Quasi-stationary distributions for one-dimensional minimal diffusion processes

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(based on a joint work with Professor Mao Y-H)

Contents

1. Stationarity and quasi-stationarity

- Consider a Markov process X_t taking values in state space $(E, \mathscr{E})$ with lifetime ζ. Then the (quasi-)stationary distribution μ is defined as

$$\mathbb{P}_\mu[X_t \in A | t < \zeta] = \mu(A),\ \ A \in \mathscr{E}.$$

- If $\zeta = \infty$ a.e. then μ is defined as a stationary distribution.
  If $\zeta < \infty$ a.e. then μ is defined as a quasi-stationary distribution (QSD).
- Equivalently,

$$\sum_i \mu_i p_{ij}(t) = \mu_j e^{-\lambda t},$$

  where

$$p_{ij}(t) = \mathbb{P}_i[X_t = j, t < \zeta].$$

From Branching process by Yaglom

- Yaglom (1947) considered the following limit problem $\forall i, j \geq 1$

$$\lim_{t\to\infty} \mathbb{P}_i[X_t = j | \tau_0 > t] = \lim_{t\to\infty} \frac{p_{ij}(t)}{\sum_{k\geq 1} p_{ik}(t)}.$$

- Let $Z_0 = 1, Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k}$, with $\xi,\ \xi_{n,k}$ i.i.d. Denote

$$f(s) = \mathbb{E}s^{\xi}, \quad g(s) = \lim_{n\to\infty} \mathbb{E}[s^{Z_n} | Z_n > 0].$$

If $m \triangleq \mathbb{E}\xi < 1$, then g(s) exists and satisfies

$$g(f(s)) = mg(s) + 1 - m.$$

In particular,

$$\lim_{n} \mathbb{P}[Z_n = i | Z_n > 0] = b_i > 0, \quad \sum_{i\geq 1} b_i = 1.$$

- A.M. Yaglom (1947). Certain limit theorems of the theory of branching processes. *Dokl. Acad. Nauk SSSR* 56,795-798(in Russian).

Branching process

- Futhermore, there exists a family of QSD's :

$$\tilde{g}(s) = 1 - (1 - g(s))^{\alpha}, \quad \alpha \in (0, 1].$$

- When $\alpha = 1$, $\widetilde{g}(s) = g(s)$.

Birth-death process: regular case

- Let X_t be a birth-death process on $\mathbb{Z}_+ = \{0, 1, 2, \cdots\} \cup \{-1\}$.
- Assume $\lim\limits_{t\to\infty} \mathbb{P}_i[X_t = -1] = 1$ (absorbed at -1 eventually) and let τ_i be the hitting time to i.
- Doorn (1991) proved that
  - if $S \triangleq \mathbb{E}_\infty \tau_0 \triangleq \lim\limits_{i\to\infty} \tau_0 = \infty$ and $\lambda \triangleq -\lim\limits_{t\to\infty} \frac{1}{t}\log\mathbb{P}_i[X_t = i] > 0$, then there exist a family of QSDs,
  - if $S < \infty$, then $\exists$! QSD.

- E. A. van Doorn (1991). Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Advances in Applied Probability*, 23(4), 683-700.

Birth-death process: explosive case

- The assumption $\lim_{t\to\infty} \mathbb{P}_i [X_t = -1] = 1$ implies the process is not explosive.
- What happens for an explosive birth-death process?
- It's known the birth-death process is explosive when $R \triangleq \mathbb{E}_0\tau_\infty = \lim_{i\to\infty} \mathbb{E}_0\tau_i < \infty$.
- If $R < \infty$, the QSD for the minimal process (before explosion) can be considered.
- When $R < \infty$ and $S = \infty$, Gao and Mao (2015) proved $\exists\,!$ QSD for the minimal birth-death process.
- When $R < \infty$ and $S < \infty$, Gao, Mao and Zhang (2017) proved $\exists\,!$ QSD for the minimal birth-death process.

