SRB Measures, Entropy, and Horseshoes for Infinite Dimensional Dynamical Systems

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1. Introduction

Mathematical Models

\[ \frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n \]

A map: \( F : X \rightarrow X. \)

Question: What is long term behavior?

\[ x(t, x_0) \rightarrow? \quad \text{as } t \rightarrow \infty. \]

\[ x_n = F^n(x_0) \rightarrow? \quad \text{as } n \rightarrow \infty. \]
Questions:

**Orbits:** \( \{x, F(x), F^2(x), \cdots \} \)

- Can we predict the behavior of \( x_n = F^n(x_0) \)?
- Given a set \( A \subset X \), how often do \( F^n(x_0) \) visit \( A \)?
- Can the behaviors be observed?
1. Introduction

Basic Problems:

Stability, instability, and complicated dynamics

Three fundamental quantities:

- Lyapunov exponents.
  - Lyapunov

- Entropy.
  - Kolmogorov, Sinai

- SRB Measure.
  - Sinai, Ruelle, Bowen
2. Lyapunov Exponents

- The Linear random dynamical system in $X$

$$\Phi(n, \omega)$$

- **Basic Problem:** Find all Lyapunov exponents

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|\Phi(n, \omega)v_k\| = \lambda_k$$

**Multiplicative Ergodic Theorem**

**Existence of Lyapunov Exponents and the associated invariant subspaces**
3. SRB Measures

Invariant Measures:

◊ Let \((X, \mathcal{F}, \mu)\) be a measurable space.

◊ Let \(F : X \to X\) be a measurable map.

• \(\mu\) is an \textbf{invariant measure} if

\[ \mu(F^{-1}(A)) = \mu(A) \]

• \(\mu\) is \textbf{ergodic} if for any \(A \subset X\),

\[ F^{-1}(A) = A \]

then \(\mu(A) = 0\) or \(\mu(A) = 1\)
3. SRB Measures

Basic Facts:

### Poincare Recurrence Theorem:

Assume

- \((X, \mathcal{F}, \mu)\) — Probability space.
- \(F : X \to X\) be a measure-preserving

Then, almost every point of \(A \in \mathcal{F}\) is recurrent.

- \(x \in A\) is recurrent if \(\exists 0 < n_1 < n_2 < \cdots\) such that
  \[F^{n_k}(x) \in A\]

- Let \(n_r(x) = \#\{k \leq n \mid F^k(x) \in A\}\).

\[
\lim_{n \to \infty} \frac{n_r(x)}{n} = ?
\]
Basic Facts:

- Let \( \phi \) be a characteristic function of \( A \).

\[
\phi(x) = \begin{cases} 
  1, & x \in A \\
  0, & x \notin A.
\end{cases}
\]

\[
\frac{n_r(x)}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k(x)).
\]

Time average of observation \( \phi \).
Basic Facts:

◊ Birkhoff Ergodic Theorem:

Let

- \((X, \mathcal{F}, \mu)\) be a probability space,
- \(F : X \to X\) be ergodic measure-preserving,
- \(\phi \in L^1_\mu(X, \mathbb{R})\), a observable function.

Then, for almost every \(x \in X\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k(x)) = \int_X \phi \, d\mu.
\]

Time average = Space average
3. SRB Measures

**SRB (Sinai-Ruelle-Bowen) Measure:**

- Let $M$ be a compact manifold.

  $$F : M \to M,$$  
  
  **diffeomorphism**

  $$\mu,$$ **invariant ergodic probability measure.**

- $\mu$ is a SRB measure if $\exists$ a **positive Lebesque measure set** $B \subset M$ such that for $\phi \in C^0(M, \mathbb{R})$

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k(x)) = \int_M \phi \, d\mu, \quad \forall \, x \in B.
\]
3. SRB Measures

Basin of Attraction

Positive Lebesque measure

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k(x)) = \int_M \phi \, d\mu, \quad \forall \, x \in B.
\]
Alternative Definition of SRB Measure:

Eckmann and Ruelle

◊ Attractor $\Lambda \subset M$ of $F$ with basin $U$.
  - $\Lambda$, compact, invariant.
  - $\cap_{n \geq 0} F^n(U) = \Lambda$.

