Heat kernel estimates for time fractional equations

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1 Introduction

**Example:** Random conductance model (RCM)

$\{\mu_e\}$: random conductance, i.i.d. on each edge $e$ of $\mathbb{Z}^d$ s.t. $\exists \alpha \in (0, 1)$

\[
\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\alpha}(1 + o(1)) \text{ as } u \to \infty. \quad (1.1)
\]

(Note that $\mathbb{E}\mu_e = \infty$.) $\{X_t\}_{t \geq 0}$: cont. time MC on $\mathbb{Z}^d$ (holding time $\exp(1)$).
Theorem 1.1 \( d \geq 2 \) (Barlow-Černý ’11) For \( d \geq 3 \),

\[ \varepsilon X_{\varepsilon^2 / \varepsilon^2 / \alpha} \xrightarrow{d} F K_{d,\alpha}(t) := B M_d(S_{\alpha}^{-1}(t)) \quad \text{P-a.s. on } D([0, \infty), \mathbb{R}^d), \]

where \( \{S_{\alpha}(t)\}_{t \geq 0} \): \( \alpha \)-stable subord. (indep. of \( \{B M_d(t)\} \)).

For \( d = 2 \) (Černý ’11), same result by replacing \( \varepsilon^{-2/\alpha} \) to \( \varepsilon^{-2/\alpha}(\log \varepsilon^{-1})^{1-1/\alpha} \).
Theorem 1.1 \(d \geq 2\) (Barlow-Černý ’11) For \(d \geq 3\),

\[
\varepsilon X_{ct/\varepsilon^{2/\alpha}} \overset{d}{\to} \text{FK}_{d,\alpha}(t) := BM_d(S_{\alpha}^{-1}(t)) \quad \text{P-a.s. on } D([0, \infty), \mathbb{R}^d),
\]

where \(\{S_{\alpha}(t)\}_{t \geq 0}\): \(\alpha\)-stable subord. (indep. of \(\{BM_d(t)\}\)).

For \(d = 2\) (Černý ’11), same result by replacing \(\varepsilon^{-2/\alpha}\) to \(\varepsilon^{-2/\alpha}(\log \varepsilon^{-1})^{1-1/\alpha}\).

\(\text{FK}_{d,\alpha}\): Fractional-kinetics process — It is no longer a Markov process!

Density of its fixed time distribution \(p(t, x)\) satisfies the fractional-kinetics equation:

\[
\frac{\partial^{\alpha}}{\partial t^{\alpha}} p(t, x) = \frac{1}{2} \Delta p(t, x).
\]

Rem 1: Limit is very different for \(d = 1\): \(Z(s) = BM(\phi^{-1}_\rho(s))\), FIN diffusion.

\(\phi_\rho(t) := \int_{\mathbb{R}} \ell(t, y) \rho(dy), \rho := \sum_i \nu_i \delta_{x_i}\): PPP with intensity \(dx \alpha \nu^{-1-\alpha} d\nu\).

Rem 2: For Bouchaud’s trap model (BTM), Theorem 1.1 is by Ben Arous-Černý ’07.
2 Unique existence of the weak solution and Heat kernel estimates for generalized FK processes

Classical case  \( 0 < \beta < 1 \)

\[
\partial_t^\beta p(t, x) = \Delta p(t, x) \quad t > 0, x \in \mathbb{R}^d.
\]

where \( \partial_t^\beta \psi(t) := \frac{d}{dt} I^{1-\beta} (\psi - \psi(0))(t) = \frac{d}{dt} I^{1-\beta} \psi(t) - \frac{\psi(0)}{t^\beta \Gamma(1-\beta)}: \) Caputo derivative

\[
I^\beta \psi(t) = \Gamma(\beta)^{-1} \int_0^t (t - s)^{\beta - 1} \psi(s) ds.
\]

HK estimates are made by PDE people (e.g. Eidelman-Kochubei ('04, JDE))

\[
E_\beta(z) = \sum_{k=1}^{\infty} z^k / \Gamma(\beta k + 1): \text{Mittag-Leffler function}
\]

\[
p(t, x) = \mathcal{F}^{-1} (E_\beta(|\xi|^2 t^\beta)) \quad \text{use Fourier analysis}
\]
2 Unique existence of the weak solution and HKE for generalized FK processes

**Classical case** \(0 < \beta < 1\)

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\]

(Q) More general case? (General space, general operator)

Motivation: • Question from industry. (Predict the progress of soil contamination.)