- Wu-Jun Gao and Yong-Hua Mao (2015). Quasi-stationary distribution for the birth-death process with exit boundary. *Journal of Mathematical Analysis and Applications* 427,114-125.
- Wu-Jun Gao, Yong-Hua Mao and Chi Zhang (2017). The birth-death processes with regular boundary: stationarity and quasi-stationarity. Preprint.

Three main problems

For the general Markov process, the study of QSD on three problems:

- Existence and uniqueness of QSD
- QSD's attraction domian
- Convergence rate to the QSD

2. 1D (minimal) diffusion process

- Consider the stochastic differential equation (SDE) on $(0,\infty)$

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad x_0 = x > 0. \tag{1}$$

  where $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion, $\sigma \in C^2((0,\infty))$, $\sigma^2(x) > 0, x \in (0,\infty)$ and $b(x) \in C^1((0,\infty))$.
- Define

$$\tau_y = \inf\{t > 0 : X_t = y\}, \quad \tau_\infty = \lim_{n\to\infty} \tau_n, \quad \tau_0 = \lim_{n\to\infty} \tau_{\frac{1}{n}}, \quad \zeta = \tau_0 \wedge \tau_\infty.$$

- Then SDE (1) has a unique solution $X^x(t)$ up to the explosion time ζ. We define $X^x(t)$ to be $\lim_{s\uparrow\zeta} X^x(s)$ for $t \geq \zeta$ on the set $\{\zeta < \infty\}$.
- $(X_t^x)_{t\geq 0}$ is called the minimal L-diffusion, where the differential operator L is defined by

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \quad a(x) = \frac{1}{2}\sigma^2(x). \tag{2}$$

Feller's boundary classification

Let

$$\Sigma(0) \triangleq \mathbb{E}_1\tau_0 \triangleq \lim_{n\to\infty} \mathbb{E}_1\tau_{\frac{1}{n}}, \quad N(0) \triangleq \mathbb{E}_0\tau_1 \triangleq \lim_{n\to\infty} \mathbb{E}_{\frac{1}{n}}\tau_1.$$

Classification of boundary 0 :

- Natural boundary: $\Sigma(0) = \infty, \quad N(0) = \infty$
- Entrance boundary: $\Sigma(0) = \infty, \quad N(0) < \infty$
- Exit boundary: $\Sigma(0) < \infty, \quad N(0) = \infty$
- Regular boundary: $\Sigma(0) < \infty, \quad N(0) < \infty$

Classification of boundary ∞.

3. Known results

- Let X_t be a diffusion process on $(0, \infty)$ given by the unique solution to the SDE
$$X_t = B_t - \int_0^t \alpha(X_s) ds,$$
where $\alpha \in C^1(0, \infty)$.
- Assume $\Sigma(0) = \mathbb{E}_1 \tau_0 < \infty$ and ∞ to be a natural boundary.
- Collet et al. (1995) concluded that $\eta \triangleq -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x[t < \tau_0]$ can only take two values. Based on their results, Zhang and He (2016) proved that when $\eta > 0$, there exist a family of QSDs.

- P. Collet, J. S. Martín, and S. Martínez (1995). Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption. *Annals of Probability*, 23(3), 1300-1314.
- H. J. Zhang and G. M. He (2016). Existence and construction of quasi-stationary distributions for one-dimensional diffusions. *Journal of Mathematical Analysis and Applications*, 434(1), 171-181.

