Harmonic moments and lower large deviations for a branching process in a random environment

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Outline

Introduction

- Current research interests on a supercritical branching process in a random environment: mainly focussed on large deviations of Z_n.
- Very linked topic: asymptotic properties of harmonic moments $\mathbb{E}\left[Z_n^{-r}\right]$ $(r > 0)$ of Z_n.

Let $(Z_n)_{n\geq 0}$ be a supercritical branching process in an independent and identically distributed random environment $\xi = (\xi_n)_{n\geq 0}$.

- We will give a precise description of the asymptotic behavior of the harmonic moments $\mathbb{E}\left[Z_n^{-r}\right]$ of order $r > 0$ as $n \to \infty$, thus exhibit a phase transition with a critical value $r_k > 0$ determined explicitly. Contrary to the constant environment case (the Galton-Watson case), this critical value is different from that for the existence of the harmonic moments of $W = \lim_{n\to\infty} Z_n/\mathbb{E}(Z_n|\xi)$.
- As main application, we give lower large deviation results for Z_n.

Outline

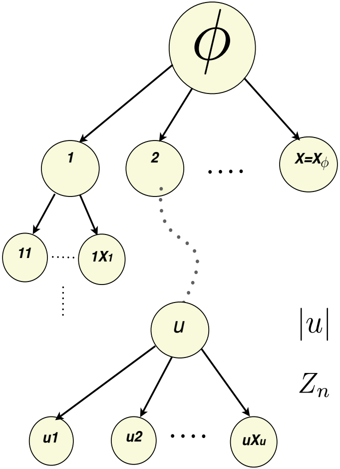
Galton-Watson process and branching process in a random environment

1. Galton-Watson process: each particle has the same deterministic offspring distribution p (on $\mathbb{N}$).
2. Branching process in a random environment (BPRE): particles of generation n have a common random offspring distribution $p(\xi_n)$ depending on the environment at time n, given the environment $\xi = (\xi_0, \xi_1, \cdots)$ which is usually supposed to be i.i.d., or stationary and ergodic

The Galton - Watson process is a branching process in a contant environment or deterministic environment (when $\xi_0 = \xi_1 = \cdots = const.$)

Description of a BPRE by a tree

- Branching Process in a Random Environment (BPRE)



$$\xi = (\xi_n)_{(n\geq 0)} \quad i.i.d.$$

$$|u| = n, P_\xi(X_u = k) = p_k(\xi_n)$$

$Z_n = {}^\sharp\{u : |u| = n\}$ – the population size of n^{th} generation

Definition of a BPRE

Z_n – the population size of the nth generation,

X_u– the number of offspring of u.

By definition,

$$Z_0 = 1, \quad Z_{n+1} = \sum_{|u|=n} X_u, \quad (n \geq 0).$$

where given ξ , $\{X_u : |u| = n\}$ are conditionally independent and have a common distribution $p(\xi_n) = \{p_k(\xi_n) : k \geq 0\}$. An alternative definition is:

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad (n \geq 0),$$

where, conditional on ξ, $X_{n,i}$ are indep. and have law $p(\xi_n)$.

Quenched and annealed laws

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is fixed. As usual, $\mathbb{P}_\xi$ is called quenched law. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathbb{P})$, where $\mathbb{P}(d\xi, dx) = \mathbb{P}_\xi(dx)\tau(d\xi)$ in the sense that for all measurable and positive g, we have

$$\int g(\xi, x)\, \mathbb{P}(d\xi, dx) = \int \int g(\xi, x) \mathbb{P}_\xi(dx)\tau(d\xi),$$

where τ is the law of the environment ξ. P is called annealed law. $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given ξ.

Classification

- For a G-W process (Z_n), with $m = \mathbb{E}Z_1 = \sum_k kp_k$, subcritical if $m < 1$, critical if $m = 1$, supercritical if $m > 1$.
- For a BPRE (Z_n), with $m_0 = \mathbb{E}_\xi Z_1 = \sum_k kp_k(\xi_0)$,
  subcritical if $\mathbb{E}\log m_0 < 0$,
  critical if $\mathbb{E}\log m_0 = 0$,
  supercritical if $\mathbb{E}\log m_0 > 0$.

If subcritical or critical, then $\mathbb{P}(Z_n \to 0) = 1$ provided that $\mathbb{P}(p_1 = 1) < 1$; if supercritical, then $\mathbb{P}(Z_n \to \infty) > 0$ provided that $\mathbb{E}[-\log(1 - p_0(\xi_0)] < \infty$.

We consider the supercritical case, and desire to give precise descriptions on the size of Z_n.

