Error bounds on augmented truncation approximations of invariant probability vectors for Markov chains

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Introduction

Let \(\{X_n, n \geq 1\}\) be a discrete-time Markov chain (DTMC) with transition matrix \(P = (P(i,j))\) on the countable state space \(E = \{0, 1, 2, \cdots\}\).

\(\triangleright\) Suppose that \(X_n\) is irreducible, positive recurrent with the unique invariant probability vector \(\pi^T = (\pi(0), \pi(1), \cdots)\).

\(\triangleright\) Usually it is not easy to compute \(\pi^T\) since the transition matrix is infinite-dimensional.

\(\triangleright\) We consider numerical computation of \(\pi^T\).
The technique of augmented truncation approximation is a powerful tool for numerically computing the invariant probability vector $\pi^T$.

The scheme of an augmented truncation approximation is specified as follows.

First we let $(n)P$ be the $(n + 1) \times (n + 1)$ northwest corner truncation of $P$.

Then we augment the truncated elements of $P$ to $(n)\tilde{P}$ to form a stochastic matrix $(n)\tilde{P}$ on the state space $(n)E = \{0, 1, \cdots, n\}$, whose stationary probability vector is assumed to exist, and is denoted by $(n)\pi^T$. 
In order to use \((n)\pi^T\) to approximate \(\pi^T\), we need to

- (i) establish the convergence of \((n)\pi^T\) to \(\pi^T\).
- (ii) provide theoretically guaranteed bounds on the difference between \((n)\pi^T\) and \(\pi^T\).

To perform the algebraic operations between the resulting matrices and original matrices, we extend \((n)\tilde{P}\) and \((n)\pi^T\) from \((n)\mathcal{E}\) to \(\mathcal{E}\).

We still adopt the same notations of \((n)\tilde{P}\) and \((n)\pi^T\).
Let $\nu_{i,n}(\cdot)$ be a probability distribution on $(n)\mathbb{E}$, which depends on $n$ and $i$. Define

$$(n)\tilde{P}(i,j) = \begin{cases} 
P(i,j) + \sum_{m>n} P(i,m)\nu_{i,n}(j), & \text{if } i \in \mathbb{E}, 0 \leq j \leq n, \\
0, & \text{if } i \in \mathbb{E}, j > n,
\end{cases}$$

and $(n)\pi^T = ((n)\pi(0), \cdots, (n)\pi(n), 0, \cdots)$.

When $\nu_{i,n}(\cdot) = \nu_n(\cdot)$ only depends on $n$, we have the linear augmentation.

For any $0 \leq k \leq n$, let $\nu_n(k) = 1$ and $\nu_n(j) = 0$ for any $j \neq k$, then we have the special $(k+1)$th column augmentation, whose transition matrix and invariant probability vector are denoted by $(n)\tilde{P}_k$ and $(n)\pi^T_k$, respectively.
For a vector $V \geq e$, $e$ is a column vector of ones, we are interested in finding the upper bound $H(n, V)$ in $V$-norm as follows

$$\|(n)\pi^T - \pi^T\|_V \leq H(n, V), \quad (1)$$

where $\|\mu^T\|_V = \sup_{|g| \leq V} |\mu^T g| = \sum_{i \in E} |\mu(i)|V(i)$ denotes the $V$-norm for the row vector $\mu^T$.

When $V \equiv e$, the $V$-norm becomes the total variation norm.

The bound (1) enables us to compute the steady performance measure $\pi^T V$, since

$$\|(n)\pi^T V - \pi^T V| \leq \|(n)\pi^T - \pi^T\|_V.$$
For convergence in total variation norm:

- **Seneta (1980):** convergence holds iff \( \{(n)\pi^T, n \geq 1\} \) is tight;
- **Gibson and Seneta (1987):** convergence holds for upper-Hessenberg matrix and Markov matrix.
- **Liu and Zhao (1995):** the censored MC is the best.
- **Tweedie (1998):** addressed the two issues well for the first and the last column augmentation when \( P \) is a Markov matrix or \( P \) is geometrically ergodic and monotone. We call his method as the **ergodicity method**.
- **L. (2010):** investigated an arbitrary augmentation and truncation bounds for polynomially ergodic and monotone MCs.
Block-monotone DTMCs

- **Li and Zhao (2000)**: the last-column-block-augmented truncation (LCBA) is the best in a certain sense.
- **Masuyama (2015)**: developed error bounds for the LCBA for geometrically ergodic chains.

