# Heat kernel estimates for time fractional equations

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13th Workshop on Markov Processes and Related Topics at Wuhan

19 July, 2017

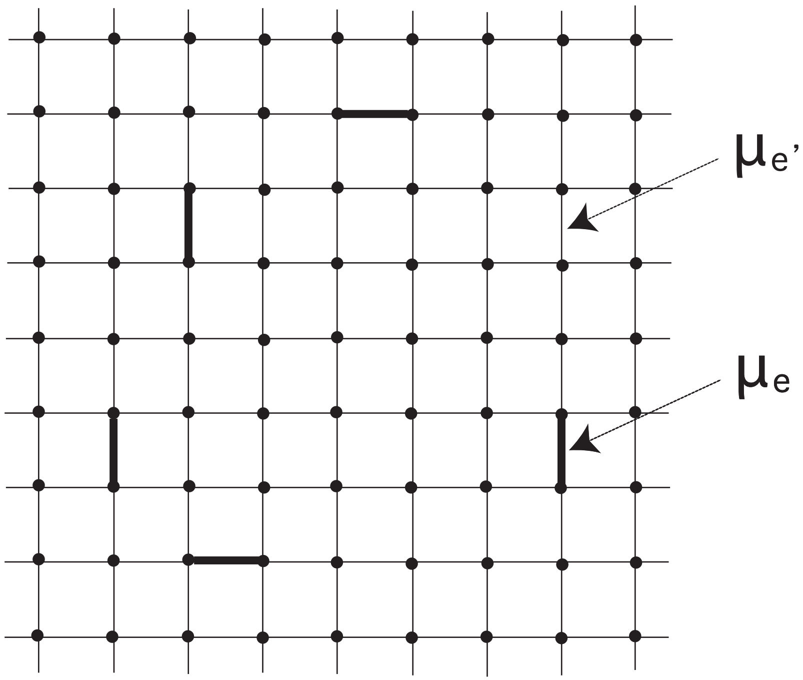
## 1 Introduction

**Example**: Random conductance model (RCM)

$\{\mu_e\}$: random conductance, *i.i.d.* on each edge $e$ of $\mathbb{Z}^d$ s.t. $\exists \alpha \in (0,1)$

$$\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\alpha}(1+o(1)) \text{ as } u \to \infty. \tag{1.1}$$

(Note that $\mathbb{E}\mu_e = \infty$.) $\{X_t\}_{t\geq 0}$: cont. time MC on $\mathbb{Z}^d$ (holding time exp(1)).



**Theorem 1.1** $d \geq 2$ (Barlow-Černý '11) *For* $d \geq 3$,

$$\varepsilon X_{ct/\varepsilon^{2/\alpha}} \xrightarrow{d} \mathbf{FK}_{d,\alpha}(t) := BM_d(S_\alpha^{-1}(t)) \quad \mathbf{P}\text{-}a.s. \ \ on \ D([0,\infty),\mathbb{R}^d),$$

*where* $\{S_\alpha(t)\}_{t\geq 0}$*:* *$\alpha$-stable subord.* *(indep. of* $\{BM_d(t)\}$*).*

*For* $d = 2$ *(Černý '11), same result by replacing* $\varepsilon^{-2/\alpha}$ *to* $\varepsilon^{-2/\alpha}(\log \varepsilon^{-1})^{1-1/\alpha}$.

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$\mathbf{FK}_{d,\alpha}$: Fractional-kinetics process — It is no longer a Markov process!

Density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation:

$$\boxed{\frac{\partial^\alpha}{\partial t^\alpha} p(t,x) = \frac{1}{2}\Delta p(t,x).}$$

**Rem 1:** Limit is very different for $d = 1$ : $Z(s) = BM(\phi_\rho^{-1}(s))$, FIN diffusion.

$\phi_\rho(t) := \int_{\mathbb{R}} \ell(t,y)\rho(dy)$, $\rho := \sum_i \nu_i \delta_{x_i}$: PPP with intensity $dx\alpha\nu^{-1-\alpha}d\nu$.