The natural martingale (W_n) and its limit W

Denote by

$$m_n = \sum_k k p_k(\xi_n)$$

the conditional mean of the offspring distri. at time n, and set

$$\Pi_0 = 1, \qquad \Pi_n = m_0 \cdots m_{n-1} \text{ for } n \geq 1.$$

Then $\Pi_n = \mathbb{E}_\xi Z_n$, and the normalized population size

$$W_n = \frac{Z_n}{\Pi_n}$$

is a nonnegative martingale, so that the limit

$$W = \lim_{n \to \infty} W_n$$

exists a.s. with $\mathbb{E}W \leq 1$. It is known that

$$\mathbb{P}(W > 0) > 0 \quad \text{iff} \quad \mathbb{E}\frac{Z_1}{m_0} \log^+ Z_1 < \infty.$$

Supercriticality and non-degeneration of W

We consider the *supercritical* case where

$$\mathbb{E}\log m_0 \in (0,\infty),$$

so that $\mathbb{P}(Z_n \to \infty) > 0$. For simplicity, let $p_k = p_k(\xi_0)$ and assume that

$$p_0 = 0 \quad a.s.$$

Assume also

$$\mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1 < \infty,$$

so that $\mathbb{P}(W > 0) > 0$. The three conditions above imply that

$$Z_n \to \infty \text{ and } W > 0 \quad a.s..$$

Historical notes

- Smith and Wilkinson (1969): i.i.d. environment, criterion for extinction.
- Athreya and Karlin (1971): stationary and ergodic environment, basic limit theorems.
- critical and subcritical cases: survival probability and conditional limit theorems, see e.g. Afanasyev, Böinghoff, Kersting & Vatutin (2014, 2012), Vatutin & Zheng (2012), Vatutin (2010).
- supercritical case: large deviations, see e.g. Grama, Liu & Miqueu (2017 SPA), Bansaye & Böinghoff (2014, 2013, 2011), Huang & Liu (2012), Bansaye & Berestycki (2009).

Objective

- Asymptotic behavior of harmonic moments of Z_n:

$$\mathbb{E}[Z_n^{-r}] \sim ? \quad r > 0.$$

Contrary to the constant environment (GW) case, our result will exhibits a phase transition with a critical value different from that for the existence of harmonic moments of $W = \lim_{n\to\infty} Z_n/\pi_n$.

- Lower deviation of Z_n:

$$\log \mathbb{P}(Z_n \leq e^{\theta n}) \sim ? \quad 0 < \theta < \mathbb{E} \log m_0$$

(recall that $\frac{\log Z_n}{n} \to \mathbb{E} \log m_0$ a.s.) Our result will improve significantly the known ones, and give a new expression for the rate function.

Outline

Harmonic moments of Z_n: $\mathbb{E}[Z_n^{-r}]$

$$\mathbb{E}\left[Z_n^{-r}\right] \underset{n\to\infty}{\sim} \quad ? \qquad r > 0$$

Let $\gamma_1 = \mathbb{P}(Z_1 = 1)$ and $r_1 > 0$ be the solution of $\mathbb{E} m_0^{-r_1} = \gamma_1$. By convention $r_1 = +\infty$ if $\mathbb{P}(Z_1 = 1) = 0$.

Theorem [Grama-Liu-Miqueu (2017)]

(Hyp) $\mathbb{E} m_0^{\varepsilon} < \infty$ for some $\varepsilon > 0$. Then

$$\begin{cases} \dfrac{\mathbb{E}\left[Z_n^{-r}\right]}{\gamma_1^n} \underset{n\to\infty}{\longrightarrow} C(r) & \text{if } r > r_1, \\ \dfrac{\mathbb{E}\left[Z_n^{-r}\right]}{n\gamma_1^n} \underset{n\to\infty}{\longrightarrow} C(r) & \text{if } r = r_1, \\ \dfrac{\mathbb{E}\left[Z_n^{-r}\right]}{\left(\mathbb{E} m_0^{-r}\right)^n} \underset{n\to\infty}{\longrightarrow} C(r) & \text{if } r < r_1; \end{cases}$$

where $C(r) \in (0, \infty)$ for which we have an integral expression.

Constant environment (GW) case: due to Ney-Vidyashankar (2003). Our proof is new and simpler.

Harmonic moments of Z_n: phase transition

From the theorem above, about the asymptotic behavior of $\mathbb{E}\left[Z_n^{-r}\right]$ there is a prase transition if and only if $\mathbb{P}(Z_1 = 1) > 0$, since

$$r_1 < +\infty \text{ if } \mathbb{P}(Z_1 = 1) > 0$$

and

$$r_1 = +\infty \text{ if } \mathbb{P}(Z_1 = 1) = 0.$$

Harmonic moments of W: $\mathbb{E}[W^{-a}]$

Contrary to the constant environment (GW) case, the result above exhibits a phase transition with a critical value different from that for the existence of harmonic moments of $W = \lim_{n\to\infty} Z_n/\Pi_n$.