Convergence in $V$-norm:

- **Tweedie (1998)**: also $V$-norm bounds for monotone and geometrically ergodic MCs.
- **L. Tang and Zhao (2015)**: the best augmentation in total variation norm is not necessarily best in $V$-norm.
- **Masuyama (2016)**: adopted *the perturbation method* to investigate the error bounds in $V$-norm for the LCBA truncation under drift conditions for continuous-time matrix-analytic models.
A Lot of literature on perturbation bounds in $V$-norm:

- L (2012, 2015, 2017),
- \ldots

We will present several types of truncation bounds through the perturbation method in this talk.

In the long run, we are motivated to develop theoretically guaranteed augmented truncation approximation algorithms for computing the invariant probability vectors for high-dimensional Markov chains.
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Bounds in terms of Poisson’s equation

Poisson’s equation

The bound is based on Poisson’s equation:

\[ (I - P)\tilde{g} = \bar{g}, \]

where \( \bar{g} = g - \pi^T g e \). The forcing function \( g \) satisfies that \( \pi^T|g| < \infty \). \( \tilde{g} \) is the solution to Poisson’s equation.

Let \( \Delta = (n)\tilde{P} - P \), from (2), we can obtain the identity

\[ (n)\pi^T \Delta \tilde{g} = (n) \pi^T [(I - P)\tilde{g}] = (n)\pi^T - \pi^T)g. \]

This identity enables us to estimate the augmented truncation bounds if we can bound \( \tilde{g} \) well.
Bounds in terms of Poisson’s equation

\[ D(V, \lambda, b, C) \]: Suppose that there exist a finite set \( C \), positive constants \( b < \infty, \lambda < 1 \) and finite column vectors \( V \geq e \) such that

\[ PV \leq \lambda V + bl_C. \]

To investigate the convergence of the augmented truncation, define

\[ \Delta_n(i, V) = \sum_{j>n} P(i,j)(V(n) + V(j)), \]
Theorem 1  Suppose that $D(\mathbf{V}, \lambda, b, C)$ holds for a non-decreasing function $\mathbf{V}$ and an atom $C = \{\alpha\}$. Then for an arbitrary augmentation

$$
\| (n) \pi^T - \pi^T \|_V \leq H_1(n, \mathbf{V}),
$$

where

$$
H_1(n, \mathbf{V}) = \kappa_1 \sum_{i=0}^{n} (n) \pi(i) \Delta_n(i, \mathbf{V}),
$$

and

$$
\kappa_1 = \frac{1 + \pi^T \mathbf{V}}{1 - \lambda} \leq \frac{1 - \lambda + b}{(1 - \lambda)^2}.
$$

Moreover, $H_1(n, \mathbf{V}) \to 0$ as $n \to \infty$, and

$$
\| (n) \pi^T - \pi^T \|_V \to 0, \; n \to \infty.
$$
Remarks:

▷ (i) The drift condition $\mathbf{D} (\mathbf{V}, \lambda, b, C)$ for an atom $C$ implies $|\tilde{g}| \leq (1 + \pi^T \mathbf{V}) \mathbf{V}$ for a function $g$ such that $|g| \leq c \mathbf{V}$.

▷ (ii) The condition that $V$ is increasing is a key to deal with an arbitrary augmentation.

▷ (iii) The results can hold in a more general setting, e.g.

◊ under $f$-modulated drift condition $P \mathbf{V} \leq \mathbf{V} - f + bl_C$;

◊ under a small set instead of an atom.
Based on Theorem 1, we can give theoretically guaranteed upper bounds on the truncation size for a given $\varepsilon$ as follows.

- **M(a)** Find $V, \lambda, b, C$ such that $D(V, \lambda, b, C)$ holds.
- **M(b)** Calculate $\kappa_1$.
- **M(c)** For a given $n$ calculate $(n)\pi^T$ and assess whether
  \[ H_1(n, V) \leq \varepsilon. \]
  (3)
- **M(d)** If (3) is not satisfied, increase the value of $n$ and repeat M(c) until (3) is satisfied.
Define a suitable residual matrix $T$ by

$$U = P - h\phi^T,$$

$h$ and $\phi^T$ are respectively non-negative bounded column vector and non-negative row vector such that $T$ is nonnegative. For a matrix $B$, define its $V$-norm by