**Rem 2:** For Bouchaud's trap model (BTM), Theorem 1.1 is by Ben Arous-Černý '07.

## 2 Unique exsitence of the weak solution and Heat kernel estimates for generalized FK processes

Classical case $0 < \beta < 1$

$$\partial_t^\beta p(t,x) = \Delta p(t,x) \qquad t > 0, x \in \mathbb{R}^d.$$

where $\partial_t^\beta \psi(t) := \frac{d}{dt} I^{1-\beta}(\psi - \psi(0))(t) = \frac{d}{dt} I^{1-\beta}\psi(t) - \frac{\psi(0)}{t^\beta \Gamma(1-\beta)}$: Caputo derivative

$I^\beta \psi(t) = \Gamma(\beta)^{-1} \int_0^t (t-s)^{\beta-1}\psi(s)ds.$

HK estimates are made by PDE people (e.g. Eidelman-Kochubei ('04, JDE))

$E_\beta(z) = \sum_{k=1}^\infty z^k/\Gamma(\beta k+1)$: Mittag-Leffler function

$p(t,x) = \mathcal{F}^{-1}(E_\beta(|\xi|^2 t^\beta))$ use Fourier analysis

## 2 Unique exsitence of the weak solution and HKE for generalized FK processes

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**(Q)** More general case? (General space, general operator)

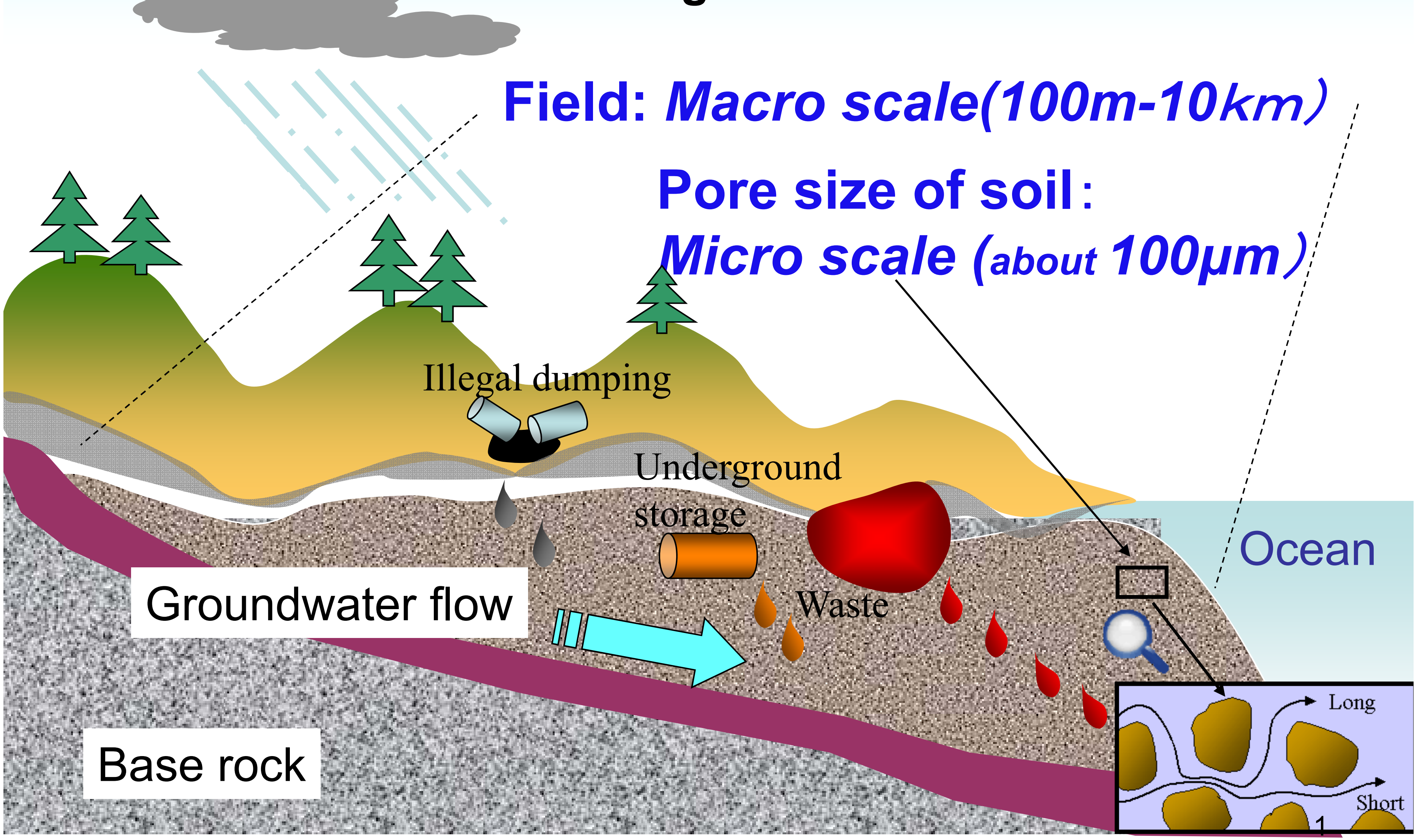
Motivtion : • Question from industory. (Predict the progress of soil contamination.)

