mathcal{S}_1^n(h, \tilde{\omega}_n) \xrightarrow{D} I_1(h)$$

for $h = 1_{\{t_0 \leq t \leq t_1, x_0 \leq x \leq x_1\}}$ as $n \longrightarrow \infty$ by Slutsky's theorem again.

Thank You!