$$\|B\|_V = \sup_{i\in E} \frac{\sum_j |B(i,j)|V(j)}{V(i)}.$$

$U(V, \lambda)$: Suppose that there exist a finite vector $V$, $V \geq e$ and a positive constant $\lambda$, $\lambda < 1$ such that $\|P\|_V < \infty$ and $\|U\|_V \leq \lambda.$
**Theorem 2** If $U(V, \lambda)$ holds for a non-decreasing function $V$ and $\phi^T V < \infty$, then for an arbitrary augmentation

$$\| (n) \pi^T - \pi^T \|_V \leq H_2(n, V),$$

where

$$H_2(n, V) = \kappa_2 \sum_{i=0}^{n} (n) \pi(i) \Delta_n(i, V)$$

and

$$\kappa_2 = \frac{1 + \pi^T V}{1 - \lambda} \leq \frac{1 - \lambda + (\phi^T V)(\pi^T h)}{(1 - \lambda)^2}.$$ 

Moreover, $H_2(n, V) \to 0$ as $n \to \infty$, and

$$\| (n) \pi^T - \pi^T \|_V \to 0, n \to \infty.$$
Remarks:

▷ 1. A starting point of the proof is

\[(n)\pi^T - \pi^T = (n)\pi^T \Delta \sum_{k=0}^{\infty} U^k (I - \Pi).\]

▷ 2. The results hold in a more general setting, e.g. for

\[U = P^m - h\phi^T\]

for some \(m \geq 1\).

▷ 3. There are various choices of the vectors \(h\) and \(\phi^T\) such that \(T\) is nonnegative.
Bounds in terms of the residual matrix

Let \( \phi^T = (P(0, 0), P(0, 1), \cdots) \) denote the first row of \( P \) and let \( h = (1, 0, 0, \cdots)^T \), then the matrix \( U = P - h\phi^T \) is obtained by setting the elements in the first row of \( P \) to zeros and keeping the other elements unchanged.

Let \( \phi^T = (1, 0, 0, \cdots) \) and \( h = (P(0, 0), P(1, 0), \cdots)^T \). Then the matrix \( U = P - h\phi^T \) is obtained by setting the elements in the first column of \( P \) to zeros and keeping the other elements unchanged.

4. Actually, if \( D(V, \lambda, b, C) \) holds for an atom \( C \), then we can take appropriate \( h \) and \( \phi^T \) such that \( U(V, \lambda) \) holds for the same \( V \) and \( \lambda \).
Ergodicity coefficient

Define the set $M_0 = \{\mu : \|\mu\|_V < \infty, \mu^T e = 0\}$. Let $V \geq e$, the $V$-norm ergodicity coefficient $\Lambda(B)$ of a matrix $B = (b_{ij})$ is defined by

$$\Lambda(B) = \sup\{\|\mu^T B\|_V : \|\mu^T\|_V \leq 1, \mu \in M_0\}.$$

The explicit representation of $\Lambda(B)$ is given by

$$\Lambda(B) = \sup_{i,j \in E} \sum_{k \in E} \frac{|b_{ik} - b_{jk}| V(k)}{V(i) + V(j)}.$$

Let $V \equiv e$, then $\Lambda(B)$ becomes the classical ergodicity coefficient $\tau(B)$. 
Theorem 3  Suppose that $V$ is a non-decreasing function, $\pi^T V < \infty$, $\|P\|_V < \infty$, and $\Lambda(P) \leq \rho < 1$. Then for an arbitrary augmentation

$$\|(n)\pi^T - \pi^T\|_V \leq H_3(n, V),$$

where

$$H_3(n, V) = \kappa_3 \sum_{i=0}^{n} (n)\pi(i)\Delta_n(i, V).$$

and

$$\kappa_3 = \frac{1}{1 - \rho}.$$

Moreover, if $D(V, \lambda, b, C)$ holds, then $\pi^T V < \infty$, $\|P\|_V < \infty$, $H_3(n, V) \to 0$ as $n \to \infty$, and

$$\|(n)\pi^T - \pi^T\|_V \to 0, n \to \infty.$$
Remarks:

1. A starting point of the proof is

\[
\| (n) \pi^T - \pi^T \|_V = \| (n) \pi^T \Delta (\sum_{t=0}^{\infty} P^t - \Pi) \|_V \\
\leq \| (n) \pi^T \Delta \|_V \cdot \sum_{t=0}^{\infty} \Lambda(P^t).
\]

2. The results hold under more general assumptions, e.g.

\[
\Lambda(P^m) \leq \rho_m < 1 \text{ for some positive integer } m.
\]
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Suppose that (i) $P$ is aperiodic, (ii) $P$ is monotone, and (iii) $P$ satisfies $D(V, \lambda, b, C)$ for $C = \{0\}$ and a non-decreasing $V$.