Diffusion process: ∞ an entrance boundary

- Let $\Sigma(0) = \mathbb{E}_1\tau_0 < \infty$, ∞ be an entrance boundary and the drift parameter α satisfy the additional conditons

$$\lim_{x\to\infty} \alpha^2(x) - \alpha'(x) = \infty, \quad -\inf_{x\in(0,\infty)} \alpha^2(x) - \alpha'(x) < \infty,$$

$$\int_0^1 y e^{-\int_1^y \alpha(z)dz} dy < \infty.$$

- Cattiaux et al. (2009) got the following results:
  - there exists a unique QSD
  - any measure ρ with compact support in $(0,\infty)$ satisfies

$$\lim_{t\to\infty} \mathbb{P}_\rho(X_t \in A | t < \zeta) = \nu(A).$$

- P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. S. Martín (2009). Quasi-stationary distributions and diffusion models in population dynamics. *Annals of Probability*, 37(5), 1926-1969.
- H. J. Zhang and G. M. He (2016). Domain of attraction of quasi-stationary distribution for one-dimensional diffusions. *Frontiers of Mathematics in China*, 11(2), 411-421.

Diffusion process: ∞ an entrance boundary

- Based on the work of Cauttiaux et al., various authors further studied the QSDs of one-dimensional diffusions with ∞ as an entrance boundary, such as Littin, Zhang and He.
- Let 0 be an exit boundary ($\mathbb{E}_1\tau_0 < \infty$, $\mathbb{E}_0\tau_1 = \infty$) and ∞ be an entrance boundary ($\mathbb{E}_1\tau_\infty = \infty$, $\mathbb{E}_\infty\tau_1 < \infty$). (There isn't any additional condition on the drift parameter α.)
- Littin (2012) proved that
  - $\exists$! QSD,
  - the unique QSD attracts initial distributions with compact support.

- Jorge Littin C. (2012). Uniqueness of quasistationary distributions and discrete spectra when ∞ is an entrance boundary and 0 is singular. *Journal of Applied Probability*, 49(3), 719-730.
- H. J. Zhang and G. M. He (2016). Domain of attraction of quasi-stationary distribution for one-dimensional diffusions. *Frontiers of Mathematics in China*, 11(2), 411-421.

Motivation

- All the diffusions considered above are absorbed at 0.
- What happens when diffusions explode eventually?
- This urges one to consider the minimal diffusion process.

Main results (1): ∞ an exit boundary

- It is known the diffusion process is explosive when $\Sigma(\infty) \triangleq \mathbb{E}_1 \tau_\infty < \infty$.
- Hypothesis (H-EE): 0 is an entrance boundary ($\mathbb{E}_0 \tau_1 < \infty,\ \mathbb{E}_1 \tau_0 = \infty$) and ∞ is an exit boundary ($\mathbb{E}_\infty \tau_1 = \infty,\ \mathbb{E}_1 \tau_\infty < \infty$).
- The generator L of the minimal diffusion process is given by

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x},$$

  where $a(x) = \frac{1}{2}\sigma^2(x)$.
- $m(x) := \frac{1}{a(x)} \exp\left[\int_1^x \frac{b(y)}{a(y)} \mathrm{d}y\right]$.

Main results(1): ∞ an exit boundary

Theorem

Under hypothesis $(\mathrm{H}-\mathrm{EE})$*, there exists a unique QSD for the minimal L-diffusion process, which is given by*

$$\nu(\mathrm{d}x) = \frac{\psi_0(x)m(x)}{\int_0^\infty \psi_0(y)m(y)\mathrm{d}y}\mathrm{d}x, \tag{3}$$

where λ_0 *is the minimal eigenvalue of generator* $-L$ *and* ψ_0 *is the eigenfunction associated to* λ_0*.*
Moreover, the QSD attracts all initial probability distributions μ *on* $(0, \infty)$*, that is, for any Borel subset* $A \subseteq (0, \infty)$

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in A|\zeta > t) = \nu(A).$$

Tools

- Duality
- Spectral theory
- Passage time between two states

Duality

- Consider the the stochastic differential equation (SDE)

$$d\widetilde{X}_t = \sigma(\widetilde{X}_t)dB_t + (\sigma(\widetilde{X}_t)\sigma'(\widetilde{X}_t) - b(\widetilde{X}_t))dt, \quad \widetilde{x}_0 = x > 0. \tag{4}$$

