Limit Theorems for Supercritical MBPRE with Linear Fractional Offspring Distributions

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The 13th Workshop on Markov Processes and Related Topics

July 20, 2017

Wuhan University

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Model

Multitype Branching Processes in Random Environments(MBPRE):

- K different types of particles
- $(\Delta, \mathcal{B}(\Delta))$ is the space of probability measures on $\mathbb{N}_0^K$,

  $(\Delta_K, \mathcal{B}(\Delta_K))$ is the K-fold product of the space $(\Delta, \mathcal{B}(\Delta))$ on itself.

  Let $\pi = (\pi^{(1)}, \cdots, \pi^{(K)})$ be a r.v. taking values in $(\Delta_K, \mathcal{B}(\Delta_K))$.

  An infinite sequence $\Pi = (\pi_0, \pi_1, \pi_2, \cdots)$ of i.i.d. copies of π is called a random environment.

  P: an i.i.d. product measure on $(\Delta_K)^{\mathbb{Z}}$.
- $\{\mathbf{Z}_n = (Z_n^{(1)}, \cdots, Z_n^{(K)})\}_{n \geq 0}$is a sequence of random K-dimensional vectors.

- If $\mathbf{Z}_0$ is independent of Π, and

$$\mathcal{L}(\mathbf{Z}_{n+1} \mid \mathbf{Z}_n = (z_1, \cdots, z_K), \Pi = (\overline{\pi}_0, \overline{\pi}_1, \cdots)) = \mathcal{L}\left(\sum_{i=1}^{K}\sum_{j=1}^{z_i} \xi_{n,j}^{(i)}(\overline{\pi}_n)\right),$$

  where $\xi_{n,1}^{(i)}, \cdots, \xi_{n,z_i}^{(i)}$ are independent K-dimensional random vectors and are identically distributed according to $\overline{\pi}_n^{(i)}$, $i = 1, \cdots, K$.

  Such $\{\mathbf{Z}_n\}$ is called a K-type Galton-Watson branching process in a random environment.

- Quenched law $\mathbb{P}(\cdot \mid \Pi)$: fix an environment

  Annealed law $\mathbb{P}$: average over environments

  $\mathbb{P}(\cdot) := \int_{(\Delta_K)^{\mathbb{Z}}} \mathbb{P}(\cdot \mid \Pi) P(d\Pi)$

- An one-to-one correspondence between probability measures and generating functions $\mathbf{f}_n(\mathbf{s}) = (f_n^{(1)}(\mathbf{s}), \cdots, f_n^{(K)}(\mathbf{s}))$, $n \geq 0$:

$$f_n^{(i)}(\mathbf{s}) = \sum_{\mathbf{t} \in \mathbb{N}_0^K} \pi_n^{(i)}(\{\mathbf{t}\})\mathbf{s}^{\mathbf{t}}$$

$f_n^{(i)}(\mathbf{s}) = f_n^{(i)}(s_1, \cdots, s_K)$ is the generating function of type i in nth generation.

- The mean matrix $M_n = M_n(\pi_n) = (M_n(i,j))_{i,j=1}^K := \left(\frac{\partial f_n^{(i)}(\mathbf{1})}{\partial s_j}\right)_{i,j=1}^K$ is a random matrix, $n \geq 0$.

  Assume that the elements of the mean matrices are positive.

- In constant environments, $\rho > 0$ is the Perron root(maximal eigenvalue) of M, $\mathbf{l}$ and $\mathbf{r}$ are the corresponding positive left and right eigenvectors. According to Perron-Frobenius Theorem, $E[\mathbf{Z}_n] = E[\mathbf{Z}_0]M^n \approx \rho^n E[\mathbf{Z}_0]S$, where $S = (r_i l_j)_{i,j=1}^K$.
- In random environments, define $X := \log \rho$.

classification

$$\begin{cases} \mathbb{E}(X) > 0, & \text{supercritical} \\ \mathbb{E}(X) = 0, & \text{critical} \\ \mathbb{E}(X) < 0, & \text{subcritical} \end{cases}$$

Backgrounds

- For single-type BPRE, the survival probability problem has been well studied(see, for example, Athreya & Karlin(1971),Kozlov (1976), Geiger et. al(2003), Afanasyev et. al(2005)).
- For multi-type BPRE, Weissner(1971), Kaplan(1974), and Tanny(1981) have got the extinction criteria.

