Limit Theorems for Supercritical MBPRE with Linear Fractional Offspring Distributions

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Model

Multitype Branching Processes in Random Environments (MBPRE):

- $K$ different types of particles
- $(\Delta, \mathcal{B}(\Delta))$ is the space of probability measures on $\mathbb{N}_0^K$,
- $(\Delta_K, \mathcal{B}(\Delta_K))$ is the $K$-fold product of the space $(\Delta, \mathcal{B}(\Delta))$ on itself.

Let $\pi = (\pi^{(1)}, \cdots, \pi^{(K)})$ be a r.v. taking values in $(\Delta_K, \mathcal{B}(\Delta_K))$.

An infinite sequence $\Pi = (\pi_0, \pi_1, \pi_2, \cdots)$ of i.i.d. copies of $\pi$ is called a random environment.

$P$: an i.i.d. product measure on $(\Delta_K)^\mathbb{Z}$.

$\{Z_n = (Z_n^{(1)}, \cdots, Z_n^{(K)})\}_{n \geq 0}$ is a sequence of random $K$-dimensional vectors.
If $Z_0$ is independent of $\Pi$, and

$$\mathcal{L}(Z_{n+1} \mid Z_n = (z_1, \cdots, z_K), \Pi = (\pi_0, \pi_1, \cdots)) = \mathcal{L}\left(\sum_{i=1}^{K} \sum_{j=1}^{\xi_{n,j}^{(i)}(\pi_n)} z_i \xi_{n,j}^{(i)}(\pi_n)\right),$$

where $\xi_{n,1}^{(i)}, \cdots, \xi_{n,z_i}^{(i)}$ are independent $K$-dimensional random vectors and are identically distributed according to $\pi_n^{(i)}, i = 1, \cdots, K$.

Such $\{Z_n\}$ is called a $K$-type Galton-Watson branching process in a random environment.

Quenched law $\mathbb{P}(\cdot \mid \Pi)$: fix an environment

Annealed law $\mathbb{P}$: average over environments

$$\mathbb{P}(\cdot) := \int_{(\Delta_K)^Z} \mathbb{P}(\cdot \mid \Pi) P(d\Pi)$$
An one-to-one correspondence between probability measures and generating functions $f_n(s) = (f_n^{(1)}(s), \cdots, f_n^{(K)}(s))$, $n \geq 0$:

$$f_n^{(i)}(s) = \sum_{t \in N_0^K} \pi_n^{(i)}(\{t\}) s^t$$

$f_n^{(i)}(s) = f_n^{(i)}(s_1, \cdots, s_K)$ is the generating function of type $i$ in $n$th generation.

The mean matrix $M_n = M_n(\pi_n) = (M_n(i, j))_{i, j=1}^K := \left( \frac{\partial f_n^{(i)}(1)}{\partial s_j} \right)_{i, j=1}^K$ is a random matrix, $n \geq 0$.

Assume that the elements of the mean matrices are positive.
In constant environments, $\rho > 0$ is the Perron root (maximal eigenvalue) of $M$, $\mathbf{l}$ and $\mathbf{r}$ are the corresponding positive left and right eigenvectors. According to Perron-Frobenius Theorem,

$$E[\mathbf{Z}_n] = E[\mathbf{Z}_0]M^n \approx \rho^n E[\mathbf{Z}_0]S,$$

where $S = (r_i l_j)_{i,j=1}^K$.

In random environments, define $X := \log \rho$.

\[ \begin{align*}
\mathbb{E}(X) > 0, & \quad \text{supercritical} \\
\mathbb{E}(X) = 0, & \quad \text{critical} \\
\mathbb{E}(X) < 0, & \quad \text{subcritical}
\end{align*} \]
For single-type BPRE, the survival probability problem has been well studied (see, for example, Athreya & Karlin (1971), Kozlov (1976), Geiger et al (2003), Afanasyev et al (2005)).

For multi-type BPRE, Weissner (1971), Kaplan (1974), and Tanny (1981) have got the extinction criteria.
A phase transition in the behavior of subcritical BPRE was first noted in Afanasyev(1980) and Dekking(1988), and it has been studied detailedly in Afanasyev(2001), Birkner et al(2005), etc.