- There exists a minimal $\widetilde{L}$-diffusion process $\left(\widetilde{X}_{t\wedge\widetilde{\tau}}\right)_{t\geq 0}$, where the lifetime $\widetilde{\zeta} = \widetilde{\tau}_0 \wedge \widetilde{\tau}_\infty$.
- $L = D_M D_S$ and $\widetilde{L} = D_{\widetilde{M}} D_{\widetilde{S}}$, where $\widetilde{M} = S$ and $\widetilde{S} = M$. More, L and $\widetilde{L}$ have the same spectrum.
-

$$X_t = \begin{cases} 0: & \textit{entrance boundary} \\ \infty: & \textit{exit boundary} \end{cases} \Longleftrightarrow \widetilde{X}_t = \begin{cases} 0: & \textit{exit boundary} \\ \infty: & \textit{entrance boundary} \end{cases}$$

Integrability of eigenfunction ψ_0

- From the above relation, we have

$$\widetilde{\psi}_0(x) = \int_0^x \psi_0(y)m(y)dy.$$

- Then $\int_0^\infty \psi_0(x)m(x)dx = \lim_{x\to\infty} \widetilde{\psi}_0(x)$.

Our aim : $\lim_{x\to\infty} \widetilde{\psi}_0(x) < \infty$.

- Use a spectral representation for $\widetilde{\psi}_0$.

Spectral representation for $\widetilde{\psi}_0(x)$

Theorem

Assume 0 is an exit boundary and ∞ is an entrance boundary. Let $\widetilde{X}_t$ be the minimal $\widetilde{L}$-diffusion and $\widetilde{\tau}_{x,y} := \inf\{t > 0 : \widetilde{X}_t = y, \widetilde{X}_0 = x\}$. Then for any $0 < y < x < \infty$, $\widetilde{\psi}_0(x) = \widetilde{\psi}_0(y)\mathbb{E}e^{\widetilde{\lambda}_0 \widetilde{\tau}_{x,y}}$.

- Cheng and Mao (2015, 2017) proved that $\lim_{x\to\infty} \mathbb{E}e^{\widetilde{\lambda}_0 \widetilde{\tau}_{x,y}} = \prod_{n=0}^{\infty} \frac{\widehat{\lambda}_{n,y}}{-\widetilde{\lambda}_0+\widehat{\lambda}_{n,y}}$ and $\sum_{n=0}^{\infty} \widehat{\lambda}_{n,y}^{-1} < \infty$.
- By Mao (2006), we get $y_0 \in (0,\infty)$ satisfying $\widehat{\lambda}_{n,y_0} > \widetilde{\lambda}_0$ for all $n \in \mathbb{N}$. $\widehat{\lambda}_{n,y}$, $n \geq 1$ are the eigenvalues of $\widetilde{X}_t$ absorbed at y from right.
- Then we have $\lim_{x\to\infty} \widetilde{\psi}_0(x) < \infty$.

- L. J. Cheng and Y. H. Mao (2015). Eigentime identity for one-dimensional diffusion processes[J]. *Journal of Applied Probability*, 52(1):224-237.
- L. J. Cheng and Y. H. Mao (2017). Passage time distribution for one-dimensional diffusion processes. Preprint.
- Y. H. Mao, (2006). On the empty essential spectrum for markov processes in dimension one. *Acta Mathematica Sinica, English Series*, 22(3), 807-812.

Attraction domain

- We have the relationship

$$\mathbb{P}_x(X_t \in (0,y], t < \zeta) = \mathbb{P}_y(\widetilde{X}_t \in [x,\infty), t < \widetilde{\zeta}) \quad x - a.s.$$

- Then we have

$$\lim_{t\to\infty} e^{\lambda_0 t}\mathbb{P}_x(X_t \in A, \zeta > t) = \nu(A)\psi_0(x)\left(\int_0^\infty \psi_0(y)m(y)\mathrm{d}y\right).$$

  for all $x > 0$ and all Borel subsets $A \subseteq (0,\infty)$.
- By Zhang and He (2016), we get $e^{\lambda_0 t}\mathbb{P}_x(\zeta > t)$ is uniformly bounded in the variables t and x. Then $\lim_{t\to\infty} \mathbb{P}_\rho(X_t \in A | t < \zeta) = \nu(A)$ exists for any probability measure ρ on $(0,\infty)$.