The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.).

• The third slide: Nature (2006, Jan.) by D. Brockmann, L. Hufnagel and T. Geisel.

# Issues Seen by Academia Engineering Researchers

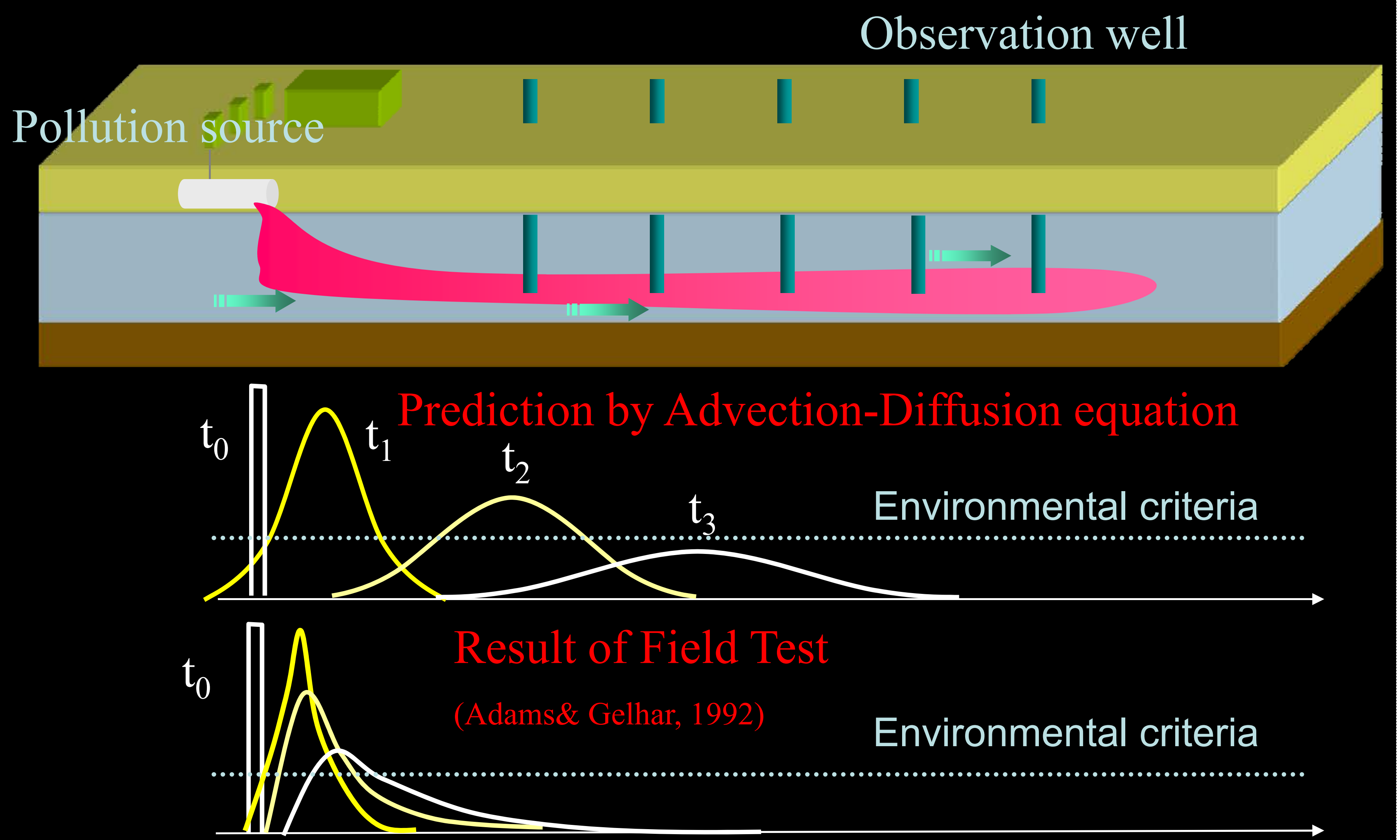
## “The Prediction of the Progress of Soil Contamination”