He used the triangle inequality

$$
\| (n) \pi^T - \pi^T \|_V \leq \| (n) \tilde{P}^k(i, \cdot) - (n) \pi^T \|_V + \| P^k(i, \cdot) - \pi^T \|_V \\
+ \| (n) \tilde{P}^k(i, \cdot) - P^k(i, \cdot) \|_V
$$

for any $i \in \mathbb{E}$ and any integer $k \geq 1$ and derived

$$
\| (n) \pi_n^T - \pi^T \|_V \leq \frac{4 \lambda^k b}{1 - \lambda} + D \sum_{i=0}^{n} (n) \pi_n(i) \Delta_n(i, V),
$$

where $D = \sum_{s=0}^{k-1} (\lambda + b)^s$.  

Tweedie (1998): ergodicity method
Example-1 Consider a M/G/1 queue with transition matrix

\[
P = \begin{pmatrix}
c_0 & a_1 & a_2 & a_3 & \ldots \\
a_{-1} & a_0 & a_1 & a_2 & \ldots \\
0 & a_{-1} & a_0 & a_1 & \ldots \\
0 & 0 & a_{-1} & a_0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Let \( a_{-1} = (1 - a) \), \( a_j = (1 - a)a^{j+1} \), \( j \geq 0 \) with \( a = 0.25 \).

Let \( V(i) = 2^i \), \( i \geq 0 \) and \( C = \{0\} \). Then we can calculate \( \lambda = \frac{3}{4} \), \( b = \frac{3}{8} \), \( \pi^T V = \frac{4}{3} \), and the difference

\[
T(n, V) = |(n)\pi_n^T V - \pi^T V|.
\]
The (Poisson-equation type) bound is given by

\[
H_2(n, V) = \frac{28}{3} \sum_{i=0}^{n} (n)\pi_n(i)\Delta_n(i, V).
\]

Choosing \( \varepsilon = 0.01 \) and applying the procedures \( M^*(a)-M^*(d) \) in page 530 of Tweedie (1998) obtains

\[
\| (n)\pi_n^T - \pi^T \|_V \leq F_2(n, V) := 0.0257 + \frac{283}{4} \sum_{i=0}^{n} (n)\pi_n(i)\Delta_n(i, V).
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( F_2(n, V) )</th>
<th>( H_2(n, V) )</th>
<th>( (n)\pi_n^T )</th>
<th>( T(n, V) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.2946</td>
<td>0.0355</td>
<td>1.3256</td>
<td>0.0077</td>
</tr>
<tr>
<td>14</td>
<td>0.0796</td>
<td>0.0071</td>
<td>1.3318</td>
<td>0.0015</td>
</tr>
<tr>
<td>18</td>
<td>0.0363</td>
<td>0.0014</td>
<td>1.3330</td>
<td>0.0003</td>
</tr>
</tbody>
</table>
Now we make a comparison of the three types of bounds in Sections 2 to 4 through two specific examples.

**Example-2** For the M/G/1 queue in Example-1,

\[ H_1(n, V) = H_2(n, V) = \frac{28}{3} \sum_{i=0}^{n} (n)\pi(i)\Delta_n(i, V) \]

\[ < H_3(n, V) = 16 \sum_{i=0}^{n} (n)\pi(i)\Delta_n(i, V). \]
Example-3. Consider a single-birth process, taken from (M.F. Chen 1999), with transition probabilities

\[ P_{00} = P_{01} = \frac{1}{2}; \quad P_{i,i+1} = \frac{1}{2}, \quad P_{ij} = \frac{1}{2i}, \quad i \geq 1, j \leq i - 1. \]

Let \( V(i) = i + 1 \) for \( i \geq 1 \) and \( V(0) = 2 \), and consider the last-column augmentation.

\[ H_1(n, V) = 6(1 + \pi^T V) \cdot [(n)\pi_n(n)(V(n) + V(n + 1) + \frac{16}{3})], \]
\[ H_2(n, V) = 3(1 + \pi^T V) \cdot [(n)\pi_n(n)(V(n) + V(n + 1))], \]
\[ H_3(n, V) = \frac{70}{23} \cdot [(n)\pi_n(n)(V(n) + V(n + 1))]. \]
Obviously,

\[ H_1(n, V) > H_2(n, V) > H_3(n, V). \]

Furthermore, since \( \pi^T V \) cannot be obtained directly, we calculate the value of \( (n)\pi_n^T V \) as an approximation.

