Sharp vertical Littlewood–Paley inequalities for heat flows in weighted L^2 spaces

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Outline

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Littlewood–Paley square functions

Let M be a complete (smooth) Riemannian manifold with volume measure vol, non-negative Laplace–Beltrami operator Δ, and gradient operator ∇. For $f \in C_c^\infty(M)$, define

- horizontal Littlewood–Paley g-function
$$g(f)(x) = \Big(\int_0^\infty t \Big| \frac{\partial}{\partial t} e^{-t\sqrt{\Delta}} f(x) \Big|^2 \, \mathrm{d}t \Big)^{1/2},$$
- vertical Littlewood–Paley $\mathcal{G}$-function
$$\mathcal{G}(f)(x) = \Big(\int_0^\infty t |\nabla e^{-t\sqrt{\Delta}} f(x)|^2 \, \mathrm{d}t \Big)^{1/2},$$
- horizontal Littlewood–Paley h-function
$$h(f)(x) = \Big(\int_0^\infty t \Big| \frac{\partial}{\partial t} e^{-t\Delta} f(x) \Big|^2 \, \mathrm{d}t \Big)^{1/2},$$
- vertical Littlewood–Paley $\mathcal{H}$-function
$$\mathcal{H}(f)(x) = \Big(\int_0^\infty |\nabla e^{-t\Delta} f(x)|^2 \, \mathrm{d}t \Big)^{1/2}.$$

L^p boundedness for $1 < p < \infty$: known results

- $g, \mathcal{G}, h, \mathcal{H}$ are all bounded in $L^p(\mathbb{R}^n)$ (see e.g. Stein's book in 1970a).
- g, h are bounded in L^p, $1 < p < \infty$, for symmetric Markov semigroups in a general context (see Stein's book in 1970b).
- For $1 < p \leq 2$, $\mathcal{H}$ and $\mathcal{G}$ are bounded in $L^p(M, \mathrm{vol})$ (see e.g. Coulhon–Duong–Li [Studia Math. 2003]).
- For $2 < p < \infty$, much stronger assumptions are need for the L^p boundedness of $\mathcal{H}, \mathcal{G}$, e.g. $|\nabla e^{-t\Delta} f|^2 \leq C e^{-t\Delta} |\nabla f|^2$ (see Coulhon–Duong [Comm. Pure Appl. Math. 2003]).

Many other results...

Weighted boundedness: A_2 conjecture

Let $M = \mathbb{R}^n$. Given an operator $S : L^2_w(\mathbb{R}^n) \to L^2_w(\mathbb{R}^n)$, prove

$$\|S(f)\|_{L^2_w(\mathbb{R}^n)} \leq C(n,S)\|w\|_{A_2(\mathbb{R}^n)}\|f\|_{L^2_w(\mathbb{R}^n)}, \tag{1}$$

where w is the 2-Muckenhoupt weight (or A_2 weight), i.e., $0 \leq w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and

$$\|w\|_{A_2(\mathbb{R}^n)} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w \, \mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1} \, \mathrm{d}x\Big) < \infty,$$

where the sup is taken over all cubes $Q \subset \mathbb{R}^n$, and

$$L^2_w(\mathbb{R}^n) := \Big\{ f : \mathbb{R}^n \to \mathbb{R} \text{ measurable} \,\Big|\, \int_{\mathbb{R}^n} |f|^2 w \, \mathrm{d}x < \infty \Big\},$$

with the norm of $f \in L^2_w(\mathbb{R}^n)$ defined by

$$\|f\|_{L^2_w(\mathbb{R}^n)} = \Big(\int_M |f(x)|^2 w(x) \, \mathrm{d}x\Big)^{1/2}.$$

Known results on the A_2 conjecture

The problem (1) was solved for

- Hardy–Littlewood maximal operator (Buckley [Trans. Amer. Math. Soc. 1993]),
- Beurling–Ahlfors operator (Petermichl–Volberg [Duke Math. J. 2002]),
- Hilbert transform (Petermichl [Amer. J. Math. 2007]),
- Riesz transform (Petermichl [Proc. Amer. Math. Soc. 2008]),
- Haar shift (Lacey–Petermichl–Reguera [Math. Ann. 2010]).

Later, Hytönen [Ann. Math. 2012] proved (1) for the general Calderón–Zygmund operator.

Recently, by establishing sharp weighted L^2 martingale inequalities, Bañuelos and Osekowski proved (1) for the Littlewood–Paley square function for heat flows in $\mathbb{R}^n$ (see arXiv:1603.07618).

