Harmonic moments and lower large deviations for a branching process in a random environment

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4. Lower deviation of $Z_n$
   - Lower deviation of $Z_n$ via harmonic moments of $Z_n$
   - Heuristic of the rate function
   - Improving the result of Bansaye and Boinghoff
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Introduction

- **Current research interests** on a supercritical branching process in a random environment: mainly focussed on large deviations of $Z_n$.
- **Very linked topic**: asymptotic properties of harmonic moments $\mathbb{E}[Z_n^{-r}]$ ($r > 0$) of $Z_n$.

Let $(Z_n)_{n \geq 0}$ be a supercritical branching process in an independent and identically distributed random environment $\xi = (\xi_n)_{n \geq 0}$.

- We will give a precise description of the asymptotic behavior of the harmonic moments $\mathbb{E}[Z_n^{-r}]$ of order $r > 0$ as $n \to \infty$, thus exhibit a phase transition with a critical value $r_k > 0$ determined explicitly. Contrary to the constant environment case (the Galton-Watson case), this critical value is different from that for the existence of the harmonic moments of $W = \lim_{n \to \infty} Z_n/\mathbb{E}(Z_n|\xi)$.
- As main application, we give lower large deviation results for $Z_n$. 

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Galton-Watson process and branching process in a random environment

1. Galton-Watson process: each particle has the same deterministic offspring distribution $p$ (on $\mathbb{N}$).

2. Branching process in a random environment (BPRE): particles of generation $n$ have a common random offspring distribution $p(\xi_n)$ depending on the environment at time $n$, given the environment $\xi = (\xi_0, \xi_1, \cdots)$ which is usually supposed to be i.i.d., or stationary and ergodic.

The Galton - Watson process is a branching process in a constant environment or deterministic environment (when $\xi_0 = \xi_1 = \cdots = \text{const.}$).
Description of a BPRE by a tree

Branching Process in a Random Environment (BPRE)

\[ \xi = (\xi_n)_{n \geq 0} \text{ i.i.d.} \]

\[ |u| = n, P_\xi(X_u = k) = p_k(\xi_n) \]

\[ Z_n = \# \{ u : |u| = n \} \text{ the population size of } n^{th} \text{ generation} \]
Definition of a BPRE

\( Z_n \) – the population size of the \( n \)th generation,
\( X_u \) – the number of offspring of \( u \).

By definition,
\[
Z_0 = 1, \quad Z_{n+1} = \sum_{|u|=n} X_u, \quad (n \geq 0).
\]

where given \( \xi \), \( \{X_u : |u| = n\} \) are conditionally independent and have a common distribution \( p(\xi_n) = \{p_k(\xi_n) : k \geq 0\} \). An alternative definition is:
\[
Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad (n \geq 0),
\]

where, conditional on \( \xi \), \( X_{n,i} \) are indep. and have law \( p(\xi_n) \).
Let $(\Gamma, P_{\xi})$ be the probability space under which the process is defined when the environment $\xi$ is fixed. As usual, $P_{\xi}$ is called quenched law. The total probability space can be formulated as the product space $(\Theta^N \times \Gamma, P)$, where $P(d\xi, dx) = P_{\xi}(dx)\tau(d\xi)$ in the sense that for all measurable and positive $g$, we have

$$\int g(\xi, x) P(d\xi, dx) = \int \int g(\xi, x)P_{\xi}(dx)\tau(d\xi),$$

where $\tau$ is the law of the environment $\xi$. $P$ is called annealed law. $P_{\xi}$ may be considered to be the conditional probability of $P$ given $\xi$. 

**Quenched and annealed laws**
Classification

- For a G-W process \((Z_n)\), with \(m = \mathbb{E}Z_1 = \sum_k kp_k\),
  subcritical if \(m < 1\), critical if \(m = 1\), supercritical if \(m > 1\).

- For a BPRE \((Z_n)\), with \(m_0 = \mathbb{E}_\xi Z_1 = \sum_k kp_k(\xi_0)\),
  subcritical if \(\mathbb{E} \log m_0 < 0\),
  critical if \(\mathbb{E} \log m_0 = 0\),
  supercritical if \(\mathbb{E} \log m_0 > 0\).

If subcritical or critical, then \(\mathbb{P}(Z_n \to 0) = 1\) provided that \(\mathbb{P}(p_1 = 1) < 1\); if supercritical, then \(\mathbb{P}(Z_n \to \infty) > 0\) provided that
\(\mathbb{E}[- \log(1 - p_0(\xi_0))] < \infty\).