Recently, there has been interest in a phase transition in supercritical BPRE, conditioned on surviving and having small values at some large generation, see Bansaye & Böinghoff(2012,2013), Nakashima(2013), Böinghoff(2014).
For the scaling limit of supercritical branching diffusions, a phase transition has been noted in Hutzenthaler(2011).
For multi-type BPRE, the asymptotic behavior of the survival probability in the subcritical case has been found in Dyakonova (2008, 2013).

**Question:** Does supercritical multi-type BPRE exist a phase transition in the probability of having small values at some large generation?
Main Results

- In supercritical($\mathbb{E}(X) > 0$) multi-type BPRE, there exists more detailed regime classifications:

\[
\begin{cases}
0 < \mathbb{E}(X e^{-X}) < \infty, & \text{strongly supercritical} \\
\mathbb{E}(X e^{-X}) = 0, & \text{intermediately supercritical} \\
\mathbb{E}(X e^{-X}) < 0, & \text{weakly supercritical}
\end{cases}
\]

- When $X \geq 0$, then $\mathbb{E}(X e^{-X}) \leq \mathbb{E}(X)$,
when $X < 0$, then $\mathbb{E}(X e^{-X}) > \mathbb{E}(X)$.
So timing the $e^{-X}$ is similar to make the part of $X < 0$ ”bigger” and the part of $X \geq 0$ ”smaller”.

An important tool: a change of measure.

For any \( n \in \mathbb{N} = \{1, 2, \cdots \} \) and any bounded measurable function \( \phi : (\Delta_K)^n \times (\mathbb{N}_0^K)^{n+1} \to \mathbb{R} \),

\[
\mathbb{E} \left[ \phi (\pi_0, \cdots, \pi_{n-1}, Z_0, \cdots, Z_n) \right] := \kappa^{-n} \mathbb{E} \left[ e^{-S_n} \phi (\pi_0, \cdots, \pi_{n-1}, Z_0, \cdots, Z_n) \right],
\]

where \( \kappa := \mathbb{E} \left[ e^{-X} \right] \).

Then \( 0 < \mathbb{E}(Xe^{-X}) < \infty \Rightarrow \mathbb{E}(X) > 0 \),

\( \mathbb{E}(Xe^{-X}) = 0 \Rightarrow \mathbb{E}(X) = 0 \).

Under \( \mathbb{P} \), strongly supercritical \( \Rightarrow \) Under \( \mathbb{P} \), supercritical

intermediately supercritical \( \Rightarrow \) critical
Condition 1: There exist a number $\alpha \in (0, 1)$ and a nonrandom positive vector $l = (l_1, \cdots, l_K)$ such that for all mean matrices $M_n, n \geq 0$,

$$\alpha \leq \frac{M_n(i_1, j_1)}{M_n(i_2, j_2)} \leq \frac{1}{\alpha}, \quad 1 \leq i_1, i_2, j_1, j_2 \leq K,$$

$$1(M_n) = 1.$$ 

i.e. all mean matrices $M_n$ have a common left eigenvector $l$, if we set $\rho_n = \rho(M_n)$, then $1M_n = \rho_n l$. 

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Condition 2: the linear fractional offspring distributions.

All the generating functions $f_n(s)$ have the form like

$$f_n(s) = 1 - \frac{(1 - s)M_n}{1 + (1 - s)\tilde{\gamma}_n}, \quad n \geq 0$$

where $\tilde{\gamma}_n = (\gamma_n, \cdots, \gamma_n)^T$ is a $K$-dimensional column random vector of the environment.

Under this condition, direct calculations with generating functions are feasible.
$\mathbb{P}_i(|Z_n| = z)$ is the probability that the $n$th generation total size is $z \geq 0$ with the initial state is one single particle of type $i$, $i = 1, \cdots, K$. 
Main Results (strongly supercritical)

(1) In the strongly supercritical case $0 < \mathbb{E}(Xe^{-X}) < \infty$.

**Theorem 1.1**

Under $0 < \mathbb{E}(Xe^{-X}) < \infty$, there is a constant $\theta_i > 0$ such that as $n \to \infty$,

$$
\mathbb{P}(|Z_n| = 1) \sim \theta_i \cdot \kappa^n.
$$

Furthermore, for $z_i \geq 0$, $i = 1, \cdots, K$, there is a constant $\theta'_i > 0$ such that as $n \to \infty$,

$$
\mathbb{P}(Z_n = (z_1, \cdots, z_K)) \sim \theta'_i \cdot \kappa^n,
$$

where $\kappa := \mathbb{E}[e^{-X}]$. 