- A phase transition in the behavior of subcritical BPRE was first noted in Afanasyev(1980) and Dekking(1988), and it has been studied detailedly in Afanasyev(2001), Birkner et. al(2005), etc.

- Recently, there has been interest in a phase transition in supercritical BPRE, conditioned on surviving and having small values at some large generation, see Bansaye & Böinghoff(2012,2013), Nakashima(2013), Böinghoff(2014).
  For the scaling limit of supercritical branching diffusions, a phase transition has been noted in Hutzenthaler(2011).

- For multi-type BPRE, the asymptotic behavior of the survival probability in the subcritical case has been found in Dyakonova(2008,2013).

- Question: Does supercritical multi-type BPRE exist a phase transition in the probability of having small values at some large generation?

Main Results

- In supercritical($\mathbb{E}(X) > 0$) multi-type BPRE, there exists more detailed regime classifications:

$$\begin{cases} 0 < \mathbb{E}(Xe^{-X}) < \infty, & \text{strongly supercritical} \\ \mathbb{E}(Xe^{-X}) = 0, & \text{intermediately supercritical} \\ \mathbb{E}(Xe^{-X}) < 0, & \text{weakly supercritical} \end{cases}$$

- When $X \geq 0$, then $\mathbb{E}(Xe^{-X}) \leq \mathbb{E}(X)$,
  when $X < 0$, then $\mathbb{E}(Xe^{-X}) > \mathbb{E}(X)$.
  So timing the e^{-X} is similar to make the part of $X < 0$ "bigger" and the part of $X \geq 0$ "smaller".

- An important tool: a change of measure.

  For any $n \in \mathbb{N} = \{1, 2, \cdots\}$ and any bounded measurable function $\phi : (\Delta_K)^n \times (\mathbb{N}_0^K)^{n+1} \to \mathbb{R}$,

$$\mathbf{E}\left[\phi\left(\pi_0, \cdots, \pi_{n-1}, \mathbf{Z}_0, \cdots, \mathbf{Z}_n\right)\right]$$
$$:= \kappa^{-n}\mathbb{E}\left[e^{-S_n}\phi\left(\pi_0, \cdots, \pi_{n-1}, \mathbf{Z}_0, \cdots, \mathbf{Z}_n\right)\right],$$

  where $\kappa := \mathbb{E}\left[e^{-X}\right]$.

- Then $0 < \mathbb{E}(Xe^{-X}) < \infty \Rightarrow \mathbf{E}(X) > 0$,

$$\mathbb{E}(Xe^{-X}) = 0 \ \Rightarrow \mathbf{E}(X) = 0.$$

  Under $\mathbb{P}$, strongly supercritical $\Rightarrow$ Under $\mathbf{P}$, supercritical

  intermediately supercritical $\Rightarrow$ critical

Condition

- Condition 1: There exist a number $\alpha \in (0,1)$ and a nonrandom positive vector $\mathbf{l} = (l_1, \cdots, l_K)$ such that for all mean matrices $M_n, n \geq 0$,

$$\alpha \leq \frac{M_n(i_1, j_1)}{M_n(i_2, j_2)} \leq \frac{1}{\alpha}, \quad 1 \leq i_1, i_2, j_1, j_2 \leq K,$$

$$\mathbf{l}(M_n) = \mathbf{l}.$$

i.e. all mean matrices M_n have a common left eigenvector $\mathbf{l}$, if we set $\rho_n = \rho(M_n)$, then $\mathbf{l} M_n = \rho_n \mathbf{l}$.