*Dr. Yuko Hatano, Department of Risk Engineering, University of Tsukuba*

# Model Prediction and Reality

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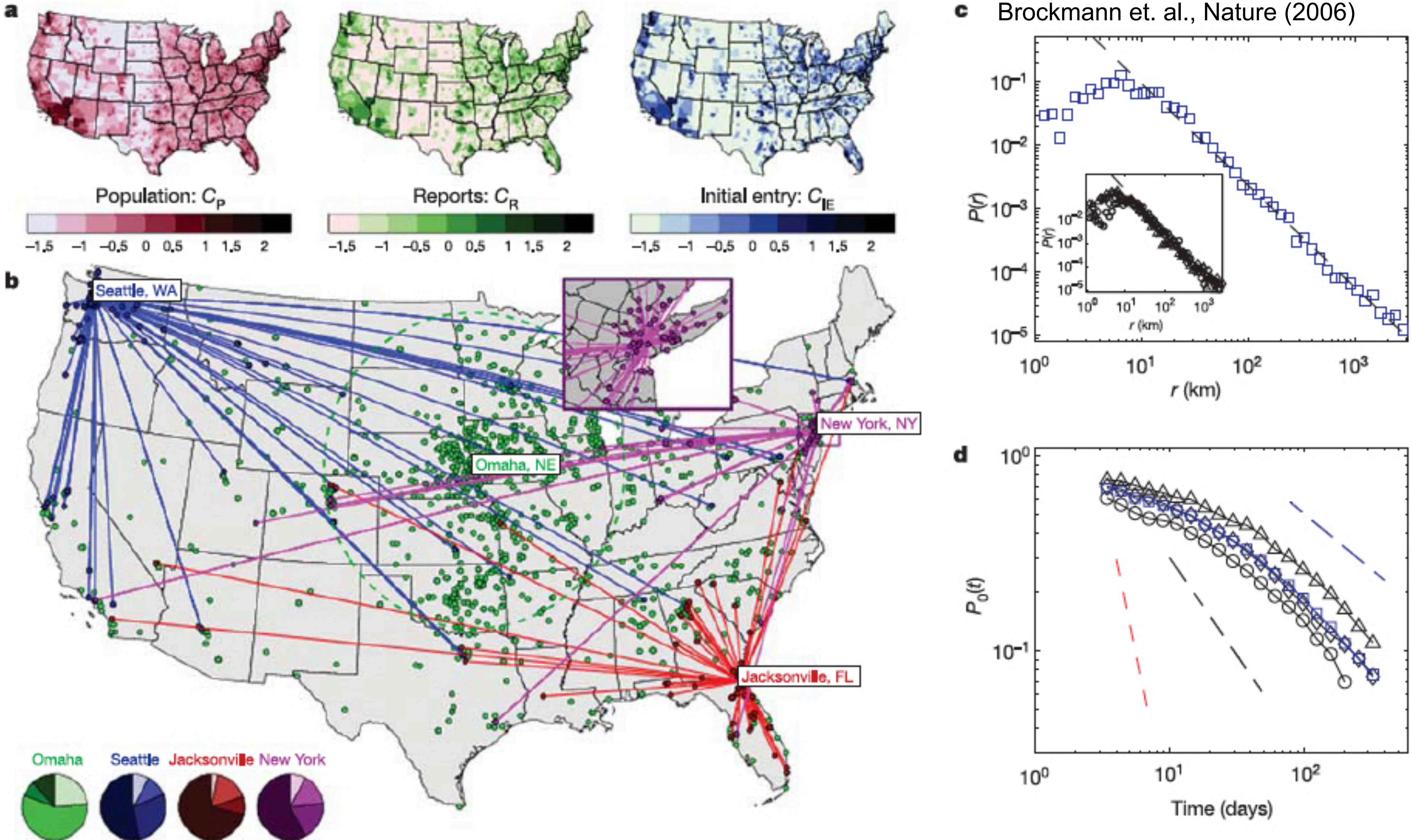
Brockmann et. al., Nature (2006)
a
Population: C_P
Reports: C_R
Initial entry: C_IE
−1.5 −1 −0.5 0 0.5 1 1.5 2
b
Seattle, WA
New York, NY
Omaha, NE
Jacksonville, FL
Omaha
Seattle
Jacksonville
New York
c
P(r)
r (km)
d
P_0(t)
Time (days)