The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.).

Issues Seen by Academia Engineering Researchers

“The Prediction of the Progress of Soil Contamination”

- Field: Macro scale (100m-10km)
- Pore size of soil: Micro scale (about 100μm)

Illegal dumping
Groundwater flow
Base rock

Dr. Yuko Hatano, Department of Risk Engineering, University of Tsukuba

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Model Prediction and Reality

Pollution source

Prediction by Advection-Diffusion equation

Result of Field Test

(Adams & Gelhar, 1992)
Result I: Weak solution  (Cf. Talk by Z.-Q. Chen yesterday)

\((F, d, \mu)\) metric meas. space

\((\mathcal{E}, \mathcal{F})\): reg. Dirichlet form on \(L^2(F, \mu)\). \(\{X_t\}\): process, \(\mathcal{L}\): generator

\(\{S_t\}_{t \geq 0}\): subordinator without drift, \(\mathbb{E}e^{-\lambda S_t} = e^{-t \phi(\lambda)}\),

\(\nu\): Lévy measure of \(S\) (i.e. \(\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) \nu(ds)\)). Let

\[ I_t^w(\psi) := \int_0^t w(t - s)(\psi(s) - \psi(0)) \, ds, \quad \partial_t^w \psi(t) = \frac{d}{dt} I_t^w(\psi), \]

where \(w(x) := \nu((x, \infty))\). \(\partial_t^w\) is a generalized Caputo derivative. Define

\[ \nu(t, x) = \mathbb{E}[f(X(S_t^{-1}))], \]

where \(S_t^{-1} = \inf\{s > 0 : S_s > t\}\). Then …
Theorem 2.1 For $\forall f \in L^2(M; \mu)$, $v(t, x) = \mathbb{E}[f(X(S_t^{-1}))]$ is a weak solution to

$$\partial_t^w v(t, x) = \mathcal{L}v(t, x) \quad \text{with} \quad v(0, x) = f(x) \quad (2.1)$$

in the following sense:

(i) $t \mapsto v(t, x)$ is continuous in $L^2(M; \mu)$ with $\|v(t, x)\|_2 \leq \|f\|_2$. Hence $I_t^w(v(\cdot, x))$ is absolutely convergent in $L^2(M; \mu)$ for $\forall t > 0$.

(ii) For $\forall g \in D(\mathcal{L})$ and $t > 0$

$$\frac{d}{dt} \int_M g(x)I_t^w(v(\cdot, x)) \mu(dx) = \int_M v(t, x)\mathcal{L}g(x) \mu(dx). \quad (2.2)$$

Conversely, if $v(t, x)$ is a weak solution to (2.1) in the sense of (i) and (ii) above with $f \in L^2(M; \mu)$, then $v(t, x) = \mathbb{E}[f(X(S_t^{-1}))]$ $\mu$-a.e. on $M$ for every $t \geq 0$. 
Result II: Heat kernel estimates

\((F, d, \mu)\) metric meas. space \(d\): geodesic metric

\((\mathcal{E}, \mathcal{F})\): reg. Dirichlet form on \(L^2(F, \mu)\), conservative. \(\{X_t\}\): process, \(\mathcal{L}\): generator

\(u(t, x, y)\) the corresponding heat kernel

(Q): What is the HK of the generalized FK processes?