Harmonic moments of W

Theorem [Grama-Liu-Miqueu (2017+)]

(Hyp): $\mathbb{E}[m_0^p] < \infty$ for $p > 0$. Then for all $a \in (0, p)$,

$$\mathbb{E}[W^{-a}] < \infty \quad \text{iff} \quad \mathbb{E}[p_1(\xi_0) m_0^a] < 1.$$

Corollary [Grama-Liu-Miqueu (2017+)]

Let $a_1 > 0$ be the solution of

$$\mathbb{E}[p_1(\xi_0) m_0^{a_1}] = 1$$

and assume that $\mathbb{E} m_0^{a_1} < \infty$. Then

$$\begin{cases} \mathbb{E}[W^{-a}] < \infty & \text{for} \quad a \in [0, a_1), \\ \mathbb{E}[W^{-a}] = \infty & \text{for} \quad a \in [a_1, \infty). \end{cases}$$

Harmonic moments of Z_n: key ideas in the proof (1)

1. Measure change via Cramér's change of the associated random walk: recall that

$$\mathbb{P}(d\xi, dx) = \mathbb{P}_\xi(dx)\tau(dx)$$

with $\tau = \tau_0^{\otimes \mathbb{N}}$ = the law of $\xi = (\xi_0, \xi_1, \cdots)$, τ_0 = the law of ξ_0. Define the new annealed law $\mathbb{P}_\lambda$ by

$$\mathbb{P}_r(d\xi, dx) = \mathbb{P}_\xi(dx)\tau_r(d\xi) \tag{3.1}$$

with $\tau_r = \tau_{0,r}^{\otimes \mathbb{N}}$, $\quad \tau_{0,r}(dx) = \frac{m(x)^{-r}}{\mathbb{E}m_0^{-r}}\tau_0(dx), \quad m(x) = \mathbb{E}(Z_1|\xi_0 = x)$. The measure change from $\mathbb{P}$ to $\mathbb{P}_r$ corresponds to Cramér's change for the random walk $S_n = \sum_{i=0}^{n-1} \log m_i$.

Harmonic moments of Z_n: key ideas in the proof (2)

2. Using the fact that

$$\frac{\mathbb{P}(Z_n = j)}{\gamma_1^n} \quad \text{is increasing in } n.$$

Harmonic moments of Z_n: key ideas in the proof (3)

3. Using the branching property

$$Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)}, \tag{3.2}$$

where conditionally on ξ, for $i \geq 1$, $\{Z_{n,i}^{(m)} : n \geq 0\}$ are i.i.d. branching processes with the shifted environment $T^m(\xi_0, \xi_1, \ldots) = (\xi_m, \xi_{m+1}, \ldots)$, and are also indep. of Z_m. This leads to the equation

$$\mathbb{E}\left[Z_{n+1}^{-r}\right] = \gamma_1^{n+1} + \sum_{j=0}^{n} b_j \gamma_1^{n-j} c_r^j, \tag{3.3}$$

where $c_r = \mathbb{E} m_0^{-r}$ and $(b_j)_{j\geq 0}$ is an increasing and bounded sequence. This relation highlights the main role played by γ_1 and c_r in the asymptotic study of $\mathbb{E}[Z_n^{-r}]$ whose behavior depends on whether $\gamma_1 < c_r$, $\gamma_1 = c_r$ or $\gamma_1 > c_r$.

Outline

Lower deviation of Z_n

$$\mathbb{P}(Z_n \le e^{\theta n}) \approx \quad ? \quad \theta \in (0, \mathbb{E}[X_1]).$$

Notation:
The associated random walk:

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad X_i = \log m_{i-1}, \quad i \ge 1.$$

Rate function for large deviations of S_n:

$$\Lambda^*(\theta) := \sup_{\lambda \in \mathbb{R}} \{\theta\lambda - \Lambda(\lambda)\}, \quad \Lambda(\lambda) := \log(\mathbb{E}[\exp(\lambda X_1)])$$

Outline

Lower deviation of Z_n via harmonic moments of Z_n

By the proceeding theorem on the harmonic moments of Z_n and a version of the Gärtener - Ellis theorem, we obtain the following lower large deviation result.