We consider the supercritical case, and desire to give precise descriptions on the size of \(Z_n\).
The natural martingale \((W_n)\) and its limit \(W\)

Denote by

\[
m_n = \sum_k kp_k(\xi_n)
\]

the conditional mean of the offspring distri. at time \(n\), and set

\[
\Pi_0 = 1, \quad \Pi_n = m_0 \cdots m_{n-1} \text{ for } n \geq 1.
\]

Then \(\Pi_n = \mathbb{E}_\xi Z_n\), and the normalized population size

\[
W_n = \frac{Z_n}{\Pi_n}
\]

is a nonnegative martingale, so that the limit

\[
W = \lim_{n \to \infty} W_n
\]

exists a.s. with \(\mathbb{E}W \leq 1\). It is known that

\[
P(W > 0) > 0 \quad \text{iff} \quad \mathbb{E}\frac{Z_1}{m_0} \log^+ Z_1 < \infty.
\]
Supercriticality and non-degeneration of $W$

We consider the supercritical case where

$$\mathbb{E} \log m_0 \in (0, \infty),$$

so that $\mathbb{P}(Z_n \to \infty) > 0$. For simplicity, let $p_k = p_k(\xi_0)$ and assume that

$$p_0 = 0 \ \ a.s.$$ 

Assume also

$$\mathbb{E} \frac{Z_1}{m_0} \log^+ Z_1 < \infty,$$

so that $\mathbb{P}(W > 0) > 0$. The three conditions above imply that

$$Z_n \to \infty \ \text{and} \ W > 0 \ \ a.s..$$
Historical notes

Objective

- Asymptotic behavior of harmonic moments of $Z_n$:

$$\mathbb{E}[Z_n^{-r}] \sim q^r \quad r > 0.$$ 

Contrary to the constant environment (GW) case, our result will exhibit a phase transition with a critical value different from that for the existence of harmonic moments of $W = \lim_{n \to \infty} Z_n / \pi_n$.

- Lower deviation of $Z_n$:

$$\log \mathbb{P}(Z_n \leq e^{\theta n}) \sim q^\theta \quad 0 < \theta < \mathbb{E} \log m_0$$

(recall that $\frac{\log Z_n}{n} \to \mathbb{E} \log m_0$ a.s.) Our result will improve significantly the known ones, and give a new expression for the rate function.
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Harmonic moments of $Z_n$: $\mathbb{E}[Z_n^{-r}]$

$$\mathbb{E} \left[ Z_n^{-r} \right] \sim_{n \to \infty} ? \quad r > 0$$
Let $\gamma_1 = \mathbb{P}(Z_1 = 1)$ and $r_1 > 0$ be the solution of $E m_0^{-r_1} = \gamma_1$. By convention $r_1 = +\infty$ if $\mathbb{P}(Z_1 = 1) = 0$.

**Theorem [Grama-Liu-Miqueu (2017)]**

(Hyp) $E m_0^\varepsilon < \infty$ for some $\varepsilon > 0$. Then

\[
\begin{align*}
\frac{\mathbb{E}[Z_n^{-r}]}{\gamma_1^n} & \quad \xrightarrow{n \to \infty} \quad C(r) \quad \text{if } r > r_1, \\
\frac{\mathbb{E}[Z_n^{-r}]}{n\gamma_1^n} & \quad \xrightarrow{n \to \infty} \quad C(r) \quad \text{if } r = r_1, \\
\frac{\mathbb{E}[Z_n^{-r}]}{(Em_0^{-r})^n} & \quad \xrightarrow{n \to \infty} \quad C(r) \quad \text{if } r < r_1;
\end{align*}
\]

where $C(r) \in (0, \infty)$ for which we have an integral expression.

**Constant environment (GW) case**: due to Ney-Vidyashankar (2003). Our proof is new and simpler.
Harmonic moments of $Z_n$: phase transition

From the theorem above, about the asymptotic behavior of $\mathbb{E} [Z_n^{-r}]$ there is a phase transition if and only if $\mathbb{P}(Z_1 = 1) > 0$, since

$$r_1 < +\infty \text{ if } \mathbb{P}(Z_1 = 1) > 0$$

and

$$r_1 = +\infty \text{ if } \mathbb{P}(Z_1 = 1) = 0.$$
Harmonic moments of $W$: $\mathbb{E}[W^{-a}]$

Contrary to the constant environment (GW) case, the result above exhibits a phase transition with a critical value different from that for the existence of harmonic moments of $W = \lim_{n \to \infty} Z_n/\Pi_n$. 
Harmonic moments of $W$