\[
v(t, x) = \mathbb{E}[f(X(S_t^{-1}))] = \int_0^\infty T_r f(x) \, d_r \mathbb{P}(S_t^{-1} \leq r) = \int_0^\infty T_r f(x) \, d_r \mathbb{P}(S_r \geq t)
\]

\[
= \int_0^\infty \int_M f(y) u(r, x, y) \, \mu(dy) \, d_r \mathbb{P}(S_r \geq t)
\]

\[
= \int_M f(y) \left( \int_0^\infty u(r, x, y) \, d_r \mathbb{P}(S_r \geq t) \right) \mu(dy).
\]

So the HK of the FK process \(p_t(x, y)\) is \(p_t(x, y) := \int_0^\infty u(r, x, y) \, d_r \mathbb{P}(S_r \geq t)\).
Suppose the corresponding heat kernel enjoys

$$u(t, x, y) \asymp t^{-d/\alpha} \Psi(d(x, y)/t^{1/\alpha})$$

for some $\Psi$ monotone decreasing

$$\Rightarrow \text{(Grigor’yyan-K ’08)} \text{ Either } \mathcal{E} \text{ is local, } \alpha \geq 2 \text{ and } \Psi(s) \asymp \exp(-s^{\alpha/(\alpha-1)}): \text{ sub-Gaussian}$$

$$\text{ or } \mathcal{E} \text{ is non-local, } \alpha > 0 \text{ and } \Psi(s) \asymp (1 + s)^{-(d+\alpha)}: \alpha\text{-stable-like.}$$

**Note:** $\alpha = 2$ and $\Psi(s) = \exp(-s^2)$ is the classical case.

Suppose $0 < \exists \beta_1 \leq \beta_2 < 1$ s.t.

$$c_1 \kappa^{\beta_1} \leq \frac{\phi(\kappa \lambda)}{\phi(\lambda)} \leq c_2 \kappa^{\beta_2} \quad (2.3)$$

for all $\lambda > 0, \kappa \geq 1$. Then ⋯
Theorem 2.2 (i) If \( d(x, y)\phi(t^{-1})^{1/\alpha} \leq 1 \), then

\[
p_t(x, y) \propto \begin{cases} 
\phi(t^{-1})^{d/\alpha} & \text{if } d < \alpha, \\
\phi(t^{-1}) \log \left( \frac{2}{d(x, y)\phi(t^{-1})^{1/\alpha}} \right) & \text{if } d = \alpha, \\
\phi(t^{-1})^{d/\alpha} \left( d(x, y)\phi(t^{-1})^{1/\alpha} \right)^{-d+\alpha} = \phi(t^{-1})/d(x, y)^{d-\alpha} & \text{if } d > \alpha.
\end{cases}
\]

(ii) Suppose \( d(x, y)\phi(t^{-1})^{1/\alpha} \geq 1 \). ● When the Dirichlet form \((\mathcal{E}, \mathcal{F})\) is local,

\[
p_t(x, y) \propto \phi(t^{-1})^{d/\alpha} \exp \left( -t\bar{\phi}_{\alpha}^{-1}((d(x, y)/t)^{\alpha}) \right),
\]

where \( \bar{\phi}_\alpha(\lambda) = \lambda^\alpha/\phi(\lambda) \), and \( \bar{\phi}_{\alpha}^{-1}(\lambda) \) is the inverse function of \( \bar{\phi}_\alpha(\lambda) \);

● When \((\mathcal{E}, \mathcal{F})\) is non-local,

\[
p_t(x, y) \propto \phi(t^{-1})^{d/\alpha}(d(x, y)\phi(t^{-1})^{1/\alpha})^{-d-\alpha} = \frac{1}{\phi(t^{-1})d(x, y)^{d+\alpha}}.
\]
Special case: \( \phi(s) = s^\beta, \; 0 < \beta < 1 \) \( \beta \)-stable subordinator

