Hunt's Hypothesis (H) for the Sum of Two Independent Lévy processes

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Outline

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2. Progress on Getoor's conjecture
3. (H) for the sum of two independent Lévy processes
   - 3.1 (H) for one-dimensional Lévy processes
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4. Open problems

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Meaning of Hunt's Hypothesis (H)

Let X be a standard Markov process on E.

Hunt's hypothesis (H): every semipolar set A of X is polar.

Intuitively speaking, (H) means that if A cannot be immediately hit by X with arbitrary starting point, then A will not be hit by X forever.

Some basic notions

Let $B \subset E$. Define $\sigma_B := \inf\{t > 0 : X_t \in B\}$.

$\mathcal{B}^* :=$ the family of all nearly Borel sets relative to X.

$B \subset E$ is called polar if there exists $C \in \mathcal{B}^*$ such that $B \subset C$ and $P^x(\sigma_C < \infty) = 0$ for every $x \in E$. B is called essentially polar if $P^x(\sigma_C < \infty) = 0$ for every m-a.e. $x \in E$, where m is the reference measure.

B is called a thin set if there exists $C \in \mathcal{B}^*$ such that $B \subset C$ and $P^x(\sigma_C = 0) = 0$ for every $x \in E$.

B is called semipolar if $B \subset \cup_{n=1}^{\infty} B_n$ for some thin sets $\{B_n\}_{n=1}^{\infty}$.

Example and counterexample

• (H) is satisfied by Brownian motion in $\mathbf{R}$: All semipolar and polar sets must be the empty set, since in this case, every point x is the regular point of the single set $\{x\}$.

In fact, all Brownian motions in any finite dimensional space satisfy (H).

• (H) is not satisfied by uniform motion to the right in $\mathbf{R}$: semipolar set is countable set but polar set is the empty set.

In this case, for every point x, the set $\{x\}$ has no regular point. But if $y < x$, we have $P^y\{\sigma_{\{x\}} < \infty\} = 1$.

Several potential principles (I)

Assume X is in duality with $\hat{X}$ on E w.r.t. σ-finite excessive measure m.

- Bounded positivity principle (P^*_α): If μ is a finite signed measure and $U^\alpha\mu$ is bounded, then $\mu U^\alpha\mu \geq 0$.

- Bounded energy principle (E^*_α): If μ is a finite measure with compact support and $U^\alpha\mu$ is bounded, then μ does not charge semipolar sets.

- Bounded maximum principle (M^*_α): If μ is a finite measure with compact support K and $U^\alpha\mu$ is bounded, then $\sup\{U^\alpha\mu(x) : x \in E\} = \sup\{U^\alpha\mu(x) : x \in K\}$.

Several potential principles (II)

• Bounded regularity principle (R_α^*): If μ is a finite measure with compact support such that $U^\alpha\mu$ is bounded, then $U^\alpha\mu$ is regular.

Assume that all α-excessive functions are lower semicontinuous. Then

$$(P_\alpha^*) \Leftrightarrow (E_\alpha^*) \Leftrightarrow (M_\alpha^*) \Leftrightarrow (R_\alpha^*) \Leftrightarrow (\mathrm{H}).$$

Blumenthal, Getoor, Rao, Fitzsimmons.

Other equivalent characterizations

- (H) holds if and only if every natural additive functional of X is a continuous additive functional(Blumenthal and Getoor).

- (H) holds if and only if fine topology and cofine topology differ by polar sets, which means that the fine closure of any set and its cofine closure differ by polar sets (Blumenthal and Getoor, Glover).

- (H) is equivalent to the dichotomy of capacity (Fitzsimmons and Kanda (1992, Ann. Probab.)).

Some sufficient conditions for (H)

- If X is symmetric, i.e. $u(x,y) = u(y,x)$, then (H) holds.

- Glover and Rao (86, Ann. Prob.): α-subordinators of general Hunt processes satisfy (H).

- The sector condition implies that every semipolar set is essentially polar: Silverstein (1978,PTRF), Fitzsimmons (2001,PA), Han-Ma-Sun (2011, Acta. Math. Sinica).