◊ Invariant Measure $\mu$ is supported on $\Lambda$.
  - $F$ has positive Lyapunov exponents.
  - $\mu$ has absolutely continuous conditional measures on unstable manifolds with respect to the Lebesgue measures.
3. SRB Measures

◊ If $\mu$ is hyperbolic

then two definitions are same

Pesin, Pugh-Sub
3. SRB Measures

Background

❤ Existence of SRB for hyperbolic systems.


◊ Bowen and Ruelle (1975), Ruelle (1976):
  Axiom A diffeomorphisms and flows.
SRB for non-uniformly hyperbolic systems.

◊ Pesin (1977): \textit{diffeomorphisms preserving smooth measures}.

◊ Benedicks and Young (1992): \textit{Hénon attractors}.

◊ Wang and Young (2008): \textit{Rank-one attractors}.

◊ …
3. SRB Measures

 mensajes

SRB for partially hyperbolic systems.

◊ Pesin and Sinai (1982).
◊ Bonatti and Viana (2000).
◊ Alves, Bonatti, and Viana (2000).
◊ Cowieson and Young (2005).
◊ ...

SRB for effectively hyperbolic systems.

◊ Climenhaga, Dolgopyat, and Pesin (2016).
3. SRB Measures

❤ SRB for 1-d dynamical systems.
◊ Misiurewicz, Jackobson, Bruin, Shen, van Strien, ...

❤ SRB for ordinary differential equations.
◊ Guckenheimer, Wechselberger, and Young (2006)

2-d singularly perturbed time-periodic ODE


2-d ODEs with a homoclinic orbit driven by a periodic forcing.
4. SRB measure for PDEs

❤ SRB for Partial Differential Equations.

♦ L, Wang, and Young, 2013, Memiors of AMS

\[
\begin{cases}
    u_t = D \Delta u + F(u, \mu), \ x \in \Omega \subset \mathbb{R}^N \\
    u = 0, \text{ or } \frac{\partial u}{\partial n} = 0, \ \text{on} \ \partial \Omega
\end{cases}
\]

(A1) \( \mu = 0 \) is a Hopf bifurcation point
Example. Brusselator System:

\[
\begin{align*}
    v_t &= d_1 v_{xx} + (b - 1)v + a^2 w + ba^{-1}v^2 + 2awv + v^2w, \\
    w_t &= d_2 w_{xx} - bv - a^2 w - (ba^{-1}v^2 + 2awv + v^2w)
\end{align*}
\]

where \( x \in (0, 1) \), \( t > 0 \) and \( v, w \) are concentrations of two chemicals. \( a, b, d_1, d_2 \) are parameters.

- Dirichlet Boundary Conditions
- Neumann Boundary Conditions
Periodically Forced Equation

\[ u' = A_\mu u + f(u, \mu) + \rho \Phi(u) P_{T,p}(t) \]

\[ P_{T,p}(t) = \begin{cases} 
\frac{1}{p} & 0 \leq t < p \\
0 & p \leq t < T.
\end{cases} \]
Theorem (L, Wang, and Young, 2013).

If \((1)\) \( \rho, \rho << 1 \) and \( C\rho^2 < \mu < 10C\rho^2 \)

\((2)\) the twist constant

\[
\tau = \left| \frac{Im(k_1(0))}{Re(k_1(0))} \right| >> 1
\]

then there exists a positive measure set \( \Lambda \subset [0, \infty) \) such that for \( T \in \Lambda \)

\[
F_T(x) = u(T, x)
\]

has a strange attractor with a SRB measure.
Properties of A Strange Attractor:

(1) Existence of SRB Measure.

∃ an invariant measure (SRB measure) \( \nu \) supported on \( \mathcal{A} \) such that for a.e. \( x \in \mathcal{B} \) and every continuous function \( \phi : \mathcal{B} \to \mathbb{R} \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F_T^i x) = \int \phi \, d\nu.
\]
(2) Statistical Properties of Observations.

