Spectral Radii of Truncated Circular Unitary Matrices

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Outline

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Random Matrix Theory

- Early study of RMT motivated by analysis of high-dimensional data: Wishart's (1928)

  Large covariance matrices whose statistical properties are mainly determined by eigenvalues and eigenvectors from the point view of a principal components analysis.

- More applications

  Heavy-nuclei atoms (Wigner, 1955), number theory (Mezzadri and Snaith, 2005), quantum mechanics (Mehta, 2005), condensed matter physics (Forrester, 2010), wireless communications (Couillet and Debbah, 2011).

- Tracy-Widom laws by Tracy and Widom (1994, 1996)

  – The largest eigenvalues of the three Hermitian matrices (Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble) converge to some special distributions that are now known as the Tracy-Widom laws.

- The Tracy-Widom laws have found their applications in the study of problems such as

  – the longest increasing subsequence; Baik et al. (1999)

  – combinatorics, growth processes, random tilings and the determinantal point processes; Tracy and Widom (2002), and Johansson (2007)

  – the largest eigenvalues in the high-dimensional statistics; Johnstone (2001, 2008), Jiang (2009)

- Some recent research focuses on the universality of the largest eigenvalues of matrices with non-Gaussian entries

  – see, for example, Tao and Vu (2011), Erdős et al. (2012)

Consider a non-Hermitian matrix **M** with eigenvalues $z_1, \cdots, z_n$:

– The largest absolute values of the eigenvalues $\max_{1 \le j \le n} |z_j|$ is refereed to as the spectral radius of **M**.

- The spectral radii of the real, complex and symplectic Ginibre ensembles are investigated by Rider (2003, 2004) and Rider and Sinclair (2014)

  – the spectral radius for the complex Ginibre ensemble converges to the Gumbel distribution.

  This indicates that non-Hermitian matrices exhibit quite different behaviors from Hermitian matrices in terms of the limiting distribution for the largest absolute values of the eigenvalues

Jiang and Qi (2017) studies the largest radii of three rotation-invariant and non-Hermitian random matrices:

– the spherical ensemble,

– the truncation of circular unitary ensemble, and

– the product of independent complex Ginibre ensembles.

- The spectral radii converge to the Gumbel distribution and some new distributions.

Outline

The circular unitary ensemble

- The circular unitary ensemble is an $n \times n$ random matrix with Haar measure on the unitary group, and it is also called Haar-invariant unitary matrix.

- Let $\mathbf{U}$ be an $n \times n$ circular unitary matrix.

  The n eigenvalues of the circular unitary matrix $\mathbf{U}$ are distributed over $\{z \in \mathcal{C} : |z| = 1\}$, where $\mathcal{C}$ is the complex plane, and their joint density function is given by

$$\frac{1}{n!(2\pi)^n} \cdot \prod_{1 \le j < k \le p} |z_j - z_k|^2;$$

  see, e.g., Hiai and Petz (2000).

Truncation

For $n > p \geq 1$, write

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{C}^* \\ \mathbf{B} & \mathbf{D} \end{pmatrix}$$

where **A**, as a truncation of **U**, is a $p \times p$ submatrix.

Let $z_1, \cdots, z_p$ be the eigenvalues of **A**. Then their density function is

$$C \cdot \prod_{1 \leq j < k \leq p} |z_j - z_k|^2 \prod_{j=1}^{p} (1 - |z_j|^2)^{n-p-1} \qquad (1)$$

where C is a normalizing constant. See, e.g., Zyczkowski and Sommers (2000).

Empirical distribution for truncation ensemble

Assume $p = p_n$ depends on n and set $c = \lim_{n\to\infty} \frac{p}{n}$.

- The empirical distribution of z_i's converges to the distribution with density proportional to $\frac{1}{(1-|z|^2)^2}$ for $|z| \le c$ if $c \in (0, 1)$. Życzkowski and Sommers (2000)
- The empirical distribution goes to the circular law and the arc law as $c = 0$ and $c = 1$, respectively. Dong et al. (2012).
  See also Diaconis and Evans (2001) and Jiang (2009, 2010) and references therein for more results.

Spectral radius for truncation ensemble

The spectral radius $\max_{1\le j\le p}|z_j|$ for the truncated circular unitary ensemble converges to the Gumbel distribution when the dimension of the truncated truncated circular unitary matrix is of the same order as the dimension of the original circular unitary matrix.