Open problem: For a general Markov process X, can we give a good sufficient condition which implies (H)?

Getoor's conjecture

About 45 years ago, Prof. R. K. Getoor conjectured that essentially all Lévy processes satisfy (H), except in certain extremely non-symmetric cases like uniform motions.

Outline

Lévy processes

$(\Omega, \mathcal{F}, P)$: probability space

$X = (X_t)_{t \geq 0}$: $\mathbf{R}^n$-valued Lévy process on $(\Omega, \mathcal{F}, P)$ with Lévy-Khintchine exponent ψ, i.e.

$$E[\exp\{i\langle z, X_t\rangle\}] = \exp\{-t\psi(z)\}, \ z \in \mathbf{R}^n, t \geq 0.$$

Lévy-Khintchine formula:

$$\psi(z) = i\langle a, z\rangle + \frac{1}{2}\langle z, Qz\rangle + \int_{\mathbf{R}^n} \left(1 - e^{i\langle z,x\rangle} + i\langle z, x\rangle 1_{\{|x|<1\}}\right) \mu(dx).$$

We also use (a, Q, μ) to stand for Lévy-Khintchine exponent ψ.

Progress on Getoor's conjecture (1)

Port and Stone (69, Ann. Math. Statist.): For the asymmetric Cauchy process on the line every x is regular for $\{x\}$. Hence only the empty set is a semipolar set and therefore (H) holds.

Blumenthal and Getoor (70): All stable processes with index $\alpha \in (0, 2)$ on the line satisfy (H).

Progress on Getoor's conjecture (2)

Kanda (76, PTRF), Forst (75, Math. Ann.): (H) holds if X has bounded continuous transition densities and $|\mathrm{Im}(\psi)| \leq M(1+\mathrm{Re}(\psi))$.

Rao (77, PTRF) gave a short proof of the Kanda-Forst theorem under the weaker condition that X has resolvent densities.

In particular, for $n > 1$ all stable processes of index $\alpha \neq 1$ satisfy (H). Kanda (78, PTRF) settled this problem for $\alpha = 1$ assuming the linear term vanishes.

Progress on Getoor's conjecture (3)

Rao (88, Proc. AMS): If all 1-excessive functions of X are lower semicontinuous and $|\mathrm{Im}(\psi)| \leq (1+\mathrm{Re}(\psi))f(1+\mathrm{Re}(\psi))$, where f is increasing on $[1,\infty)$ such that

$$\int_N^\infty \frac{1}{zf(z)}dz = \infty$$

for any $N \geq 1$, then X satisfies (H).

Examples for f: $f(x) \equiv M(constant), \;\; f(x) = \ln x, \;\; f(x) = \ln\ln x \cdots$.

Progress on Getoor's conjecture (4)

(SY): It has resolvent densities w.r.t. the Lebesgue measure and is symmetric.

(KF): It has resolvent densities w.r.t. the Lebesgue measure and the Kanda-Forst condition holds.

(R): It has resolvent densities w.r.t. the Lebesgue measure and Rao's condition holds.

$$(SY) \Rightarrow (KF) \Rightarrow (R) \Rightarrow (H).$$

Progress on Getoor's conjecture (5)

(ND): Q is non-degenerate.

(S): $\mu(\mathbf{R}^n \backslash \sqrt{Q}\mathbf{R}^n) < \infty$, and the solution condition holds, i.e. the equation $\sqrt{Q}y = -a - \int_{\{x \in \mathbf{R}^n \backslash \sqrt{Q}\mathbf{R}^n:\, |x|<1\}} x\mu(dx)$ has at least one solution $y \in \mathbf{R}^n$.

(SP): It has bounded continuous transition densities, and the Lévy process X and its symmetrization $X - \bar{X}$ have the same polar sets.

$$\begin{array}{ccccccccc} (ND) & \Rightarrow & (KF) & \Rightarrow & (R) & \Rightarrow & (H) & \Leftarrow & (SP) \\ & & & & & & \Uparrow & & \\ & & & & & & (S) & & \end{array}$$

Hu-Sun (2012): Hunt's hypothesis (H) and Getoor's conjecture for Levy processes, *SPA*, **122**, 2319-2328.