- Center Limit Theorem holds

For every Hölder continuous function \( \phi : B \rightarrow \mathbb{R} \), if \( \int \phi \, d\nu = 0 \), then

\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi \circ F_T^i
\]

converges in distribution to the normal distribution.
(3) Chaotic Behavior.

\[ F_T|_A \text{ has a horseshore} \]

(4) Physical Measure.

Absolute continuity of stable foliations.
Geometric Mechanism in Production of Chaos
Geometric Mechanism in Production of Chaos

t=0+
t=1
t=3
5. Partially Hyperbolic Systems

♡ **Infinite Dimensional Dynamical Systems.**

- $X$ - Hilbert Space, or Banach Space

Assume:

- $f : X \to X$ is a $C^2$ map.
- $f$ has a partially hyperbolic attractor $\Lambda$ with basin $U$.

**Theorem** (Lian, Liu, and L, 2015, 2016).

*There exists at least one SRB measure.*
5. Partially Hyperbolic Systems

 gratuitement Hyperbolic Attractor \(-\Lambda\).

(1) **Invariant Splitting:** For each \(x \in \Lambda\),

- \(X = E^u_x \oplus E^{cs}_x\);
- \(Df_x E^u_x = E^u_{f(x)}, \quad Df_x E^{cs}_x \subset E^{cs}_{f(x)};\)
- \(E^u_x\) and \(E^{cs}_x\) are \(C^0\) on \(x\) and \(\dim E^u_x > 0\),

(2) **Partial Hyperbolicity:**

- \(|Df_x \eta| \leq |\eta|, \quad \forall \ \eta \in E^{cs}_x.\)
- \(|Df_x \xi| \geq e^{\lambda_0} |\xi|, \quad \forall \ \xi \in E^u_x,\)
6. Entropy and Horseshoe

.measure-theoretic (metric) entropy.

Kolmogorov (1950’s), Sinai.

It measures the rate of increase in dynamical complexity as the system evolves with time.

.topological entropy.

Adler, Konheim and McAndrew (1965)

It measures the exponential growth rate of the number of distinguishable orbits as time advances.

.variational principle.

\[ h_{\text{top}} = \sup \{ h_\mu : \mu \in \mathcal{P}_f(X) \} \]
6. Entropy and Horseshoe

- **Problem:**
  
  What is the implication of positive entropy of a dynamical system?

- **Sinai, 1964**

  An ergodic measure-preserving map $T$ on a probability space $(X; \mathcal{F}; \mu)$.

  If its measure theoretic entropy is positive, then $T$ contains a factor which is semi-conjugate to a shift map.
Finite Dimensional Dynamical Systems


Let $M$ - 2D, compact, $C^\infty$ Riemannian manifold, $f \in Diff^{1+\alpha}(M)$.

If $h_{top}(f) > 0$, then $\exists k \in \mathbb{N}$ and a closed $f^k$-invariant set $\Gamma$ such that $f^k|_\Gamma$ has Horseshoe of two symbols.

$$\cdots -k \ 0 \ k \ 2k \ 3k \ 4k \ \cdots$$

$$\cdots 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ \cdots$$

bi-infinite sequence of 2 symbols
Without assuming any hyperbolicity, Glasner, Kolyada, and Maass showed that

**Theorem.** [BGKM, J. Reine Angew. Math. 2002]

Let $X$ be a compact metric space and

$$T : X \to X$$

be a homeomorphism.

If $h_{\text{top}}(T) > 0$, then $(X,T)$ is chaotic in the sense of Li-Yorke.
Definition of Chaos. (Li-Yorke, 1975)

\[ \exists \kappa > 0, E \subset X \text{ which is a union of countably many Cantor sets,} \]

such that for every pair \( x_1, x_2 \) of distinct points in \( E \), we have

\[
\liminf_{n \to +\infty} d(\phi(n, \omega)(x_1), \phi(n, \omega)(x_2)) = 0,
\]

\[
\limsup_{n \to +\infty} d(\phi(n, \omega)(x_1), \phi(n, \omega)(x_2)) \geq \kappa.
\]

More some results about positive entropy and chaos: See S.Num, ETDS 2003; W.Huang and X.Ye, ISR 2006; W.Huang, CMP 2008;
• Dynamical Systems in Hilbert Spaces

◊ Nonuniformly hyperbolic differentiable maps in a Hilbert space.