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When the size of $Z_n$ is conditioned on a finite interval, we get that

**Theorem 1.2**

Assume $0 < \mathbb{E}(X e^{-X}) < \infty$. As $n \to \infty$ and for every $c \in \mathbb{N}$ and $1 \leq k \leq c$,

$$
\mathbb{P}(\mathbb{P}_i(\mid Z_n \mid = k \mid 1 \leq \mid Z_n \mid \leq c) \to \frac{1}{c},
$$

i.e. the limiting distribution is uniform on $\{1, \cdots, c\}$. 
For fixed $k \leq n$, denote the right eigenvector of $M_n \cdots M_k$ that corresponds to the Perron root of the product by $r(M_{n,k})$. Under Condition 1 there exists a random vector $r = (r_1, \ldots, r_K)$ such that $r(M_{n,k}) \to r$ uniformly as $n \to \infty$.

**Theorem 1.3**

Under $0 < \mathbb{E}(X e^{-X}) < \infty$, there is a probability distribution $u = (u_z)_{z \in \mathbb{N}}$ such that for all $t \in (0, 1)$,

$$
\lim_{n \to \infty} \mathbb{P}(i) (|Z_{nt}| = z \mid |Z_n| = 1) = u_z.
$$

Moreover, $u_z$ does not depend on $t$ and is given by

$$
u_z = z \underbrace{\mathbb{E} \left[ \left( \sum_{i=1}^{K} r_i \mathbb{P}(i)(|Z_\infty| = 0 \mid \Pi) \right)^{z-1} \sum_{j=1}^{K} \frac{r_j}{l_j} \mathbb{P}(j)(|Z_\infty| > 0 \mid \Pi)^2 \right]}_{Z.
$$
(2) In the intermediately supercritical case $\mathbb{E}(X e^{-X}) = 0$.

In this situation, we require some regularity of the distribution of $X$ and the offspring.

- **Assumption 1:**
  
  The distribution of $X := \log \rho$ belongs to the domain of attraction of some stable law with index $a \in (0, 2]$.

- **Assumption 2:**
  
  There is an $\varepsilon > 0$ such that $\mathbb{E} [ | \log(\sum_{i=1}^{K} (1 - \pi^{(i)}(0))) |^{a+\varepsilon} ] < \infty$. 

Theorem 2.1

Assume $\mathbb{E}(X e^{-X}) = 0$. Then under Assumptions 1 and 2, there is a constant $\delta_i > 0$ such that as $n \to \infty$,

$$
\mathbb{P}(i)(|Z_n| = 1) \sim \delta_i \cdot \tilde{\kappa}^n \mathbb{P}(L_n \geq 0).
$$

Furthermore, for $z_i \geq 0$, $i = 1, \cdots, K$, there is a constant $\delta'_i = \delta'_i(z_1, \cdots, z_K) > 0$ such that as $n \to \infty$,

$$
\mathbb{P}(i)(Z_n = (z_1, \cdots, z_K)) \sim \delta'_i \cdot \tilde{\kappa}^n \mathbb{P}(L_n \geq 0),
$$

where $\tilde{\kappa} := \mathbb{E}[e^{-X}]$.

Define $L_n := \min\{S_0, \cdots, S_n\}$.

Note that though $\tilde{\kappa}$ and $\kappa$ in Theorem 1.1 have the same form, they have different values.
Theorem 2.2

Assume $\mathbb{E}(X e^{-X}) = 0$ and Assumptions 1 and 2. As $n \to \infty$ and for every $c \in \mathbb{N}$ and $1 \leq k \leq c$,

$$\mathbb{P}_{(i)}(\{Z_n\} = k \mid 1 \leq \{Z_n\} \leq c) \to \frac{1}{c},$$

i.e. the limiting distribution is uniform on $\{1, \cdots, c\}$. 
Theorem 2.3

Assume $\mathbb{E}(Xe^{-X}) = 0$ and Assumptions 1 and 2. There is a probability distribution $\nu = (\nu_z)_{z \in \mathbb{N}}$ such that for all $t \in (0, 1)$

$$\lim_{n \to \infty} \mathbb{P}(i)(|Z_{\tau_{\lfloor nt \rfloor}, n}| = z | |Z_n| = 1) = \nu_z,$$

where

$$\nu_z = \sum_{z_1 + \ldots + z_K = z} \frac{z!}{z_1! \cdots z_K!} \mathbb{E}^{+}[r_1^{z_1} \cdots r_K^{z_K} \sum_{i=1}^{K} \frac{z_i}{l_i} \mathbb{P}(i)(|Z_\infty| > 0 | \Pi)^2 \cdot \mathbb{P}(i)(|Z_\infty| = 0 | \Pi)^{z_i - 1} \prod_{j \neq i} \mathbb{P}(j)(|Z_\infty| = 0 | \Pi)^{z_j}].$$