Main results (2): ∞ a regular boundary

- Hypothesis (H-ER): 0 is an entrance boundary ($\mathbb{E}_0\tau_1 < \infty,\ \mathbb{E}_1\tau_0 = \infty$) and ∞ is a regular boundary ($\mathbb{E}_\infty\tau_1 < \infty,\ \mathbb{E}_1\tau_\infty < \infty$).

Theorem

Under hypothesis (H − ER), *there exists a unique QSD for the minimal L-diffusion process, which is given by*

$$\nu(\mathrm{d}x) = \frac{\psi_0(x)m(x)}{\int_0^\infty \psi_0(y)m(y)\mathrm{d}y}\mathrm{d}x, \tag{5}$$

where λ_0 is the minimal eigenvalue of generator $-L$ and ψ_0 is the eigenfunction associated to λ_0.
Moreover, the QSD attracts all initial probability distributions μ on $(0,\infty)$, that is, for any Borel subset $A \subseteq (0,\infty)$,

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in A|\zeta > t) = \nu(A).$$

Regular boundary

QSD's spectral representation:

Theorem

Under hypothesis (H − ER), *we have*

$$\nu(dx) = \frac{\prod_{n=1}^{\infty} \frac{-\lambda_0+\lambda_{n,x}}{\lambda_{n,x}} m(x)dx}{\int_0^{\infty} \prod_{n=1}^{\infty} \frac{-\lambda_0+\lambda_{n,x}}{\lambda_{n,x}} m(x)dx},$$

where $\lambda_{n,x}$ *is the eigenvalue of* $-L|_{(0,x)}$ *with state* x *as an absorbing boundary.*

Tools

- Spectral theory
- Spectral representation of the moment generating function

Spectral theory

- According to Cheng and Mao (2015), we conclude that

Theorem

Under hypothesis (H − ER), $-L$ *has a purely discrete spectrum. The eigenvalues*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \tag{6}$$

are simple. Let ψ_n *be an eigenfunction associated to* λ_n*, then the sequence* $\{\psi_n : n \geq 0\}$ *is an orthonormal basis of* $L^2(m)$*.* ψ_0 *can be chosen to be strictly positive on* $(0, \infty)$*.*

- According to Hölder inequality, we can get $\psi_0(x) \in L^1(m)$
- This gives QSD and its attraction domain.

- L. J. Cheng and Y. H. Mao (2015). Eigentime identity for one-dimensional diffusion processes[J]. *Journal of Applied Probability*, 52(1):224-237.

Spectral representation for QSD

- Our aim : spectral representation for ψ_0.

- By the iteration method, we get $\psi_0(x) = \psi_0(y)\mathbb{E}_x e^{\lambda_0 \tau_y}$, for all $0 < x < y < \infty$.
- Cheng and Mao (2017) proved that under hypothesis (H-ER)

$$\mathbb{E}_x e^{\lambda_0 \tau_y} = \frac{\prod_{n=1}^{\infty} \frac{\lambda_{n,y}}{-\lambda_0 + \lambda_{n,y}}}{\prod_{n=1}^{\infty} \frac{\lambda_{n,x}}{-\lambda_0 + \lambda_{n,x}}}, \quad \forall\, 0 < x < y < \infty,$$

  where $\lambda_{n,x}$ is the eigenvalue of $-L|_{(0,x)}$ with state x as a absorbing boundary and $\lambda_0 = \lim_{x\to\infty} \lambda_{0,x}$.
- This gives the spectral representation for ψ_0, moreover for QSD.

Thank you!