Result I: Weak solution (Cf. Talk by Z.-Q. Chen yesterday)

$(F, d, \mu)$ metric meas. space

$(\mathcal{E}, \mathcal{F})$: reg. Dirichlet form on $L^2(F, \mu)$. $\{X_t\}$: process, $\mathcal{L}$: generator

$\{S_t\}_{t\geq 0}$: subordinator without drift, $\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}$,

$\nu$: Lévy measure of $S$ (i.e. $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s})\, \nu(ds)$). Let

$$I_t^w(\psi) := \int_0^t w(t-s)(\psi(s) - \psi(0))\, ds, \quad \partial_t^w \psi(t) = \frac{d}{dt} I_t^w(\psi),$$

where $w(x) := \nu((x, \infty))$. $\partial_t^w$ is a *generalized Caputo derivstive*. Define

$$v(t, x) = \mathbb{E}[f(X(S_t^{-1}))],$$

where $S_t^{-1} = \inf\{s > 0 : S_s > t\}$. Then $\cdots$

**Theorem 2.1** *For* $\forall f \in L^2(M;\mu)$, $v(t,x) = \mathbb{E}[f(X(S_t^{-1}))]$ *is a weak solution to*

$$\partial_t^w v(t,x) = \mathcal{L}v(t,x) \quad \text{with } v(0,x) = f(x) \tag{2.1}$$

*in the following sense:*

(i) $t \mapsto v(t,x)$ *is continuous in* $L^2(M;\mu)$ *with* $\|v(t,x)\|_2 \leq \|f\|_2$. *Hence* $I_t^w(v(\cdot,x))$ *is absolutely convergent in* $L^2(M;\mu)$ *for* $\forall t > 0$.

(ii) *For* $\forall g \in D(\mathcal{L})$ *and* $t > 0$

$$\frac{d}{dt}\int_M g(x) I_t^w(v(\cdot,x))\,\mu(dx) = \int_M v(t,x)\mathcal{L}g(x)\,\mu(dx). \tag{2.2}$$

*Conversely, if* $v(t,x)$ *is a weak solution to* (2.1) *in the sense of* (i) *and* (ii) *above with* $f \in L^2(M;\mu)$, *then* $v(t,x) = \mathbb{E}[f(X(S_t^{-1}))]$ $\mu$*-a.e. on* $M$ *for every* $t \geq 0$.