– Jiang and Qi (2017)

Theorem 1

Assume that $z_1, \cdots, z_p$ have density as in (1) and there exist constants $h_1, h_2 \in (0,1)$ such that $h_1 < \frac{p_n}{n} < h_2$ for all $n \geq 2$. Then $(\max_{1\leq j\leq p} |z_j| - A_n)/B_n$ converges weakly to the Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where
$A_n = c_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$,
$B_n = \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$,

$$c_n = \left(\frac{p_n - 1}{n-1}\right)^{1/2}, \quad b_n = b\left(\frac{nc_n^2}{1-c_n^2}\right), \quad a_n = a\left(\frac{nc_n^2}{1-c_n^2}\right)$$

with

$$a(y) = (\log y)^{1/2} - (\log y)^{-1/2}\log(\sqrt{2\pi}\log y), \quad b(y) = (\log y)^{-1/2}$$

for $y > 3$.

Outline

- We consider heavily truncated and lightly truncated circular unitary ensembles and investigate the limiting distribution of the spectral radii for those truncated circular unitary ensembles. Our results complement that in Jiang and Qi (2017).
- Consider the $p_n \times p_n$ submatrix $\mathbf{A}$, truncated from a $n \times n$ circular unitary matrix $\mathbf{U}$
- Denote the p_n eigenvalues as $z_1, \cdots, z_{p_n}$ with the joint density function given by (1).

$$p_n \to \infty \text{ and } \frac{p_n}{n} \to 0 \text{ as } n \to \infty; \tag{2}$$

$$\frac{n - p_n}{(\log n)^3} \to \infty \text{ and } \frac{n - p_n}{n} \to 0 \text{ as } n \to \infty; \tag{3}$$

$$n - p_n \to \infty \text{ and } \frac{n - p_n}{\log n} \to 0 \text{ as } n \to \infty; \tag{4}$$

$$n - p_n = k \geq 1 \text{ is fixed integer.} \tag{5}$$

Theorem 2

Under condition (2) or (3), $(\max_{1\leq j\leq p}|z_j| - A_n)/B_n$ converges weakly to the Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where A_n and B_n are defined as in Theorem 1.

Theorem 3

Under condition (4), $(\max_{1\le j\le p}|z_j| - A_n)/B_n$ converges weakly to the Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where $A_n = (1 - a_n/n)^{1/2}$ and $B_n = a_n/(2nk_n)$, where a_n is given by

$$\frac{1}{(k_n-1)!}\int_0^{a_n} t^{k_n-1}e^{-t}dt = \frac{k_n}{n}.$$

where $k_n = n - p_n$.

Theorem 4

Under condition (5), $\frac{2n^{1+1/k}}{((k+1)!)^{1/k}}(\max_{1\leq j\leq p}|z_j|-1)$ *converges weakly to the reverse Weibull distribution* $W_k(x)$ *defined as*

$$W_k(x)=\begin{cases}\exp(-(-x)^k), & x\leq 0;\\ 1, & x>0.\end{cases}$$

Gap:
It is obvious that the case when $k_n=n-p_n$ is of order between $\log n$ and $(\log n)^3$ has not been covered in Theorems 1 to 4.

We conjecture that $\max_{1\leq j\leq p_n}|z_j|$, after properly normalized, converges in distribution to the Gumbel distribution in this case.

Outline

Independence of radius

Lemma 5

Assume the density function of $(Z_1, \cdots, Z_n) \in \mathbb{C}^n$ *is proportional to* $\prod_{1 \le j < k \le n} |z_j - z_k|^2 \cdot \prod_{j=1}^n \varphi(|z_j|)$, *where* $\varphi(x) \ge 0$ *for all* $x \ge 0$. *Let* $Y_1, \cdots, Y_n$ *be independent r.v.'s such that the density of* Y_j *is proportional to* $y^{2j-1}\varphi(y)I(y \ge 0)$ *for each* $1 \le j \le n$. *Then,*

$$g(|Z_1|, \cdots, |Z_n|) \stackrel{d}{=} g(Y_1, \cdots, Y_n)$$

for any symmetric function $g(y_1, \cdots, y_n)$.

We have

$$\max_{1 \le j \le n} |z_j| \stackrel{d}{=} \max_{1 \le j \le n} Y_j.$$

Thank you!

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