The aim

Let (M, d) be a complete separable metric space endowed with a non-negative Radon measure with full support. The triple (M, d, μ) is called a metric measure space.

Motivated by Bañuelos–Osekowski [arXiv:1603.07618], we are going to establish the weighted L^2 Littlewood–Paley inequalities for heat flows in the $\mathrm{RCD}^*(0, N)$ space (M, d, μ).

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Absolutely continuous curves

From now on, the metric measure space (M, d, μ) is fixed.

Let $q \in [1, \infty)$. A curve $\gamma : [0, 1] \to M$ is said q-absolutely continuous provided there exists $g \in L^q([0, 1])$ such that

$$d(\gamma(t), \gamma(s)) \leq \int_s^t g(r)\,\mathrm{d}r, \quad \forall\ 0 \leq s < t \leq 1.$$

For a q-absolutely continuous curve $\gamma : [0, 1] \to M$, it can be proved the metric slope

$$\lim_{\delta \to 0} \frac{d(\gamma(r + \delta), \gamma(r))}{|\delta|}$$

exists for a.e. r and belongs to $L^q([0, 1])$, denoted by $|\dot{\gamma}_r|$.

For every $\gamma \in C([0, 1], M)$, we use the notation $\int_0^1 |\dot{\gamma}_r|^q\,\mathrm{d}r$, which may be $+\infty$ if γ is not absolutely continuous.

Test plans

For $t \in [0,1]$, the evaluation map $e_t : C([0,1], M) \to M$ is defined by

$$e_t(\gamma) = \gamma(t), \quad \forall\ \gamma \in C([0,1], M).$$

Definition

A probability measure π on $C([0,1], M)$ is called a test plan if, there exists a positive constant C such that

$$(e_t)_\sharp \pi \le C\mu, \quad \text{for any } t \in [0,1],$$

and

$$\int \int_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\pi(\gamma) < \infty.$$

Sobolev spaces

Definition

The Sobolev class $S^2(M)$ is the space of all Borel functions $h : M \to \mathbb{R}$, for which there exists a non-negative function $f \in L^2(M)$ such that, for each test plan π, it holds

$$\int |h(\gamma_1) - h(\gamma_0)| \, \mathrm{d}\pi(\gamma) \leq \int \int_0^1 f(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\pi(\gamma), \qquad (2)$$

where f is called a weak upper gradient for h.

For each $h \in S^2(M)$, there exists a unique minimal function f in the μ-a.e. sense such that (2) holds, which is denoted $|\nabla h|_*$ and is called the minimal weak upper gradient of h.
Define the Sobolev space $W^{1,2}(M) = S^2(M) \cap L^2(M)$, which is a Banach space with the norm

$$\|f\|_{W^{1,2}(M)} := \left(\|f\|_{L^2(M)} + \||\nabla f|_*\|^2_{L^2(M)} \right)^{1/2}.$$

Riemannian curvature-dimension conditions

In general, $(W^{1,2}(M), \|\cdot\|_{W^{1,2}(M)})$ is not a Hilbert space.

Definition (Erbar– Kuwada–Sturm, Invent. Math. 2015)

Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. We say that a metric measure space (M, d, μ) is an $\mathrm{RCD}^*(K, N)$ space if it is a $\mathrm{CD}^*(K, N)$ space and $W^{1,2}(M)$ is Hilbert.

See Bacher–Sturm [J. Funct. Anal. 2010] for the $\mathrm{CD}^*(K, N)$.

The $\mathrm{RCD}(K, \infty)$ is first introduced by Ambrosio–Gigli–Savaré [Duke Math. J. 2014] and Ambrosio–Gigli–Mondino–Rajala [Trans. Amer. Math. Soc. 2015].

Examples of RCD spaces

- Euclidean spaces endowed with the Lebesgue measure,
- complete Riemannian manifolds with Ricci curvature bounded from below,
- measured Gromov–Hausdorff limits of a sequence of Riemannian manifolds with Ricci curvature bounded from below,
- Alexandrov spaces with curvature bounded from below,
- Separable Hilbert spaces endowed with a log-concave probability measure ($N = \infty$).