## Result II: Heat kernel estimates

$(F, d, \mu)$ metric meas. space $d$: geodesic metric

$(\mathcal{E}, \mathcal{F})$: reg. Dirichlet form on $L^2(F, \mu)$, conservative. $\{X_t\}$: process, $\mathcal{L}$: generator

$u(t, x, y)$ the corresponding heat kernel

(Q) What is the HK of the generalized FK processes?

$$\begin{aligned} v(t,x) = \mathbb{E}[f(X(S_t^{-1}))] &= \int_0^\infty T_r f(x)\, d_r\mathbb{P}(S_t^{-1} \le r) = \int_0^\infty T_r f(x)\, d_r\mathbb{P}(S_r \ge t) \\ &= \int_0^\infty \int_M f(y)u(r,x,y)\, \mu(dy)\, d_r\mathbb{P}(S_r \ge t) \\ &= \int_M f(y) \left( \int_0^\infty u(r,x,y)\, d_r\mathbb{P}(S_r \ge t) \right) \mu(dy). \end{aligned}$$

So the HK of the FK process $p_t(x,y)$ is $p_t(x,y) := \int_0^\infty u(r,x,y)\, d_r\mathbb{P}(S_r \ge t)$.

Suppose the corresponding heat kernel enjoys

$$u(t,x,y) \asymp t^{-d/\alpha}\Psi(d(x,y)/t^{1/\alpha})$$

for some $\Psi$ monotone decreasing

$\Rightarrow$ (Grigor'yan-K '08) Either $\mathcal{E}$ is local, $\alpha \geq 2$ and $\Psi(s) \asymp \exp(-s^{\alpha/(\alpha-1)})$: sub-Gaussian

or $\mathcal{E}$ is non-local, $\alpha > 0$ and $\Psi(s) \asymp (1+s)^{-(d+\alpha)}$: $\alpha$-stable-like.

**Note**: $\alpha = 2$ and $\Psi(s) = \exp(-s^2)$ is the classical case.

Suppose $0 < \exists \beta_1 \leq \beta_2 < 1$ s.t.

$$c_1 \kappa^{\beta_1} \leq \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \leq c_2 \kappa^{\beta_2} \tag{2.3}$$

for all $\lambda > 0$, $\kappa \geq 1$. Then $\cdots$

**Theorem 2.2** *(i) If* $d(x,y)\phi(t^{-1})^{1/\alpha} \le 1$, *then*

$$p_t(x,y) \asymp \begin{cases} \phi(t^{-1})^{d/\alpha} & \text{if } d < \alpha, \\ \phi(t^{-1}) \log\left(\dfrac{2}{d(x,y)\phi(t^{-1})^{1/\alpha}}\right) & \text{if } d = \alpha, \\ \phi(t^{-1})^{d/\alpha}\left(d(x,y)\phi(t^{-1})^{1/\alpha}\right)^{-d+\alpha} = \phi(t^{-1})/d(x,y)^{d-\alpha} & \text{if } d > \alpha. \end{cases}$$

*(ii) Suppose* $d(x,y)\phi(t^{-1})^{1/\alpha} \ge 1$. • *When the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *is local,*

$$p_t(x,y) \asymp \phi(t^{-1})^{d/\alpha} \exp\Big(-t\bar{\phi}_\alpha^{-1}((d(x,y)/t)^\alpha)\Big), \tag{2.4}$$

*where* $\bar{\phi}_\alpha(\lambda) = \lambda^\alpha/\phi(\lambda)$, *and* $\bar{\phi}_\alpha^{-1}(\lambda)$ *is the inverse function of* $\bar{\phi}_\alpha(\lambda)$

• *When* $(\mathcal{E},\mathcal{F})$ *is non-local,*

$$p_t(x,y) \asymp \phi(t^{-1})^{d/\alpha}(d(x,y)\phi(t^{-1})^{1/\alpha})^{-d-\alpha} = \frac{1}{\phi(t^{-1})d(x,y)^{d+\alpha}}.$$