- Condition 2: the linear fractional offspring distributions.
  All the generating functions $\mathbf{f}_n(\mathbf{s})$ have the form like

$$\mathbf{f}_n(\mathbf{s}) = \mathbf{1} - \frac{(\mathbf{1}-\mathbf{s})M_n}{1+(\mathbf{1}-\mathbf{s})\widetilde{\boldsymbol{\gamma}}_n}, \quad n \geq 0$$

  where $\widetilde{\boldsymbol{\gamma}}_n = (\gamma_n, \cdots, \gamma_n)^T$ is a K-dimensional column random vector of the environment.

  Under this condition, direct calculations with generating functions are feasible.

$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = z)$ is the probability that the nth generation total size is $z \geq 0$ with the initial state is one single particle of type i, $i = 1, \cdots, K$.

Main Results(strongly supercritical)

(1)In the strongly supercritical case $0 < \mathbb{E}(Xe^{-X}) < \infty$.

Theorem 1.1

Under $0 < \mathbb{E}(Xe^{-X}) < \infty$, there is a constant $\theta_i > 0$ such that as $n \to \infty$,

$$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = 1) \sim \theta_i \cdot \kappa^n.$$

Furthermore, for $z_i \geq 0$, $i = 1, \cdots, K$, there is a constant $\theta_i^{'} > 0$ such that as $n \to \infty$,

$$\mathbb{P}_{(i)}(\mathbf{Z}_n = (z_1, \cdots, z_K)) \sim \theta_i^{'} \cdot \kappa^n,$$

where $\kappa := \mathbb{E}\left[e^{-X}\right]$.

When the size of $\mathbf{Z}_n$ is conditioned on a finite interval, we get that

Theorem 1.2

Assume $0 < \mathbb{E}(Xe^{-X}) < \infty$. As $n \to \infty$ and for every $c \in \mathbb{N}$ and $1 \le k \le c$,

$$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = k \mid 1 \le |\mathbf{Z}_n| \le c) \to \frac{1}{c},$$

i.e. the limiting distribution is uniform on $\{1, \cdots, c\}$.

For fixed $k \leq n$, denote the right eigenvector of $M_n \cdots M_k$ that corresponds to the Perron root of the product by $\mathbf{r}(M_{n,k})$. Under Condition 1 there exists a random vector $\mathbf{r} = (r_1, \cdots, r_K)$ such that $\mathbf{r}(M_{n,k}) \to \mathbf{r}$ uniformly as $n \to \infty$.

Theorem 1.3

Under $0 < \mathbb{E}(Xe^{-X}) < \infty$, there is a probability distribution $u = (u_z)_{z \in \mathbf{N}}$ such that for all $t \in (0, 1)$,

$$\lim_{n \to \infty} \mathbb{P}_{(i)}(|\mathbf{Z}_{\lfloor nt \rfloor}| = z \mid |\mathbf{Z}_n| = 1) = u_z.$$

Moreover, u_z does not depend on t and is given by

$$u_z = z\mathbf{E}\left[\left(\sum_{i=1}^{K} r_i \mathbb{P}_{(i)}(|\mathbf{Z}_\infty| = 0 \mid \Pi)\right)^{z-1} \sum_{j=1}^{K} \frac{r_j}{l_j} \mathbb{P}_{(j)}(|\mathbf{Z}_\infty| > 0 \mid \Pi)^2\right].$$

Main Results(intermediately supercritical)

(2)In the intermediately supercritical case $\mathbb{E}(Xe^{-X}) = 0$.

In this situation, we require some regularity of the distribution of X and the offspring.

- Assumption 1:

  The distribution of $X := \log \rho$ belongs to the domain of attraction of some stable law with index $a \in (0, 2]$.

- Assumption 2:

  There is an $\varepsilon > 0$ such that $\mathbf{E}[|\log(\sum_{i=1}^{K}(1 - \pi^{(i)}(0)))|^{a+\varepsilon}] < \infty$.