Heat flows

Let (M, d, μ) *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$, $N \in [1, \infty)$.
For any $f \in W^{1,2}(M)$, define

$$\mathrm{D}(f) = \int_M |\nabla f|_*^2 \, \mathrm{d}\mu.$$

For $f, g \in W^{1,2}(M)$, let

$$\mathrm{D}(f, g) := \frac{1}{4}[\mathrm{D}(f + g) - \mathrm{D}(f - g)] = \int_M \Gamma(f, g) \, \mathrm{d}\mu,$$

where

$$\Gamma(f, g)(x) := \lim_{\epsilon \downarrow 0} \frac{|\nabla(g + \epsilon f)|_*^2(x) - |\nabla g|_*^2(x)}{2\epsilon}, \quad \text{for } \mu\text{-a.e. } x \in X,$$

Then, $(\mathrm{D}, W^{1,2}(M))$ is a strongly local and regular Dirichlet form, and denote $(P_t)_{t \geq 0}$ and Δ the corresponding heat flow and generator, respectively.

Outline

The p-heat weight

Definition

Let $w : M \to [0, \infty]$ be a locally integrable function. For $p \in (1, \infty)$, we say that w is a p-heat weight, denoted by $w \in A_p^{heat}(M)$, if

$$\|w\|_{A_p^{heat}(M)} := \left\| P_t w (P_t w^{-1/(p-1)})^{p-1} \right\|_{L^\infty(M \times [0,\infty), \mu \times \mathcal{L}^1)} < \infty,$$

where $\mathcal{L}^1$ is the Lebesgue measure restricted on $[0, \infty)$.

Note that, by Hölder's inequality, for any $1 < p \le q < \infty$,

$$A_p^{heat}(M) \subset A_q^{heat}(M).$$

For $f \in C_c(M)$ and $x \in M$, define the Littlewood–Paley $\mathcal{H}$-function and $\mathcal{H}_*$-function by

$$\mathcal{H}(f)(x) = \Big(\int_0^\infty |\nabla P_t f|_*^2(x)\, \mathrm{d}t \Big)^{1/2},$$

and

$$\mathcal{H}_*(f)(x) = \Big(\int_0^\infty \int_M |\nabla P_t f|_*^2(y) p_t(x,y)\, \mathrm{d}\mu(y) \mathrm{d}t \Big)^{1/2},$$

where $(p_t)_{t \geq 0}$ is the heat kernel corresponding to $(P_t)_{t \geq 0}$.

Sharp weighted L^2 Littlewood–Paley inequalities

Theorem

Let (M, d, μ) *be an* $\mathrm{RCD}^*(0, N)$ *space with* $N \in [1, \infty)$ *and* $w \in A_2^{heat}(M)$*. Suppose that* $\limsup_{r \to \infty}[\mu(B(o, r))/r^N] > 0$*, for some* $o \in M$*. Then, for every* $f \in C_c(M)$*,*

$$\|f\|_{L^2_w(M,\mu)} \leq (320\|w\|_{A_2^{heat}(M)})^{1/2}\|\mathcal{H}_*(f)\|_{L^2_w(M,\mu)},$$

$$\|\mathcal{H}_*(f)\|_{L^2_w(M,\mu)} \leq 2^{5/4}\|w\|_{A_2^{heat}(M)}\|f\|_{L^2_w(M,\mu)},$$

$$\|\mathcal{H}(f)\|_{L^2_w(M,\mu)} \leq 2^{7/4}\|w\|_{A_2^{heat}(M)}\|f\|_{L^2_w(M,\mu)},$$

where $\|f\|^p_{L^p_w(M,\mu)} := \int_M |f|^p w \, \mathrm{d}\mu$.

Idea of proof: express the square functions as conditional expectations of quadratic variations for martingales and then apply the sharp martingale inequalities; see Bañuelos–Osekowski (2016).

Tool 1: space-time martingales

Let $\big((Z_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in M\setminus\mathcal{N}}\big)$ be the μ-symmetric Hunt process corresponding to (D, $W^{1,2}(X)$), where $\mathcal{N}$ is a properly exceptional set. Then, under the RCD condition,

$$\mathbb{P}^x\big(t \mapsto Z_t \text{ is continuous for } t \in (0,\infty)\big) = 1, \quad \forall\, x \in M.$$

Lemma

Suppose (M, d, μ) is an $\mathrm{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1,\infty)$. Fix $T > 0$ and $f \in C_c(M)$. Then $(\mathcal{M}(f)_t)_{0\leq t\leq T}$ with

$$\mathcal{M}(f)_t = P_{T-t}f(Z_t) - P_T f(Z_0)$$

is a uniformly integrable martingales with continuous path, and moreover, for any $t \in [0, T]$, the quadratic variation

$$\langle \mathcal{M}(f)\rangle_t = 2\int_0^t |\nabla P_{T-r} f|_*^2(Z_r)\,\mathrm{d}r.$$

See Bañuelos–Méndez-Hernández (2003).