Special case: $\phi(s) = s^\beta$, $0 < \beta < 1$ $\beta$-stable subodinator

In that case (2.4) is

$$p_t(x,y) \asymp t^{-\beta d/\alpha} \exp\left( (d(x,y) t^{-\beta/\alpha})^{\alpha/(\alpha-\beta)} \right).$$

**Rem 1**: We have more general version: under vol. doubling, more general shape of HK.

**Rem 2**: Under (2.3), w.l.o.g. we may assume $\phi$ is a complete Bernstein function.

**Proposition 2.3 (Key Proposition)**

$$\mathbb{P}(S_r \geq t) \asymp r\phi(t^{-1}) \qquad \textit{if} \quad r\phi(t^{-1}) << 1,$$

$$\mathbb{P}(S_r \leq t) \asymp \exp(-t(\phi')^{-1}(t/r)) \qquad \textit{if} \quad r\phi(t^{-1}) >> 1.$$

**Rem 3**: Roughly, one can interpret Theorem 2.2 by taking $t \to 1/\phi(t^{-1})$.

Thank you!

## 3 Limit is very different when $d = 1$.

**Theorem 3.1** $d = 1$ (Fontes-Isopi-Newman '02)

$$\varepsilon X_{c_* t/\varepsilon^{1+1/\alpha}} \xrightarrow{d} Z(t) \quad \textit{under} \quad \mathbf{P} \times P_0^\xi.$$

**Definition 3.2** *FIN diffusion is defined by* $Z(s) = BM(\phi_\rho^{-1}(s))$, $s \in [0, \infty)$,

*where* $\phi_\rho(t) := \int_{\mathbb{R}} \ell(t, y)\rho(dy)$ *where* $\ell(\cdot, \cdot)$ *is the local time of BM,*

$\rho := \sum_i \nu_i \delta_{x_i}$ *where* $(x_i, \nu_i) \in \mathbb{R} \times \mathbb{R}_+$ *is distributed by PPP with intensity* $dx \alpha \nu^{-1-\alpha} d\nu$.

— Atoms of $\rho$ are dense in $\mathbb{R}$ a.e.

$$\boxed{\frac{\partial}{\partial t} p(t, x) = \frac{\partial^2}{\partial \rho \partial x} p(t, x)}$$

$\rho$ plays the role of speed measure.

**Theorem 3.5** (Barlow-Černý '10) *Let* $d \geq 3$, $\alpha \in (0,1)$, *and* $\{X_t\}_{t\geq 0}$ *be the Markov chain of RCM that satisfies (1.1). Then*

$$\varepsilon X_{t/\varepsilon^{2/\alpha}} \xrightarrow{d} c \cdot \mathbf{FK}_{d,\alpha} \quad \text{under } P_\omega^0, \ \mathbb{P}\text{-a.s. on } D([0,\infty),\mathbb{R}^d) \text{ with } J_1\text{-topology.}$$

Explanation:

Recall that scaled VSRW converges to BM, and CSRW is a time change of VSRW:

$$\text{Clock process} \quad \tilde{A}_t := \int_0^t \mu_{Y_s} ds = \int_0^t \mu_0(T_{Y_s}\omega) ds, \qquad X_t = Y_{\tilde{A}_t^{-1}}.$$

In fact, (using transience of RW)

$$(n^{-1}Y(n^2\cdot), n^{-2/\alpha}\tilde{A}_{n^2\cdot}) \to (c_1 B_d, c_2 V_\alpha) \ \text{ weakly under } P_\omega^0, \ \mathbb{P}\text{-a.s.}$$

$$\Rightarrow n^{-1}X(n^{2/\alpha}t) = n^{-1}Y(\tilde{A}^{-1}_{n^{2/\alpha}t}) = n^{-1}Y(n^2(F_t^n)^{-1}) \to c_1 B_d((c_2 V_\alpha)^{-1})$$

$$(\tilde{A}^{-1}_{n^{2/\alpha}t} = \inf\{s : \tilde{A}_s > n^{2/\alpha}t\} = n^2 \inf\{s : F_t^n := n^{-2/\alpha}\tilde{A}_{n^2 s} > t\} = n^2(F_t^n)^{-1})$$