$\tau_{k, n} = min\{k \leq i \leq n : S_i = min\{S_k, \ldots, S_n\}\}$ denotes the time of the first minimum between generations $k$ and $n.$
Proofs

By iterating,

\[ 1 - f_0(f_1 \cdots (f_{n-1}(s))) \]

\[ = \frac{(1 - s) M_{n-1} \cdots M_0}{1 + (1 - s) M_{n-1} \cdots M_1 \tilde{\gamma}_0 + (1 - s) M_{n-1} \cdots M_2 \tilde{\gamma}_1 + \cdots + (1 - s) \tilde{\gamma}_{n-1}} \]

Under Condition 1, the product of \( M_n \) converges.

We introduce the associated random walk:

let \( X_i = \ln \rho_{i-1}, \quad \eta_i = \frac{\gamma_{i-1} |\mathbf{1}|}{\rho_{i-1}}, \quad i \geq 1, \)

\( S_0 = 0, \ S_n = X_1 + \cdots + X_n, \ n \geq 1. \)
\[ H_n := \frac{\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k} \left( 1 + O(\beta^{n-k-1}) \right)}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k} \left( 1 + O(\beta^{n-k-1}) \right)}, \quad \beta = \frac{1 - \alpha^2}{1 + \alpha^2} \in (0, 1). \]

Then given the environment, the relation is valid for \( z \geq 1 \):

\[
\mathbb{P}(i)\left(|Z_n| = z \mid \Pi \right) = \frac{e^{-S_n}}{l_i (1 + O(\beta^n))} \frac{\mathbb{P}(i)\left(|Z_n| > 0 \mid \Pi \right)}{\mathbb{P}(i)\left(|Z_n| > 0 \mid \Pi \right)^2 H_n^{z-1}} \text{ a.s.}
\]
Proof of Theorem 1.1

By the change of measure,

\[
P_{(i)}(\lvert Z_n \rvert = 1) = \mathbb{E} \left[ \frac{e^{-S_n}}{l_i (1 + O(\beta^n))} P_{(i)}(\lvert Z_n \rvert > 0 \mid \Pi)^2 \right] = \frac{\kappa^n}{l_i (1 + O(\beta^n))} \mathbb{E} \left[ P_{(i)}(\lvert Z_n \rvert > 0 \mid \Pi)^2 \right]
\]

\[
\frac{P_{(i)}(\lvert Z_n \rvert = 1)}{\kappa^n} \to \frac{1}{l_i} \mathbb{E} \left[ P_{(i)}(\lvert Z_{\infty} \rvert > 0 \mid \Pi)^2 \right]
\]

It’s proved in Kaplan(1974) that under the assumption, \( \mathbb{E}(X) > 0 \) implies that the process survives with a positive probability, i.e.

\[
\lim_{n \to \infty} P_{(i)}(\lvert Z_n \rvert > 0) = P_{(i)}(\lvert Z_{\infty} \rvert > 0) > 0.
\]

Then we conclude the result.
Proof of Theorem 1.2

Again by the change of measure,

\[ P(i)(|Z_n| = z) = \frac{\kappa^n}{l_i(1 + O(\beta n))} \mathbb{E}[P(i)(|Z_n| > 0 | \Pi)^2 H_n^{z-1}] \]

Noting that under \( P \), \( S_n \to \infty \) \( a.s. \) and therefore \( H_n \to 1 \) \( a.s. \).