Theorem [Grama-Liu-Miqueu (2017+)]

(Hyp): $\mathbb{E}[m_0^p] < \infty$ for $p > 0$. Then for all $a \in (0, p)$,

$$\mathbb{E}[W^{-a}] < \infty \quad \text{iff} \quad \mathbb{E}[p_1(\xi_0)m_0^a] < 1.$$  

Corollary [Grama-Liu-Miqueu (2017+)]

Let $a_1 > 0$ be the solution of

$$\mathbb{E}[p_1(\xi_0)m_0^{a_1}] = 1$$

and assume that $\mathbb{E}m_0^{a_1} < \infty$. Then

$$\begin{cases} 
\mathbb{E}[W^{-a}] < \infty & \text{for } a \in [0, a_1), \\
\mathbb{E}[W^{-a}] = \infty & \text{for } a \in [a_1, \infty). 
\end{cases}$$
Harmonic moments of $Z_n$: key ideas in the proof (1)

1. Measure change via Cramér’s change of the associated random walk: recall that

\[ \mathbb{P}(d\xi, dx) = \mathbb{P}_\xi(dx) \tau(dx) \]

with \( \tau = \tau_0 \otimes \mathbb{N} \) = the law of \( \xi = (\xi_0, \xi_1, \cdots) \), \( \tau_0 \) = the law of \( \xi_0 \). Define the new annealed law \( \mathbb{P}_\lambda \) by

\[ \mathbb{P}_r(d\xi, dx) = \mathbb{P}_\xi(dx) \tau_r(d\xi) \tag{3.1} \]

with \( \tau_r = \tau_{0,r} \otimes \mathbb{N} \), \( \tau_{0,r}(dx) = \frac{m(x)^{-r}}{\mathbb{E}m_0^{-r}} \tau_0(dx) \), \( m(x) = \mathbb{E}(Z_1|\xi_0 = x) \). The measure change from \( \mathbb{P} \) to \( \mathbb{P}_r \) corresponds to Cramér’s change for the random walk \( S_n = \sum_{i=0}^{n-1} \log m_i \).
2. Using the fact that

\[ \frac{P(Z_n = j)}{\gamma_1^n} \]

is increasing in \( n \).
Harmonic moments of $Z_n$: key ideas in the proof (3)

3. Using the branching property

$$Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)}, \quad (3.2)$$

where conditionally on $\xi$, for $i \geq 1$, $\{Z_{n,i}^{(m)} : n \geq 0\}$ are i.i.d. branching processes with the shifted environment $T^m(\xi_0, \xi_1, \ldots) = (\xi_m, \xi_{m+1}, \ldots)$, and are also indep. of $Z_m$. This leads to the equation

$$\mathbb{E} \left[ Z_{n+1}^{-r} \right] = \gamma_1^{n+1} + \sum_{j=0}^{n} b_j \gamma_1^{n-j} c_r^j, \quad (3.3)$$

where $c_r = \mathbb{E} m_0^{-r}$ and $(b_j)_{j \geq 0}$ is an increasing and bounded sequence. This relation highlights the main role played by $\gamma_1$ and $c_r$ in the asymptotic study of $\mathbb{E} [Z_n^{-r}]$ whose behavior depends on whether $\gamma_1 < c_r$, $\gamma_1 = c_r$ or $\gamma_1 > c_r$. 
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Lower deviation of $Z_n$

\[ \mathbb{P}(Z_n \leq e^{\theta n}) \approx \ ? \quad \theta \in (0, \mathbb{E}[X_1]). \]

Notation:
The associated random walk:

\[ S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad X_i = \log m_{i-1}, \quad i \geq 1. \]

Rate function for large deviations of $S_n$:

\[ \Lambda^*(\theta) := \sup_{\lambda \in \mathbb{R}} \{ \theta \lambda - \Lambda(\lambda) \}, \quad \Lambda(\lambda) := \log(\mathbb{E}[\exp(\lambda X_1)]) \]
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By the proceeding theorem on the harmonic moments of $Z_n$ and a version of the Gärtener - Ellis theorem, we obtain the following lower large deviation result.
Lower deviation of $Z_n$ via harmonic moments of $Z_n$

Let $\gamma_1 = \mathbb{P}(Z_1 = 1)$ and $r_1$ be the solution of $\mathbb{E}m_0^{-r_1} = \gamma_1$.