Tool 2: probabilistic representation of LPS functions

Lemma

Suppose (M, d, μ) *is an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$*. Let* $T > 0$*,* $f \in C_c(M)$ *and* $x \in M$*. Define*

$$\mathcal{H}_{*,T}(f)(x) = \Big(\int_0^T \int_M |\nabla P_t f|_*^2(y) p_t(x,y)\, \mathrm{d}\mu(y)\mathrm{d}t \Big)^{1/2}.$$

Then $\lim_{T\to\infty} \mathcal{H}_{*,T}(f)(x) = \mathcal{H}_*(f)(x)$*, and*

$$\begin{aligned}\mathcal{H}_{*,T}(f)^2(x) &= \int_M \mathbb{E}_y \Big[\int_0^T |\nabla P_{T-r} f|_*^2(Z_r)\,\mathrm{d}r \Big| Z_T = x \Big] p_T(x,y)\,\mathrm{d}\mu(y) \\ &= \frac{1}{2} \int_M \mathbb{E}_y \Big[\langle \mathcal{M}(f) \rangle_T \Big| Z_T = x \Big] p_T(x,y)\,\mathrm{d}\mu(y).\end{aligned}$$

Varopoulos [J. Funct. Anal. 1980] first used the idea for general Poisson semigroups.

Tool 3: sharp weighted martingale inequalities

Lemma (Bañuelos–Osekowski)

Fix $T > 0$. Let $X = (X_t)_{t\geq 0}$ be an adapted, real valued and uniformly integrable martingale with continuous path, and $Y = (Y_t)_{t\geq 0}$ be a non-negative and uniformly integrable martingale with continuous path. Suppose that Y satisfies

$$\|Y\|_{A_2^{mart}} := \sup_{0\leq t\leq T} \left\| \mathbb{E}\left(\frac{Y_t}{Y_T} \middle| \mathcal{F}_t \right) \right\|_{L^\infty(\mathbb{P})} < \infty,$$

Then

$$\|X_T\|_{L^2(\mathbb{Q})} \leq \left(80\|Y\|_{A_2^{mart}}\right)^{1/2} \|\langle X\rangle_T^{1/2}\|_{L^2(\mathbb{Q})},$$
$$\|\langle X\rangle_T^{1/2}\|_{L^2(\mathbb{Q})} \leq 2^{7/4}\|Y\|_{A_2^{mart}}\|X_T\|_{L^2(\mathbb{Q})},$$

where $d\mathbb{Q} = Y_T d\mathbb{P}$.

In application, $X = \mathcal{M}(f)$ and $Y_T = w(Z_T)$.

Tool 4: heat kernel estimates

Lemma (Jiang–L.–Zhang, Potential Anal. 2016)

Let (M, d, μ) *be an* $\mathrm{RCD}^*(0, N)$ *space with* $N \in [1, \infty)$*. Then, there exists a positive constant* C *depending on* N *such that*

$$p_t(x, y) \geq \frac{1}{C\mu(B(x, \sqrt{t}))} \exp\left\{ -\frac{d^2(x, y)}{3t} \right\},$$

$$p_t(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} \exp\left\{ -\frac{d^2(x, y)}{5t} \right\},$$

for any $t > 0$ *and any* $x, y \in M$*.*

Sharp weighted L^2 Littlewood–Paley inequalities

Theorem

Let (M, d, μ) *be an* $\mathrm{RCD}^*(0, N)$ *space with* $N \in [1, \infty)$ *and* $w \in A_2^{heat}(M)$. *Suppose that* $\limsup_{r\to\infty}[\mu(B(o, r))/r^N] > 0$, *for some* $o \in M$. *Then, for every* $f \in C_c(M)$,

$$\|f\|_{L^2_w(M,\mu)} \le (320\|w\|_{A_2^{heat}(M)})^{1/2}\|\mathcal{H}_*(f)\|_{L^2_w(M,\mu)},$$

$$\|\mathcal{H}_*(f)\|_{L^2_w(M,\mu)} \le 2^{5/4}\|w\|_{A_2^{heat}(M)}\|f\|_{L^2_w(M,\mu)},$$

$$\|\mathcal{H}(f)\|_{L^2_w(M,\mu)} \le 2^{7/4}\|w\|_{A_2^{heat}(M)}\|f\|_{L^2_w(M,\mu)},$$

where $\|f\|^p_{L^p_w(M,\mu)} := \int_M |f|^p w \, \mathrm{d}\mu$.