Theorem 2.1

Assume $\mathbb{E}(Xe^{-X}) = 0$. Then under Assumptions 1 and 2, there is a constant $\delta_i > 0$ such that as $n \to \infty$,

$$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = 1) \sim \delta_i \cdot \tilde{\kappa}^n \mathbf{P}(L_n \geq 0).$$

Furthermore, for $z_i \geq 0$, $i = 1, \cdots, K$, there is a constant $\delta_i' = \delta_i'(z_1, \cdots, z_K) > 0$ such that as $n \to \infty$,

$$\mathbb{P}_{(i)}(\mathbf{Z}_n = (z_1, \cdots, z_K)) \sim \delta_i' \cdot \tilde{\kappa}^n \mathbf{P}(L_n \geq 0),$$

where $\tilde{\kappa} := \mathbb{E}\left[e^{-X}\right]$.

Define $L_n := \min\{S_0, \cdots, S_n\}$.

Note that though $\tilde{\kappa}$ and κ in Theorem 1.1 have the same form, they have different values.

Theorem 2.2

Assume $\mathbb{E}(Xe^{-X}) = 0$ and Assumptions 1 and 2. As $n \to \infty$ and for every $c \in \mathbb{N}$ and $1 \le k \le c$,

$$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = k \mid 1 \le |\mathbf{Z}_n| \le c) \to \frac{1}{c},$$

i.e. the limiting distribution is uniform on $\{1, \cdots, c\}$.

Theorem 2.3

Assume $\mathbb{E}(Xe^{-X}) = 0$ and Assumptions 1 and 2. There is a probability distribution $v = (v_z)_{z\in\mathbf{N}}$ such that for all $t \in (0,1)$

$$\lim_{n\to\infty} \mathbb{P}_{(i)}(|\mathbf{Z}_{\tau_{\lfloor nt\rfloor,n}}| = z \mid |\mathbf{Z}_n| = 1) = v_z,$$

where

$$v_z = \sum_{z_1+\cdots+z_K=z} \frac{z!}{z_1!\cdots z_K!}\mathbf{E}^+[r_1^{z_1}\cdots r_K^{z_K}\sum_{i=1}^{K}\frac{z_i}{l_i}\mathbb{P}_{(i)}(|\mathbf{Z}_\infty| > 0 \mid \Pi)^2 \cdot\mathbb{P}_{(i)}(|\mathbf{Z}_\infty| = 0 \mid \Pi)^{z_i-1}\prod_{j\neq i}\mathbb{P}_{(j)}(|\mathbf{Z}_\infty| = 0 \mid \Pi)^{z_j}].$$

$\tau_{k,n} = min\{k \le i \le n : S_i = min\{S_k, \cdots, S_n\}\}$ denotes the time of the first minimum between generations k and n.

Proofs

By iterating,

$$\mathbf{1} - \mathbf{f}_0 \left(\mathbf{f}_1 \cdots (\mathbf{f}_{n-1}(\mathbf{s})) \right)$$

$$= \frac{(\mathbf{1}-\mathbf{s})\, M_{n-1} \cdots M_0}{1 + (\mathbf{1}-\mathbf{s})\, M_{n-1} \cdots M_1 \widetilde{\boldsymbol{\gamma}}_0 + (\mathbf{1}-\mathbf{s})\, M_{n-1} \cdots M_2 \widetilde{\boldsymbol{\gamma}}_1 + \cdots + (\mathbf{1}-\mathbf{s})\, \widetilde{\boldsymbol{\gamma}}_{n-1}}$$

Under Condition 1, the product of M_n converges.

We introduce the associated random walk:

let $X_i = \ln \rho_{i-1}, \quad \eta_i = \dfrac{\gamma_{i-1}|\mathbf{l}|}{\rho_{i-1}}$, $i \geq 1$,

$$S_0 = 0,\ S_n = X_1 + \cdots + X_n,\ n \geq 1.$$

$$H_n := \frac{\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k} \left(1 + O(\beta^{n-k-1})\right)}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k} \left(1 + O(\beta^{n-k-1})\right)}, \quad \beta = \frac{1-\alpha^2}{1+\alpha^2} \in (0,1).$$

