Error bounds on augmented truncation approximations of invariant probability vectors for Markov chains

Liu Yuanyuan (刘源远)

School of Mathematics and Statistics,
Central South University (中南大学), Changsha

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Introduction

Let $\{X_n, n \geq 1\}$ be a discrete-time Markov chain (DTMC) with transition matrix $P = (P(i,j))$ on the countable state space $\mathbb{E} = \{0, 1, 2, \cdots\}$.

▷ Suppose that X_n is irreducible, positive recurrent with the unique invariant probability vector $\boldsymbol{\pi}^T = (\pi(0), \pi(1), \cdots)$.

▷ Usually it is not easy to compute $\boldsymbol{\pi}^T$ since the transition matrix is infinite-dimensional.

▷ We consider numerical computation of $\boldsymbol{\pi}^T$.

▷ The technique of augmented truncation approximation is a powerful tool for numerically computing the invariant probability vector π^T.

▷ The scheme of an augmented truncation approximation is specified as follows.

◇ First we let ${}_{(n)}P$ be the $(n+1) \times (n+1)$ northwest corner truncation of P.

◇ Then we augment the truncated elements of P to ${}_{(n)}P$ to form a stochastic matrix ${}_{(n)}\tilde{P}$ on the state space ${}_{(n)}\mathbb{E} = \{0, 1, \cdots, n\}$, whose stationary probability vector is assumed to exist, and is denoted by ${}_{(\boldsymbol{n})}\boldsymbol{\pi}^{\boldsymbol{T}}$.

In order to use $_{(\boldsymbol{n})}\boldsymbol{\pi}^T$ to approximate $\boldsymbol{\pi}^T$, we need to

▷ (i) establish the convergence of $_{(\boldsymbol{n})}\boldsymbol{\pi}^T$ to $\boldsymbol{\pi}^T$.

▷ (ii) provide theoretically guaranteed bounds on the difference between $_{(\boldsymbol{n})}\boldsymbol{\pi}^T$ and $\boldsymbol{\pi}^T$.

To perform the algebraic operations between the resulting matrices and original matrices, we extend $_{(n)}\tilde{P}$ and $_{(\boldsymbol{n})}\boldsymbol{\pi}^T$ from $_{(n)}\mathbb{E}$ to $\mathbb{E}$.

We still adopt the same notations of $_{(n)}\tilde{P}$ and $_{(\boldsymbol{n})}\boldsymbol{\pi}^T$.

Let $\nu_{i,n}(\cdot)$ be a probability distribution on ${}_{(n)}\mathbb{E}$, which depends on n and i. Define

$$
{}_{(n)}\tilde{P}(i,j) = \begin{cases} P(i,j) + \sum_{m>n} P(i,m)\nu_{i,n}(j), & \text{if } i \in \mathbb{E}, 0 \le j \le n, \\ 0, & \text{if } i \in \mathbb{E}, j > n, \end{cases}
$$

and ${}_{(n)}\boldsymbol{\pi}^T = ({}_{(n)}\pi(0), \cdots, {}_{(n)}\pi(n), 0, \cdots)$.

When $\nu_{i,n}(\cdot) = \nu_n(\cdot)$ only depends on n, we have the linear augmentation.

For any $0 \le k \le n$, let $\nu_n(k) = 1$ and $\nu_n(j) = 0$ for any $j \ne k$, then we have the special $(k+1)$th column augmentation, whose transition matrix and invariant probability vector are denoted by ${}_{(n)}\tilde{P}_k$ and ${}_{(n)}\boldsymbol{\pi}_k^T$, respectively.