The following table lists the values of the bounds \( H_i(n, V) \), \( i = 1, 2, 3 \) and \( (n)\pi_n^T V \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( H_1(n, V) )</th>
<th>( H_2(n, V) )</th>
<th>( H_3(n, V) )</th>
<th>( (n)\pi_n^T V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>0.5933</td>
<td>0.1949</td>
<td>0.0282</td>
<td>2.7774</td>
</tr>
<tr>
<td>15</td>
<td>0.2070</td>
<td>0.0693</td>
<td>0.0100</td>
<td>2.7778</td>
</tr>
</tbody>
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Let \( \{X(t), t \geq 0\} \) be a CTMC on \( E = \{0, 1, 2, \cdots\} \) with the generator \( Q = Q(i, j) \), which is positive recurrent with the invariant probability vector \( \pi^T \).

Define the augmented truncation of \( Q \) as follows:

\[
\tilde{Q}^{(n)}(i,j) = \begin{cases} 
Q(i,j) + \sum_{m>n, m \neq i} Q(i,m)\nu_{i,n}(j), & i \in E, 0 \leq j \leq n, \\
Q(i,i), & i = j \geq n + 1, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \nu_{i,n}(\cdot) \) is a probability vector on \( (n)E \) that depends on \( n \) and \( i \). Let \( (n)\pi^T \) be the invariant probability vector of \( \tilde{Q}^{(n)} \).
If \( Q \) is bounded, i.e., \( \sup_{i \in E} Q(i) < \infty \), using the technique of uniformization and similar arguments in L (2012), most of the results in Sections 2-4 can be extended to CTMCs.

Uniformization is invalid when \( Q \) is unbounded. Theorems 2 and 3 cannot be extended to CTMCs. However, Theorem 1 can be done as follows.

For CTMCs, Poisson’s equation is given by

\[
Q\tilde{g} = -\tilde{g}.
\]

where \( \tilde{g} = g - \pi^T g e \). We have the following identity

\[
(n)\pi^T \Delta \tilde{g} = -(n)\pi^T (Q\tilde{g}) = (n)\pi^T - \pi^T g.
\]
D'(\(V, \lambda, b, C\)): Suppose that there exist a finite set \(C\), positive constants \(\lambda, b\), finite column vectors \(V \geq e\) such that

\[
QV \leq -\lambda V + bl_C.
\]

**Theorem 4** If \(D'(\(V, \lambda, b, C\))\) holds for \(C = \{i_0\}\) and a non-decreasing function \(V\), then for an arbitrary augmentation

\[
\|(n)\pi^T - \pi^T\|_V \leq H_4(n, V), \tag{4}
\]

where

\[
H_4(n, V) = \kappa_4 \sum_{i=0}^{n} (n)\pi(i)\Delta_n(i, V)
\]

and

\[
\kappa_4 = \frac{1 + \pi^T V}{\lambda} \leq \frac{\lambda + b}{\lambda^2}.
\]
Remarks

1. The corresponding bound under $f$-modulated drift condition can be derived, which improves Theorem 2.1 in Masuyama (2017) by relaxing the condition $\pi^T V < \infty$ and drops the factor $\pi^T V$ in his bound.

2. Theorem 4 parallels to Theorem 1 for DTMCs. However, we cannot expect that the upper bound always converges to zero as $n$ tends to $\infty$ for CTMCs, which will be clarified by the subsequent example.
Example-4  Consider a special continuous-time birth-death process with the same birth and death rates: $b_0 = 1$, and $b_i = a_i = i\gamma$, $i \geq 1$, where $\gamma > 1$. We can compute that $\pi(n) = \frac{\pi(0)}{n^\gamma}$. Performing the last column augmentation, we have

$$(n)\pi_n(n) = \frac{\pi(n)}{\sum_{i=0}^{n} \pi(i)} = \frac{1}{n^\gamma(\sum_{i=1}^{n} \frac{1}{i\gamma} + 1)}.$$ 

Applying Theorem 4 obtains

$$\sum_{i=0}^{n} (n)\pi_n(i)\Delta_n(i, V) = \frac{1}{\sum_{i=1}^{n} \frac{1}{i\gamma} + 1}(V(j) + V(n)).$$

For any $V \geq e$, $H_4(n, V)$ does not converge to zero.
Thank you for your attention!