Then given the environment, the relation is valid for $z \geq 1$:

$$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = z \mid \Pi) = \frac{e^{-S_n}}{l_i \left(1 + O(\beta^n)\right)} \mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2 H_n^{z-1} \quad a.s.$$

Proof of Theorem 1.1

By the change of measure,

$$\begin{aligned}\mathbb{P}_{(i)}(|\mathbf{Z}_n| = 1) &= \mathbb{E}\left[\frac{e^{-S_n}}{l_i\left(1+O(\beta^n)\right)}\mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2\right] \\ &= \frac{\kappa^n}{l_i\left(1+O(\beta^n)\right)}\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2\right]\end{aligned}$$

$$\frac{\mathbb{P}_{(i)}(|\mathbf{Z}_n| = 1)}{\kappa^n} \to \frac{1}{l_i}\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_\infty| > 0 \mid \Pi)^2\right]$$

It's proved in Kaplan(1974) that under the assumption, $\mathbf{E}(X) > 0$ implies that the process survives with a positive probability, i.e.

$$\lim_{n\to\infty} \mathbf{P}_{(i)}(|\mathbf{Z}_n| > 0) = \mathbf{P}_{(i)}(|\mathbf{Z}_\infty| > 0) > 0.$$

Then we conclude the result.

Proof of Theorem 1.2

Again by the change of measure,

$$\mathbb{P}_{(i)}(|\mathbf{Z}_n| = z) = \frac{\kappa^n}{l_i\,(1 + O(\beta^n))}\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2 H_n^{z-1}\right]$$

Noting that under $\mathbf{P}$, $S_n \to \infty \quad a.s.$ and therefore $H_n \to 1 \quad a.s.$

By the dominated convergence theorem, for every $z \in \mathbb{N}$,

$$\lim_{n\to\infty} \kappa^{-n}\mathbb{P}_{(i)}(|\mathbf{Z}_n| = z) = \frac{1}{l_i}\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_\infty| > 0 \mid \Pi)^2\right] = \theta_i.$$

The limit does not depend on z. So we get

$$\lim_{n\to\infty} \mathbb{P}_{(i)}(|\mathbf{Z}_n| = k \mid 1 \le |\mathbf{Z}_n| \le c) = \lim_{n\to\infty} \frac{\mathbb{P}_{(i)}(|\mathbf{Z}_n| = k)}{\sum_{j=0}^{c} \mathbb{P}_{(i)}(|\mathbf{Z}_n| = j)} = \frac{1}{c}.$$

Proof of intermediately supercritical case

Under $\mathbf{P}$, $\mathbf{E}(X) = 0$, S_n is a recurrent random walk,

$$\limsup_{n\to\infty} S_n = +\infty, \quad \liminf_{n\to\infty} S_n = -\infty.$$

Define $L_n := \min\{S_0, \cdots, S_n\}$, $L_{k,n} := \min_{0\le j\le n-k}\{S_{k+j} - S_k\}$, $0 \le k \le n$,

$$M_n := \max\{S_0, \cdots, S_n\},$$

$$\tau_n := \min\{0 \le i \le n : S_i = L_n\}.$$

$$V(x) = \begin{cases} 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \le x, M_k < 0) & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$ is a renewal function.

For any bounded measurable random variable Y_n, $\mathbf{E}^+ Y_n = \mathbf{E}[Y_n V(S_n); L_n \ge 0]$.

Under $\mathbf{P}^+$, S_n can be viewed as a random walk conditioned to never hit the strictly negative half line.