For a vector $\boldsymbol{V} \geq \boldsymbol{e}$, $\boldsymbol{e}$ is a column vector of ones, we are interested in finding the upper bound $H(n, \boldsymbol{V})$ in V-norm as follows

$$\|{}_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \leq H(n, \boldsymbol{V}), \tag{1}$$

where $\|\boldsymbol{\mu}^T\|_{\boldsymbol{V}} = \sup_{\boldsymbol{g}:|\boldsymbol{g}|\leq \boldsymbol{V}} |\boldsymbol{\mu}^T \boldsymbol{g}| = \sum_{i\in\mathbb{E}} |\mu(i)|V(i)$ denotes the $\boldsymbol{V}$-norm for the row vector $\boldsymbol{\mu}^T$.

When $\boldsymbol{V} \equiv \boldsymbol{e}$, the $\boldsymbol{V}$-norm becomes the total variation norm.

The bound (1) enables us to compute the steady performance measure $\boldsymbol{\pi}^T\boldsymbol{V}$, since

$$|{}_{(n)}\boldsymbol{\pi}^T\boldsymbol{V} - \boldsymbol{\pi}^T\boldsymbol{V}| \leq \|{}_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}}.$$

For convergence in total variation norm:

▷ Seneta (1980): convergence holds iff $\{_{(n)}\boldsymbol{\pi}^T, n \geq 1\}$ is tight;

▷ Gibson and Seneta (1987): convergence holds for upper-Hessenberg matrix and Markov matrix.

▷ Liu and Zhao (1995): the censored MC is the best.

▷ Tweedie (1998): addressed the two issues well for the first and the last column augmentation when P is a Markov matrix or P is geometrically ergodic and monotone. We call his method as the *ergodicity method*.

▷ L. (2010): investigated an arbitrary augmentation and truncation bounds for polynomially ergodic and monotone MCs.

▷ Block-monotone DTMCs

◇ Li and Zhao (2000): the last-column-block-augmented truncation (LCBA) is the best in a certain sense.
◇ Masuyama (2015): developed error bounds for the LCBA for geometrically ergodic chains.

Convergence in V-norm:

▷ Tweedie (1998): also V-norm bounds for monotone and geometrically ergodic MCs.

▷ L. Tang and Zhao (2015): the best augmentation in total variation norm is not necessarily best in V-norm.

▷ Masuyama (2016): adopted *the perturbation method* to investigate the error bounds in V-norm for the LCBA truncation under drift conditions for continuous-time matrix-analytic models.

A Lot of literature on perturbation bounds in V-norm:

- ▷ Katashov (1986, 1996), Mouhoubi and Aissani (2010)
- ▷ Mitrophanov (2004, 2006), Heidergott etal. (2010)
- ▷ L (2012, 2015, 2017),
- ▷ $\cdots$

We will present several types of truncation bounds through *the perturbation method* in this talk.

In the long run, we are motivated to develop theoretically guaranteed augmented truncation approximation algorithms for computing the invariant probability vectors for high-dimensional Markov chains.

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Poisson's equation

The bound is based on Poisson's equation:

$$(I - P)\tilde{\boldsymbol{g}} = \bar{\boldsymbol{g}}, \tag{2}$$

where $\bar{\boldsymbol{g}} = \boldsymbol{g} - \boldsymbol{\pi}^T \boldsymbol{g}\boldsymbol{e}$. The forcing function $\boldsymbol{g}$ satisfies that $\boldsymbol{\pi}^T|\boldsymbol{g}| < \infty$. $\tilde{\boldsymbol{g}}$ is the solution to Poisson's equation.

Let $\Delta = {}_{(n)}\tilde{P} - P$, from (2), we can obtain the identity

$$_{(\boldsymbol{n})}\boldsymbol{\pi}^T \Delta \tilde{\boldsymbol{g}} = _{(\boldsymbol{n})} \boldsymbol{\pi}^T [(I - P)\tilde{\boldsymbol{g}}] = (_{(\boldsymbol{n})}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T)\boldsymbol{g}.$$

This identity enables us to estimate the augmented truncation bounds if we can bound $\tilde{\boldsymbol{g}}$ well.