**Theorem [Grama-Liu-Miqueu (2017)]**

(Hyp) $\mathbb{E}m_0^\varepsilon < \infty$ for some $\varepsilon > 0$. Then for all $\theta \in \left(0, \mathbb{E}[X_1]\right)$,

$$\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{P}\left(Z_n \leq e^{\theta n}\right) = \chi^*(\theta) \in (0, \infty),$$

$$\chi^*(\theta) = \sup_{\lambda \leq 0} \{\lambda \theta - \chi(\lambda)\} = \begin{cases} -r_1 \theta - \log \gamma_1 & \text{if } \theta < \theta_1, \\ \Lambda^*(\theta) & \text{if } \theta_1 < \theta, \end{cases}$$

where

$$\chi(\lambda) = \begin{cases} \log \gamma_1 & \text{if } \lambda \leq \lambda_1, \\ \Lambda(\lambda) & \text{if } \lambda \in [\lambda_1, 0], \end{cases}$$

and

$$\theta_1 = \Lambda'(-r_1).$$
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What makes the population small?

Branching V.S random walk

- **Influence of** $\mathbb{P}(Z_1 = 1)$. Since

  \[
  Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i},
  \]

  $X_{n,i} = 1$ implies $Z_{n+1} = Z_n$, so that $Z_{n+1}$ remains to be small when $Z_n$ is small. This indicates that $\mathbb{P}(X_{n,i} = 1) = \mathbb{P}(Z_1 = 1)$ should play a rule for $Z_n$ to be small.

- **Influence of the random walk.** Since

  \[
  Z_n = W_ne^{S_n} \sim We^{S_n},
  \]

  small values of $S_n$ implies small values of $Z_n$. The small values of $S_n$ is described by the rate function $\Lambda^*$. So the probability for small...
Typical trajectories of $\{Z_n \leq e^{\theta n}\}$

$$\text{cost}$$

$$(1 - t)n\Lambda^*(\theta/(1 - t))$$

$$-nt \log \mathbb{P}(Z_1 = 1)$$

Optimal time $t_\theta$
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Result of Bansaye and Boinghoff (2013)

Theorem [Bansaye and Böinghoff (2013)]

\[(Hyp): \mathbb{E}[m'_t] < \infty \text{ for all } t > 0. \text{ Then for } 0 < \theta < \mathbb{E}X_1,\]

\[-\frac{1}{n} \log \mathbb{P}(Z_n \leq e^{\theta n}) \xrightarrow{n \to \infty} I(\theta)\]

\[I(\theta) := \inf_{t \in [0,1]} \{-t \log \mathbb{P}(Z_1 = 1) + (1 - t)\Lambda^*(\theta/(1 - t))\}\]

\[= \begin{cases} 
\rho \left(1 - \frac{\theta}{\theta^*_1}\right) + \frac{\theta}{\theta^*_1} \Lambda^*(\theta^*_1), & 0 < \theta \leq \theta^*_1 \\
\Lambda^*(\theta), & \theta^*_1 < \theta < \mathbb{E}X_1
\end{cases}\]

where \(\rho = -\log \mathbb{P}(Z_1 = 1)\), \(\theta^*_1\) is the unique solution in \((0, \mathbb{E}X_1)\) of

\[\frac{\rho - \Lambda^*_1(\theta^*_1)}{\theta^*_1} = \inf_{0 < \theta \leq \mathbb{E}[X_1]} \frac{\rho - \Lambda^*(\theta)}{\theta}\]
Remarks

- Our result improves that of Bansaye and Boinghoff (2013) by relaxing the moment condition $\mathbb{E}[m_t^t] < \infty$ for all $t > 0$ to $\mathbb{E}[m_0^\varepsilon] < \infty$ for some $\varepsilon > 0$.

- We give a new expression of the rate function.
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Cramé’s moderate deviation expansion: for $0 \leq x = o(\sqrt{n})$, as $n \to \infty$,

$$\mathbb{P}\left( \frac{\log Z_n - n\mathbb{E}X_1}{\sigma \sqrt{n}} > x \right) = \exp \left\{ \frac{x^3}{\sqrt{n}} L\left( \frac{x}{\sqrt{n}} \right) \right\} \left[ 1 + O \left( \frac{1 + x}{\sqrt{n}} \right) \right]$$

(5.1)

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$, $L(.)$ is Cramér’s series. See Grama-Liu-Miqueu (2017 SPA)

asymptotic properties of the distribution of $Z_n$:

$$\mathbb{P}(Z_n = j) \sim ?$$

See Grama-Liu-Miqueu (2017+, Ann. IHP, in revision)
Thank you!

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