Lemma 1(Dyakonova,2007)

Suppose that the condition hold. Let Y_n, $n \in \mathbb{N}_0$, be a sequence of uniformly bounded random variables. If this sequence converges $\mathbf{P}^+ - a.s.$ to a random variable Y_∞, then

$$\mathbf{E}[Y_n \mid L_n \geq 0] \to \mathbf{E}^+ Y_\infty, \quad n \to \infty.$$

Lemma 2(Dyakonova,2007)

Suppose that the condition hold. Let V_n, $n \in \mathbb{N}_0$, be a sequence of uniformly bounded r.v.s which for every $k \geq 0$ satisfy the relation $\mathbf{E}(V_n; |\mathbf{Z}_k| > 0, L_{k,n} \geq 0 \mid \mathcal{F}_k) = \mathbf{P}(L_n \geq 0)(V_{k,\infty} + o(1)) \quad \mathbf{P} - \mathbf{a.s.}$ for some sequence of r.v.s $V_{1,\infty}, V_{2,\infty}, \cdots$ Then

$$\mathbf{E}(V_n; |\mathbf{Z}_{\tau_n}| > 0) = \mathbf{P}(L_n \geq 0) \left(\sum_{k=0}^{\infty} \mathbf{E}(V_{k,\infty}; \tau_k = k) + o(1) \right).$$

Proof of Theorem 2.1

Since $\mathbb{P}_{(i)}(|\mathbf{Z}_n| = 1) = \dfrac{\kappa^n}{l_i\,(1+O(\beta^n))}\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2\right]$,
we need to estimate $\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2\right]$.
Let $V_n = \mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi) \cdot \mathbb{1}_{\{\mathbf{Z}_0=\mathbf{e}_i, |\mathbf{Z}_n|>0\}}$, then by Lemma 1, we have a r.v. $V_{k,\infty}^{(i)}$ s.t.

$$\mathbf{E}[V_n \mid L_n \geq 0] \overset{n\to\infty}{\to} V_{k,\infty}^{(i)}.$$

We can testify that the r.v. $V_{k,\infty}^{(i)}$ satisfies the condition of Lemma 2, so

$$\mathbf{E}\left[\mathbb{P}_{(i)}(|\mathbf{Z}_n| > 0 \mid \Pi)^2\right] \overset{n\to\infty}{\sim} \mathbf{P}(L_{k,n} \geq 0) \sum_{k=0}^{\infty} \mathbf{E}(V_{k,\infty}^{(i)}; \tau_k = k).$$

Thus by $\mathbf{P}(L_{k,n} \geq 0) \overset{n\to\infty}{\sim} \mathbf{P}(L_n \geq 0)$ and define $\delta_i = \dfrac{\sum_{k=0}^{\infty} \mathbf{E}(V_{k,\infty}^{(i)}; \tau_k = k)}{l_i}$,
we conclude the result.

References

Athreya, K. B. and Karlin, S. (1971) On branching processes with random environment. I. Extinction probabilities. *Ann. Math. Statist.* **42**, 1499-1520.

Weissner, E. (1971) Multitype branching processes in random environments. *J. Appl. Prob.* **8**, 17-31.

Kaplan, N. (1974) Some results about multi-dimensional branching processes with random environments. *Ann. Prob.* **2**, 441-455.

Tanny, D. (1981) On multitype branching processes in a random environment. *Adv. in Appl. Prob.* **13**, 464-497.

Birkner, M., Geiger, J. and Kersting, G. (2005) Branching processes in random environment - a view on critical and subcritical cases. *Interacting Stochastic Systems of High Complexity.* Springer Berlin Heidelberg, 265-291.

Böinghoff, C. (2014) Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions. *Stochastic Process. Appl.* **124**, 3553-3577.

- Dyakonova, E. (2007) Critical multitype branching processes in a random environment. *Discrete Math. Appl.* **17**, 587-606.
- Dyakonova, E. (2008) On subcritical multi-type branching process in random environment. *DMTCS*, 397-404.
- Dyakonova, E. (2013) Multitype subcritical branching processes in a random environment. *Proceedings of the Steklov Institute of Maathematics* **282**, 80-89.
- Afanasyev, V. I., Geiger, J., Kersting, G. and Vatutin, V. A. (2005) Criticality for branching processes in random environment. *Ann. Prob.* **33**, 645-673.
- Kesten, H. and Spitzer, F. (1984) Convergence in distribution of products of random matrices. *Z. Wahrsch. verw. Geb.* **67**, 363-386.

Thank you!