D $(\boldsymbol{V}, \lambda, b, C)$: Suppose that there exist a finite set C, positive constants $b < \infty, \lambda < 1$ and finite column vectors $\boldsymbol{V} \geq \boldsymbol{e}$ such that

$$P\boldsymbol{V} \leq \lambda \boldsymbol{V} + bI_C.$$

To investigate the convergence of the augmented truncation, define

$$\Delta_n(i, \boldsymbol{V}) = \sum_{j>n} P(i,j)(V(n) + V(j)),$$

Theorem 1 Suppose that **D** $(\boldsymbol{V}, \lambda, b, C)$ holds for a non-decreasing function $\boldsymbol{V}$ and an atom $C = \{\alpha\}$. Then for an arbitrary augmentation

$$\|{}_{(\boldsymbol{n})}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \leq H_1(n, \boldsymbol{V}),$$

where

$$H_1(n, \boldsymbol{V}) = \kappa_1 \sum_{i=0}^{n} {}_{(n)}\pi(i)\Delta_n(i, \boldsymbol{V}),$$

and

$$\kappa_1 = \frac{1 + \boldsymbol{\pi}^T \boldsymbol{V}}{1 - \lambda} \leq \frac{1 - \lambda + b}{(1 - \lambda)^2}.$$

Moreover, $H_1(n, \boldsymbol{V}) \to 0$ as $n \to \infty$, and

$$\|{}_{(\boldsymbol{n})}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \to 0, n \to \infty.$$

Remarks:

▷ (i) The drift condition $\mathbf{D}\,(\boldsymbol{V}, \lambda, b, C)$ for an atom C implies $|\tilde{g}| \leq (1 + \boldsymbol{\pi}^T \boldsymbol{V})V$ for a function g such that $|g| \leq cV$.

▷ (ii) The condition that V is increasing is a key to deal with an arbitrary augmentation.

▷ (iii) The results can hold in a more general setting, e.g.

◇ under f-modulated drift condition $P\boldsymbol{V} \leq \boldsymbol{V} - \boldsymbol{f} + bI_C$;

◇ under a small set instead of an atom.

Based on Theorem 1, we can give theoretically guaranteed upper bounds on the truncation size for a given ε as follows.

- ▷ M(a) Find $\boldsymbol{V}$, λ, b, C such that $\mathbf{D}\ (\boldsymbol{V}, \lambda, b, C)$ holds.
- ▷ M(b) Calculate κ_1.
- ▷ M(c) For a given n calculate ${}_{(\boldsymbol{n})}\boldsymbol{\pi}^T$ and assess whether

$$H_1(n, \boldsymbol{V}) \leq \varepsilon. \tag{3}$$

▷ M(d) If (3) is not satisfied, increase the value of n and repeat M(c) until (3) is satisfied.

Residual matrix

Define a suitable residual matrix T by

$$U = P - \boldsymbol{h}\phi^T,$$

$\boldsymbol{h}$ and ϕ^T are respectively non-negative bounded column vector and non-negative row vector such that T is nonnegative. For a matrix B, define its $\boldsymbol{V}$-norm by

$$\|B\|_{\boldsymbol{V}} = \sup_{i \in E} \frac{\sum_j |B(i,j)| V(j)}{V(i)}.$$

$\mathbf{U}(\boldsymbol{V}, \lambda)$: Suppose that there exist a finite vector $\boldsymbol{V}$, $\boldsymbol{V} \geq \boldsymbol{e}$ and a positive constant λ, $\lambda < 1$ such that $\|P\|_{\boldsymbol{V}} < \infty$ and $\|U\|_{\boldsymbol{V}} \leq \lambda$.