In that case (2.4) is

\[
pt(x, y) \asymp t^{-\beta d/\alpha} \exp \left( (d(x, y)t^{-\beta/\alpha})^{\alpha/(\alpha-\beta)} \right).
\]

**Rem 1:** We have more general version: under vol. doubling, more general shape of HK.

**Rem 2:** Under (2.3), w.l.o.g. we may assume \( \phi \) is a complete Bernstein function.

**Proposition 2.3 (Key Proposition)**

\[
\mathbb{P}(S_r \geq t) \asymp r \phi(t^{-1}) \quad \text{if} \quad r \phi(t^{-1}) \ll 1,
\]

\[
\mathbb{P}(S_r \leq t) \asymp \exp(-t(\phi')^{-1}(t/r)) \quad \text{if} \quad r \phi(t^{-1}) \gg 1.
\]

**Rem 3:** Roughly, one can interpret Theorem 2.2 by taking \( t \to 1/\phi(t^{-1}) \).
Thank you!
Limit is very different when $d = 1$.

**Theorem 3.1** $d = 1$ (Fontes-Isopi-Newman ’02)

$$
\varepsilon X_{c_{*t}/\varepsilon^{1+1/\alpha}} \xrightarrow{d} Z(t) \text{ under } \mathbf{P} \times P_0^\xi.
$$

**Definition 3.2** FIN diffusion is defined by $Z(s) = BM(\phi_{\rho}^{-1}(s))$, $s \in [0, \infty)$,

where $\phi_{\rho}(t) := \int_\mathbb{R} \ell(t, y)\rho(dy)$ where $\ell(\cdot, \cdot)$ is the local time of BM,

$\rho := \sum_i \nu_i \delta_{x_i}$ where $(x_i, \nu_i) \in \mathbb{R} \times \mathbb{R}_+$ is distributed by PPP with intensity $dx_\alpha \nu^{-1-\alpha}d\nu$.

— Atoms of $\rho$ are dense in $\mathbb{R}$ a.e.

$$
\frac{\partial}{\partial t} p(t, x) = \frac{\partial^2}{\partial \rho \partial x} p(t, x)
$$

$\rho$ plays the role of speed measure.
**Theorem 3.5** (Barlow-Černý ’10) Let \( d \geq 3 \), \( \alpha \in (0, 1) \), and 
\[ \{X_t\}_{t \geq 0} \] be the Markov chain of RCM that satisfies (1.1). Then 
\[ \varepsilon X_{t/\varepsilon^{2/\alpha}} \xrightarrow{d} c \cdot \mathbf{F}K_{d,\alpha} \quad \text{under } P^0, \mathbb{P}\text{-a.s. on } D([0, \infty), \mathbb{R}^d) \text{ with } J_1\text{-topology.} \]

**Explanation:**

Recall that scaled VSRW converges to BM, and CSRW is a time change of VSRW:

Clock process \( \tilde{A}_t := \int_0^t \mu_{Y_s} ds = \int_0^t \mu_0(T_{Y_s} \omega) ds \), \( X_t = Y_{\tilde{A}_t^{-1}}. \)

In fact, (using transience of RW)

\[ (n^{-1}Y(n^2 \cdot), n^{-2/\alpha} \tilde{A}_{n^2}) \rightarrow (c_1B_d, c_2V_\alpha) \text{ weakly under } P^0, \mathbb{P}\text{-a.s.} \]

\[ \Rightarrow n^{-1}X(n^{2/\alpha}t) = n^{-1}Y(\tilde{A}_{n^2/\alpha_t}^{-1}) = n^{-1}Y(n^2(F^n_t)^{-1}) \rightarrow c_1B_d((c_2V_\alpha)^{-1}) \]

\[ (\tilde{A}_{n^2/\alpha_t}^{-1} = \inf\{s : \tilde{A}_s > n^{2/\alpha}t\} = n^2\inf\{s : F^n_t := n^{-2/\alpha} \tilde{A}_{n^2s} > t\} = n^2(F^n_t)^{-1}) \]