An example

Theorem

Let M be a complete noncompact Riemannian manifold with dimension≥ 2. Suppose $\mathrm{Ric} \geq 0$ and $w \in A_2^{heat}(M)$. Then, all the inequalities in the Theorem above hold true.

The results were obtained by Bañuelos and Osekowski [arXiv:1603.07618] without noncompactness but under an additionally assumption

$$\sup_{x \in M} p_t(x,x) = c_t \to 0, \quad \text{as } t \to \infty. \tag{3}$$

However, in the noncompact setting, (3) is available, since for any $t > 0$ and $x \in M$, we have the Li–Yau (1986) heat kernel bounds

$$\frac{C_1}{\mu(B(x,\sqrt{t}))} \leq p_t(x,x) \leq \frac{C_2}{\mu(B(x,\sqrt{t}))},$$

and Yau's result (1976) that $\mu(B(x,\sqrt{t})) \geq c\sqrt{t}$.

Outline

Let (M, d, μ) be a metric measure space and $w : M \to [0, \infty]$ be a locally integrable function. For $p \in (1, \infty)$, we say that w is a p-Muckenhoupt weight, denoted by $w \in A_p(M)$, if

$$\|w\|_{A_p(M)} := \sup_B \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right) \left(\frac{1}{\mu(B)} \int_B w^{-1/(p-1)} \, d\mu \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset M$.

Theorem

Let (M, d, μ) be an $\mathrm{RCD}^*(0, N)$ space with $N \in [1, \infty)$ and $1 < p < \infty$. Then, there exist positive constants c_1 and c_2 depending on N such that

$$c_1 \|w\|_{A_p^{heat}(M)} \leq \|w\|_{A_p(M)} \leq c_2 \|w\|_{A_p^{heat}(M)}. \tag{4}$$

The same conclusion (4) for $p = 2$ was obtained in $\mathbb{R}^2$ by Petermichl and Volberg [Duke Math. J. 2002].

An immediate corrollary

Corollary

Let (M, d, μ) *be an* $\mathrm{RCD}^*(0, N)$ *space with* $N \in [1, \infty)$ *and* $w \in A_2(M)$*. Suppose that*

$$\limsup_{r \to \infty} \frac{\mu(B(o, r))}{r^N} > 0,$$

for some $o \in M$*. Then, for every* $f \in C_c(M)$*, there exists a constant* $C > 0$ *such that*

$$\|f\|_{L^2_w(M,\mu)} \leq C \|w\|^{1/2}_{A_2(M)} \|\mathcal{H}_*(f)\|_{L^2_w(M,\mu)},$$

$$\|\mathcal{H}_*(f)\|_{L^2_w(M,\mu)} \leq C \|w\|_{A_2(M)} \|f\|_{L^2_w(M,\mu)},$$

$$\|\mathcal{H}(f)\|_{L^2_w(M,\mu)} \leq C \|w\|_{A_2(M)} \|f\|_{L^2_w(M,\mu)}.$$

Outline

A remark on sub-Riemannian manifolds

Let M be a smooth connected manifold endowed with a smooth measure μ, and d a metric canonically associated with a smooth second order diffusion operator L on M with real coefficients such that $L1 = 0$ and

$$\int_M fLg\,\mathrm{d}\mu = \int_M gLf\,\mathrm{d}\mu, \quad \int_M fLf\,\mathrm{d}\mu \leq 0,$$

for every $f, g \in C_c^\infty(M)$. More precisely, for any $x, y \in M$,

$$d(x,y) = \sup\{|\phi(x) - \phi(y)| : \phi \in C^\infty(M),\, \Gamma(\phi,\phi) \leq 1\},$$

where Γ is the *carré du champ*

$$\Gamma(f,g) = \frac{1}{2}\{L(fg) - fLg - gLf\}, \quad f, g \in C^\infty(M).$$

Similarly results should be established on sub-Riemannian manifolds satisfying the generalized curvature-dimension condition $\mathrm{CD}(0, \rho_2, \kappa, m)$ with $\rho_2 > 0$, $\kappa \geq 0$ and $2 \leq m < \infty$, in the sense of Baudoin–Garofalo [J. Eur. Math. Soc. 2017].

$\triangleleft$ **Thanks!** $\triangleright$