Theorem 2 If $\mathbf{U}(\boldsymbol{V}, \lambda)$ holds for a non-decreasing function $\boldsymbol{V}$ and $\phi^T \boldsymbol{V} < \infty$, then for an arbitrary augmentation

$$\|{}_{(\boldsymbol{n})}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \leq H_2(n, \boldsymbol{V}),$$

where

$$H_2(n, \boldsymbol{V}) = \kappa_2 \sum_{i=0}^{n} {}_{(n)}\pi(i)\Delta_n(i, \boldsymbol{V})$$

and

$$\kappa_2 = \frac{1 + \boldsymbol{\pi}^T \boldsymbol{V}}{1 - \lambda} \leq \frac{1 - \lambda + (\phi^T \boldsymbol{V})(\boldsymbol{\pi}^T \boldsymbol{h})}{(1 - \lambda)^2}.$$

Moreover, $H_2(n, \boldsymbol{V}) \to 0$ as $n \to \infty$, and

$$\|{}_{(\boldsymbol{n})}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \to 0, n \to \infty.$$

Remarks:

▷ 1. A starting point of the proof is

$$ {}_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T = {}_{(n)}\boldsymbol{\pi}^T \Delta \sum_{k=0}^{\infty} U^k (I - \Pi). $$

▷ 2. The results hold in a more general setting, e.g. for $U = P^m - \boldsymbol{h}\phi^T$ for some $m \geq 1$.

▷ 3. There are various choices of the vectors $\boldsymbol{h}$ and ϕ^T such that T is nonnegative.

$\diamond$ Let $\phi^T = (P(0,0), P(0,1), \cdots)$ denote the first row of P and let $\boldsymbol{h} = (1, 0, 0, \cdots)^T$, then the matrix $U = P - \boldsymbol{h}\phi^T$ is obtained by setting the elements in the first row of P to zeros and keeping the other elements unchanged.

$\diamond$ Let $\phi^T = (1, 0, 0, \cdots)$ and $\boldsymbol{h} = (P(0,0), P(1,0), \cdots)^T$. Then the matrix $U = P - \boldsymbol{h}\phi^T$ is obtained by setting the elements in the first column of P to zeros and keeping the other elements unchanged.

$\triangleright$ 4. Actually, if $\mathbf{D}(\boldsymbol{V}, \lambda, b, C)$ holds for an atom C, then we can take appropriate $\boldsymbol{h}$ and ϕ^T such that $\mathbf{U}(\boldsymbol{V}, \lambda)$ holds for the same $\boldsymbol{V}$ and λ.

Ergodicity coefficient

Define the set $M_0 = \{\boldsymbol{\mu} : \|\boldsymbol{\mu}\|_{\boldsymbol{V}} < \infty, \boldsymbol{\mu}^T \boldsymbol{e} = 0\}$. Let $\boldsymbol{V} \geq \boldsymbol{e}$, the $\boldsymbol{V}$-norm ergodicity coefficient $\Lambda(B)$ of a matrix $B = (b_{ij})$ is defined by

$$\Lambda(B) = \sup\{\|\boldsymbol{\mu}^T B\|_{\boldsymbol{V}} : \|\boldsymbol{\mu}^T\|_{\boldsymbol{V}} \leq 1, \boldsymbol{\mu} \in M_0\}.$$

The explicit representation of $\Lambda(B)$ is given by

$$\Lambda(B) = \sup_{i,j \in \mathbb{E}} \frac{\sum_{k \in \mathbb{E}} |b_{ik} - b_{jk}| V(k)}{V(i) + V(j)}.$$

Let $\boldsymbol{V} \equiv \boldsymbol{e}$, then $\Lambda(B)$ becomes the classical ergodicity coefficient $\tau(B)$.

Theorem 3 Suppose that $\boldsymbol{V}$ is a non-decreasing function, $\boldsymbol{\pi}^T \boldsymbol{V} < \infty$, $\|P\|_{\boldsymbol{V}} < \infty$, and $\Lambda(P) \leq \rho < 1$. Then for an arbitrary augmentation

$$\|_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \leq H_3(n, \boldsymbol{V}),$$

where

$$H_3(n, \boldsymbol{V}) = \kappa_3 \sum_{i=0}^{n} {}_{(n)}\pi(i)\Delta_n(i, \boldsymbol{V}).$$

and

$$\kappa_3 = \frac{1}{1-\rho}.$$

Moreover, if **D** $(\boldsymbol{V}, \lambda, b, C)$ holds, then $\boldsymbol{\pi}^T \boldsymbol{V} < \infty$, $\|P\|_{\boldsymbol{V}} < \infty$, $H_3(n, \boldsymbol{V}) \to 0$ as $n \to \infty$, and

$$\|_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \to 0, n \to \infty.$$

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Remarks:

▷ 1. A starting point of the proof is

$$\begin{aligned}\|_{(n)}\pi^T - \pi^T\|_v &= \|_{(n)}\pi^T \Delta(\sum_{t=0}^{\infty} P^t - \Pi)\|_v \\ &\leq \|_{(n)}\pi^T \Delta\|_v \cdot \sum_{t=0}^{\infty} \Lambda(P^t).\end{aligned}$$

▷ 2. The results hold under more general assumptions, e.g.

$\Lambda(P^m) \leq \rho_m < 1$ for some positive integer m.

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Tweedie (1998): ergodicity method

Suppose that (i) P is aperiodic, (ii) P is monotone, and (iii) P satisfies $\mathbf{D}(\boldsymbol{V}, \lambda, b, C)$ for $C = \{0\}$ and a non-decreasing $\boldsymbol{V}$.

He used the triangle inequality

$$\begin{aligned} \|{}_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} &\leq \|{}_{(n)}\tilde{P}^k(i,\cdot) - {}_{(n)}\boldsymbol{\pi}^T\|_{\boldsymbol{V}} + \|P^k(i,\cdot) - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \\ &+ \|{}_{(n)}\tilde{P}^k(i,\cdot) - P^k(i,\cdot)\|_{\boldsymbol{V}} \end{aligned}$$

for any $i \in \mathbb{E}$ and any integer $k \geq 1$ and derived

$$\|{}_{(n)}\boldsymbol{\pi}_n{}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \leq \frac{4\lambda^k b}{1-\lambda} + D\sum_{i=0}^{n} {}_{(n)}\pi_n(i)\Delta_n(i,\boldsymbol{V}),$$

where $D = \sum_{s=0}^{k-1}(\lambda + b)^s$.

Example-1 Consider a M/G/1 queue with transition matrix

$$P = \begin{pmatrix} c_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \cdots \\ 0 & a_{-1} & a_0 & a_1 & \cdots \\ 0 & 0 & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $a_{-1} = (1-a)$, $a_j = (1-a)a^{j+1}, j \geq 0$ with $a = 0.25$.

Let $V(i) = 2^i$, $i \geq 0$ and $C = \{0\}$. Then we can calculate $\lambda = \frac{3}{4}$, $b = \frac{3}{8}$, $\boldsymbol{\pi}^T \boldsymbol{V} = \frac{4}{3}$, and the difference

$$T(n, \boldsymbol{V}) = |_{(\boldsymbol{n})}\boldsymbol{\pi}_{\boldsymbol{n}}^T \boldsymbol{V} - \boldsymbol{\pi}^T \boldsymbol{V}|.$$

The (Poisson-equation type) bound is given by

$$H_2(n, \boldsymbol{V}) = \frac{28}{3} \sum_{i=0}^{n} {}_{(n)}\pi_n(i) \Delta_n(i, \boldsymbol{V}).$$

Choosing $\varepsilon = 0.01$ and applying the procedures M*(a)-M*(d) in page 530 of Tweedie (1998) obtains

$$\begin{aligned} &\|{}_{(\boldsymbol{n})}\boldsymbol{\pi}_{\boldsymbol{n}}^{T} - \boldsymbol{\pi}^{T}\|_{\boldsymbol{V}} \\ &\le F_2(n, \boldsymbol{V}) := 0.0257 + \frac{283}{4} \sum_{i=0}^{n} {}_{(n)}\pi_n(i) \Delta_n(i, \boldsymbol{V}). \end{aligned}$$

n	$F_2(n, \boldsymbol{V})$	$H_2(n, \boldsymbol{V})$	${}_{(\boldsymbol{n})}\boldsymbol{\pi}_{\boldsymbol{n}}^{T}\boldsymbol{V}$	$T(n, \boldsymbol{V})$
10	0.2946	0.0355	1.3256	0.0077
14	0.0796	0.0071	1.3318	0.0015
18	0.0363	0.0014	1.3330	0.0003

Now we make a comparison of the three types of bounds in Sections 2 to 4 through two specific examples.