Lower deviation of Z_n via harmonic moments of Z_n

Let $\gamma_1 = \mathbb{P}(Z_1 = 1)$ and r_1 be the solution of $\mathbb{E}m_0^{-r_1} = \gamma_1$

Theorem [Grama-Liu-Miqueu (2017)]

(Hyp) $\mathbb{E}m_0^{\varepsilon} < \infty$ for some $\varepsilon > 0$. Then for all $\theta \in (0, \mathbb{E}[X_1])$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}\left(Z_n \leq e^{\theta n}\right) = \chi^*(\theta) \in (0,\infty),$$

$$\chi^*(\theta) = \sup_{\lambda\leq 0}\{\lambda\theta - \chi(\lambda)\} = \begin{cases} -r_1\theta - \log\gamma_1 & \text{if} \quad \theta < \theta_1, \\ \Lambda^*(\theta) & \text{if} \quad \theta_1 < \theta, \end{cases}$$

where

$$\chi(\lambda) = \begin{cases} \log\gamma_1 & \text{if } \lambda \leq \lambda_1, \\ \Lambda(\lambda) & \text{if } \lambda \in [\lambda_1, 0], \end{cases}$$

and

$$\theta_1 = \Lambda'(-r_1).$$

Outline

What makes the population small ?

Branching V.S random walk

- **Influence of $\mathbb{P}(Z_1 = 1)$.** Since

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i},$$

$X_{n,i} = 1$ implies $Z_{n+1} = Z_n$, so that Z_{n+1} remains to be small when Z_n is small. This indicates that $\mathbb{P}(X_{n,i} = 1) = \mathbb{P}(Z_1 = 1)$ should play a rule for Z_n to be small.

- **Influence of the random walk.** Since

$$Z_n = W_n e^{S_n} \sim W e^{S_n},$$

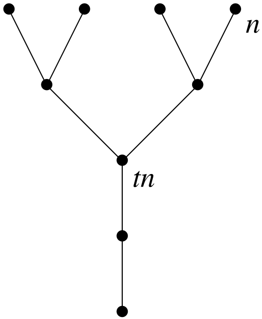
small values of S_n implies small values of Z_n. The small values of S_n is described by the rate function Λ^*. So the probability for small

Typical trajectories of $\{Z_n \leq e^{\theta n}\}$

cost

$(1-t)n\Lambda^*(\theta/(1-t))$

$-nt \log \mathbb{P}(Z_1 = 1)$



associated random walk

branching randomness

Optimal time t_θ

Outline

Result of Bansaye and Boinghoff (2013)

Theorem [Bansaye and Böinghoff (2013)]

(Hyp): $\mathbb{E}[m_0^t] < \infty$ for all $t > 0$. Then for $0 < \theta < \mathbb{E}X_1$,

$$-\frac{1}{n}\log \mathbb{P}(Z_n \leq e^{\theta n}) \underset{n\to\infty}{\longrightarrow} I(\theta)$$

$$\begin{aligned} I(\theta) &:= \inf_{t\in[0,1]}\{-t\log \mathbb{P}(Z_1 = 1) + (1-t)\Lambda^*(\theta/(1-t))\} \\ &= \begin{cases} \rho\left(1-\frac{\theta}{\theta_1^*}\right) + \frac{\theta}{\theta_1^*}\Lambda^*(\theta_1^*), & 0 < \theta \leq \theta_1^* \\ \Lambda^*(\theta), & \theta_1^* < \theta < \mathbb{E}X_1 \end{cases} \end{aligned}$$

where $\rho = -\log \mathbb{P}(Z_1 = 1)$, θ_1^* is the unique solution in $(0, \mathbb{E}X_1)$ of

$$\frac{\rho - \Lambda_1^*(\theta_1^*)}{\theta_1^*} = \inf_{0<\theta\leq\mathbb{E}[X_1]} \frac{\rho - \Lambda^*(\theta)}{\theta}$$

Remarks

- Our result improves that of Bansaye and Boinghoff (2013) by relaxing the moment condition $\mathbb{E}[m_0^t] < \infty$ for all $t > 0$ to $\mathbb{E}[m_0^\varepsilon] < \infty$ for some $\varepsilon > 0$.
- We give a new expression of the rate function.

Outline

Related topics

- Cramé's moderate deviation expansion: for $0 \leq x = o(\sqrt{n})$, as $n \to \infty$,

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}X_1}{\sigma\sqrt{n}} > x\right)}{1 - \Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}} \mathscr{L}\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right] \tag{5.1}$$

  where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$, $L(.)$ is Cramér's series.
  See Grama-Liu-Miqueu (2017 SPA)

- asymptotic properties of the distribution of Z_n:

$$\mathbb{P}(Z_n = j) \sim ?$$

  See Grama-Liu-Miqueu (2017+, Ann. IHP, in revision)

Thank you !

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