Lian - Young, Ann. Henri Poincaré, 2011

◊ Nonuniformly hyperbolic semiflow in a Hilbert space.

Lian - Young, JAMS, 2012

Positive entropy implies the existence of horseshoes.
6. Entropy and Horseshoe

- Problem:

How to characterize the chaotic behavior of orbits **topologically or geometrically** (in terms of horseshoe) in the presence of **ONLY positive entropy**?

without assuming any hyperbolic structures.
5. Entropy and Horseshoe

Setting:

Diamond Infinite Dimensional RDS.

\[ \phi(n, \omega, x), \quad n \geq 0 \]

Diamond Random Invariant Set

\[ A \subset \Omega \times X, \quad \text{measurable} \]

\[ \phi(n, \omega)A(\omega) = (A(\theta_n\omega)) \quad P - a.s., \]

where \( A(\omega) = \{ x \in X \mid (\omega, x) \in A \} \)
5. Entropy and Horseshoe

**Theorem:** (Huang-L, 2017, CPAM)

Let $\mathcal{A}(\omega)$ be a random invariant set

If the topological entropy is positive, i.e.,

$$h_{\text{top}}(\phi, \mathcal{A}) > 0,$$

then

(1) the dynamics of $\phi$ restricted to $\mathcal{A}$ is chaotic;

(2) the dynamics of $(\phi, \mathcal{A})$ has a weak horseshoe.
Let $\mathcal{A}$ be a global attractor of deterministic PDEs.

$$\frac{du}{dt} = Au + F(t, u)$$

If the topological entropy is positive, i.e.,

$$h_{\text{top}}(u, \mathcal{A}) > 0,$$

then, $(u, \mathcal{A})$ has a full horseshoe.
Corollary

Let \( A \) be a global attractor of deterministic PDEs.

\[
\frac{du}{dt} = Au + F(t, u)
\]

If the topological entropy is positive, i.e.,

\[
h_{\text{top}}(u, A) > 0,
\]

then, \((u, A)\) has a full horseshoe.
A Horseshoe with two symbols:

∃ an infinite subsequence of integers with positive density in \( \mathbb{N} \): \( 0 < n_1 < n_2, \ldots, < n_k < \ldots \)

such that for any infinite sequence of 2 symbols

\[
\begin{array}{ccccccc}
  & n_1 & n_2 & n_3 & \ldots & n_k & \ldots \\
2 & 1 & 2 & \ldots & 1 & \ldots
\end{array}
\]
5. Entropy and Horseshoe

✿ Pesin Entropy Formula:

\[ h_\mu(f) = \int \sum_{\lambda_i > 0} \lambda_i \dim E_i \, d\mu \]

Entropy = Sum of Positive Lyapunov Exponents

The Pesin formula holds if and only if $\mu$ is a SRB measure.

Ledrappier and Young, Li and Shu
Final Remark:

- What is chaos?
- How to describe a chaotic systems?
- Sensitivity with respect to initial data.
  Positive Lyapunov exponents.
  Instability can generate random phenomena.
  Poincare, Birkhoff, ...
- Topologic horseshoe.
  Smale
- Positive measure-theoretic entropy.
  Kolmogorov, Sinai
- **SRB Measure.** Sinai, Ruelle, Bowen

  Describe DS in terms of the average or statistical properties of their “typical” orbits.

  Physical measure.

  Positive Lyapunov exponents, a.e.

- **Li-Yorke Chaos**

  \[ \exists \kappa > 0, \ E \subset X \text{ which is a union of countably many Cantor sets, such that for } x_1 \neq x_2 \in E \]

  \[
  \liminf_{n \to +\infty} d(\phi^n(x_1), \phi^n(x_2)) = 0, \\
  \limsup_{n \to +\infty} d(\phi^n(x_1), \phi^n(x_2)) \geq \kappa.
  \]
Thank you!