Example-2 For the M/G/1 queue in Example-1,

$$\begin{aligned} & H_1(n, \boldsymbol{V}) \\ & = H_2(n, \boldsymbol{V}) = \frac{28}{3} \sum_{i=0}^{n} {}_{(n)}\pi(i) \Delta_n(i, \boldsymbol{V}) \\ & < H_3(n, \boldsymbol{V}) = 16 \sum_{i=0}^{n} {}_{(n)}\pi(i) \Delta_n(i, \boldsymbol{V}). \end{aligned}$$

Example-3. Consider a single-birth process, taken from (M.F. Chen 1999), with transition probabilities

$$P_{00} = P_{01} = \frac{1}{2}; \quad P_{i,i+1} = \frac{1}{2}, P_{ij} = \frac{1}{2i}, i \geq 1, j \leq i-1.$$

Let $V(i) = i+1$ for $i \geq 1$ and $V(0) = 2$, and consider the last-column augmentation.

$$H_1(n, V) = 6(1 + \boldsymbol{\pi}^T \boldsymbol{V}) \cdot [_{(n)}\pi_n(n)(V(n) + V(n+1) + \frac{16}{3})],$$
$$H_2(n, \boldsymbol{V}) = 3(1 + \boldsymbol{\pi}^T \boldsymbol{V}) \cdot [_{(n)}\pi_n(n)(V(n) + V(n+1))],$$
$$H_3(n, \boldsymbol{V}) = \frac{70}{23} \cdot [_{(n)}\pi_n(n)(V(n) + V(n+1))].$$

Obviously,

$$H_1(n, \boldsymbol{V}) > H_2(n, \boldsymbol{V}) > H_3(n, \boldsymbol{V}).$$

Furthermore, since $\boldsymbol{\pi}^T \boldsymbol{V}$ can not be obtained directly, we calculate the value of ${}_{(\boldsymbol{n})}\boldsymbol{\pi}_{\boldsymbol{n}}^T \boldsymbol{V}$ as an approximation.

The following table lists the values of the bounds $H_i(n, \boldsymbol{V})$, $i = 1, 2, 3$ and ${}_{(\boldsymbol{n})}\boldsymbol{\pi}_{\boldsymbol{n}}^T \boldsymbol{V}$.

n	$H_1(n, \boldsymbol{V})$	$H_2(n, \boldsymbol{V})$	$H_3(n, \boldsymbol{V})$	${}_{(\boldsymbol{n})}\boldsymbol{\pi}_{\boldsymbol{n}}^T \boldsymbol{V}$
13	0.5933	0.1949	0.0282	2.7774
15	0.2070	0.0693	0.0100	2.7778

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Let $\{X(t), t \geq 0\}$ be a CTMC on $\mathbb{E} = \{0, 1, 2, \cdots\}$ with the generator $Q = Q(i,j)$, which is positive recurrent with the invariant probability vector $\boldsymbol{\pi}^T$.

Define the augmented truncation of Q as follows:

$$
{}_{(n)}\tilde{Q}(i,j) = \begin{cases} Q(i,j) + \sum_{m>n, m\neq i} Q(i,m)\nu_{i,n}(j), & i \in \mathbb{E}, 0 \leq j \leq n, \\ Q(i,i), & i = j \geq n+1, \\ 0, & \text{otherwise}, \end{cases}
$$

where $\nu_{i,n}(\cdot)$ is a probability vector on ${}_{(n)}\mathbb{E}$ that depends on n and i. Let ${}_{(\boldsymbol{n})}\boldsymbol{\pi}^T$ be the invariant probability vector of ${}_{(n)}\tilde{Q}$.