By the dominated convergence theorem, for every \( z \in \mathbb{N}, \)

\[ \lim_{n \to \infty} \kappa^{-n} P(i)(|Z_n| = z) = \frac{1}{l_i} \mathbb{E}[P(i)(|Z_\infty| > 0 | \Pi)^2] = \theta_i. \]

The limit does not depend on \( z \). So we get

\[ \lim_{n \to \infty} P(i)(|Z_n| = k \mid 1 \leq |Z_n| \leq c) = \lim_{n \to \infty} \frac{P(i)(|Z_n| = k)}{\sum_{j=0}^{c} P(i)(|Z_n| = j)} = \frac{1}{c}. \]
Proof of intermediately supercritical case

Under $\mathbf{P}$, $\mathbf{E}(X) = 0$, $S_n$ is a recurrent random walk,

$$
\lim_{n \to \infty} \sup S_n = +\infty, \quad \lim_{n \to \infty} \inf S_n = -\infty.
$$

Define $L_n := \min\{S_0, \cdots, S_n\}$, $L_{k,n} := \min_{0 \leq j \leq n-k} \{S_{k+j} - S_k\}$, $0 \leq k \leq n$,

$M_n := \max\{S_0, \cdots, S_n\}$,

$\tau_n := \min\{0 \leq i \leq n : S_i = L_n\}$.

$$
V(x) = \begin{cases} 
1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \leq x, M_k < 0) & \text{if } x \geq 0 \\
0 & \text{if } x < 0
\end{cases}
$$

is a renewal function.

For any bounded measurable random variable $Y_n$, $\mathbf{E}^+Y_n = \mathbf{E}[Y_nV(S_n); L_n \geq 0]$.

Under $\mathbf{P}^+$, $S_n$ can be viewed as a random walk conditioned to never hit the strictly negative half line.
Lemma 1 (Dyakonova, 2007)

Suppose that the condition hold. Let $Y_n, n \in \mathbb{N}_0$, be a sequence of uniformly bounded random variables. If this sequence converges $\mathbb{P}^+ - a.s.$ to a random variable $Y_\infty$, then

$$
\mathbb{E}[Y_n \mid L_n \geq 0] \rightarrow \mathbb{E}^+ Y_\infty, \quad n \rightarrow \infty.
$$

Lemma 2 (Dyakonova, 2007)

Suppose that the condition hold. Let $V_n, n \in \mathbb{N}_0$, be a sequence of uniformly bounded r.v.s which for every $k \geq 0$ satisfy the relation

$$
\mathbb{E}(V_n; |Z_k| > 0, L_{k,n} \geq 0 \mid \mathcal{F}_k) = \mathbb{P}(L_n \geq 0)(V_{k,\infty} + o(1)) \quad \mathbb{P} - a.s.
$$

for some sequence of r.v.s $V_{1,\infty}, V_{2,\infty}, \cdots$. Then

$$
\mathbb{E}(V_n; |Z_{\tau_n}| > 0) = \mathbb{P}(L_n \geq 0) \left( \sum_{k=0}^{\infty} \mathbb{E}(V_{k,\infty}; \tau_k = k) + o(1) \right).
$$
Proof of Theorem 2.1

Since \( P_i(\{Z_n = 1\}) = \frac{k^n}{l_i (1 + O(\beta^n))} E \left[ P_i(\{Z_n > 0 \mid \Pi\}^2) \right] \),

we need to estimate \( E \left[ P_i(\{Z_n > 0 \mid \Pi\}^2) \right] \).

Let \( V_n = P_i(\{Z_n > 0 \mid \Pi\}) \cdot 1_{\{Z_0 = e_i, |Z_n| > 0\}} \), then by Lemma 1, we have a r.v. \( V_{k,\infty}^{(i)} \) s.t.

\[
E[V_n \mid L_n \geq 0] \xrightarrow{n \to \infty} V_{k,\infty}^{(i)}.
\]

We can testify that the r.v. \( V_{k,\infty}^{(i)} \) satisfies the condition of Lemma 2, so

\[
E \left[ P_i(\{Z_n > 0 \mid \Pi\}^2) \right] \xrightarrow{n \to \infty} P(L_{k,n} \geq 0) \sum_{k=0}^{\infty} E(V_{k,\infty}^{(i)}; \tau_k = k).
\]

Thus by \( P(L_{k,n} \geq 0) \xrightarrow{n \to \infty} P(L_n \geq 0) \) and define \( \delta_i = \frac{\sum_{k=0}^{\infty} E(V_{k,\infty}^{(i)}; \tau_k = k)}{l_i} \),

we conclude the result.


Thank you!