- If Q is bounded, i.e, $\sup_{i \in \mathbb{E}} Q(i) < \infty$, using the technique of uniformization and similar arguments in L (2012), most of the results in Sections 2-4 can be extended to CTMCs.
- Uniformization is invalid when Q is unbounded. Theorems 2 and 3 cannot be extended to CTMCs. However, Theorem 1 can be done as follows .
- For CTMCs, Poisson's equation is given by

$$Q\tilde{\boldsymbol{g}} = -\bar{\boldsymbol{g}}.$$

where $\bar{\boldsymbol{g}} = \boldsymbol{g} - \boldsymbol{\pi}^T \boldsymbol{g}\boldsymbol{e}$. We have the following identity

$${}_{(n)}\boldsymbol{\pi}^T \Delta\tilde{\boldsymbol{g}} = -{}_{(n)}\boldsymbol{\pi}^T (Q\tilde{\boldsymbol{g}}) = ({}_{(n)}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T)\boldsymbol{g}.$$

$\mathbf{D}'(\boldsymbol{V}, \lambda, b, C)$: Suppose that there exist a finite set C, positive constants λ, b, finite column vectors $\boldsymbol{V} \geq \boldsymbol{e}$ such that

$$Q\boldsymbol{V} \leq -\lambda \boldsymbol{V} + bI_C.$$

Theorem 4 If **D'** $(\boldsymbol{V}, \lambda, b, C)$ holds for $C = \{i_0\}$ and a non-decreasing function $\boldsymbol{V}$, then for an arbitrary augmentation

$$\|_{(\boldsymbol{n})}\boldsymbol{\pi}^T - \boldsymbol{\pi}^T\|_{\boldsymbol{V}} \leq H_4(n, V), \tag{4}$$

where

$$H_4(n, V) = \kappa_4 \sum_{i=0}^{n} {}_{(n)}\pi(i)\Delta_n(i, \boldsymbol{V})$$

and

$$\kappa_4 = \frac{1 + \boldsymbol{\pi}^T \boldsymbol{V}}{\lambda} \leq \frac{\lambda + b}{\lambda^2}.$$

Remarks

▷ 1. The corresponding bound under f-modulated drift condition can be derived, which improves Theorem 2.1 in Masuyama (2017) by relaxing the condition $\boldsymbol{\pi}^T \boldsymbol{V} < \infty$ and drops the factor $\boldsymbol{\pi}^T \boldsymbol{V}$ in his bound.

▷ 2. Theorem 4 parallels to Theorem 1 for DTMCs. However, we cannot expect that the upper bound always converges to zero as n tends to ∞ for CTMCs, which will be clarified by the subsequent example.

Example-4 Consider a special continuous-time birth-death process with the same birth and death rates: $b_0 = 1$, and $b_i = a_i = i^\gamma$, $i \geq 1$, where $\gamma > 1$. We can compute that $\pi(n) = \frac{\pi(0)}{n^\gamma}$. Performing the last column augmentation, we have

$$_{(n)}\pi_n(n) = \frac{\pi(n)}{\sum_{i=0}^{n} \pi(i)} = \frac{1}{n^\gamma(\sum_{i=1}^{n} \frac{1}{i^\gamma} + 1)}.$$

Applying Theorem 4 obtains

$$\sum_{i=0}^{n} {}_{(n)}\pi_n(i)\Delta_n(i, \boldsymbol{V}) = \frac{1}{\sum_{i=1}^{n} \frac{1}{i^\gamma} + 1}(V(j) + V(n)).$$

For any $\boldsymbol{V} \geq \boldsymbol{e}$, $H_4(n, V)$ does not converge to zero.

Thank you for your attention!