Progress on Getoor's conjecture (6)

(EKFR): It has resolvent densities w.r.t. the Lebesgue measure and the following (EKFR) condition holds:

There are two measurable functions ψ_1 and ψ_2 on $\mathbf{R}^n$ such that $\mathrm{Im}(\psi) = \psi_1 + \psi_2$, and

$$\begin{cases} |\psi_1| \leq A(z)f(A(z)), \quad (A(z) = 1 + \mathrm{Re}\psi(z)), \\ \int_{\mathbf{R}^n} \frac{|\psi_2(z)|}{(1+\mathrm{Re}\psi(z))^2 + (\mathrm{Im}\psi(z))^2} dz < \infty, \end{cases}$$

where f is an increasing function on $[1, \infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for any $N \geq 1$.

$$(R) \Rightarrow (EKFR) \Rightarrow (H).$$

Hu-Sun-Zhang (2015): New results on Hunt's hypothesis (H) for Lévy processes, *PA*, **42**, 585-605.

Progress on Getoor's conjecture (7)

(C^0): It has resolvent densities w.r.t. the Lebesgue measure and for any finite measure ν on $\mathbf{R}^n$ of finite 1-energy,

$$\int_{\mathbf{R}^n} \frac{1}{B(z)\log(2+B(z))[\log\log(2+B(z))]} |\hat{\nu}(z)|^2 dz < \infty,$$

where $B(z) = |1 + \psi(z)|$.

$(C^{B/A})$: It has resolvent densities w.r.t. the Lebesgue measure and there exists a constant $C > 0$ such that

$$B(z) \le CA(z)\log(2+B(z))[\log\log(2+B(z))], \forall z \in \mathbf{R}^n.$$

$$(KF) \Rightarrow (C^{B/A}) \Rightarrow (C^0) \Rightarrow (H).$$

Hu-Sun (2016): Further study on Hunt's hypothesis (H) for Levy processes, *SCM*, **59**, 2205-2226.

Progress on Getoor's conjecture (8)

$$
\begin{array}{ccccccccc}
 & & (SY) & & & & & & \\
 & & \Downarrow & & & & & & \\
(ND) & \Rightarrow & (KF) & \Rightarrow & (C^{B/A}) & \Rightarrow & (C^0) & & \\
 & & \Downarrow & & & & \Downarrow & & \\
 & & (R) & \Rightarrow & (EKFR) & \Rightarrow & (H) & \Leftarrow & (SP) \\
 & & & & & & \Uparrow & & \\
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\end{array}
$$

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Outline

3.1 (H) for one-dimensional Lévy processes

◇ Motivation

◇ Main results

◇ An example

Motivation-I

Let $X = (X_t)_{t\geq 0}$ be a one-dimensional Lévy process with exponent ψ and (a, σ, μ). If $\int (|x| \wedge 1)\mu(dx) < \infty$, we write

$$\psi(z) = ia'z + \frac{1}{2}\sigma z^2 + \int_{\mathbf{R}} \left(1 - e^{i\langle z,x\rangle}\right) \mu(dx).$$

Define

$$C = \{x \in \mathbf{R} : P\{X_t = x \text{ for some } t > 0\} > 0\},$$

and the following cases:

A. $\sigma > 0$.

B. $\sigma = 0$; $\int (|x| \wedge 1)\mu(dx) = +\infty$.

C. $\sigma = 0$; $\int (|x| \wedge 1)\mu(dx) < +\infty$. Now we further decompose it into the following three subcases:

- C_1. $a' = 0$,
- C_2. $a' > 0$, μ's support is $\mathbf{R}^+ = \{x \in \mathbf{R} | x > 0\}$.
- C_3. $a' > 0$, μ charges $\mathbf{R}^- = \{x \in \mathbf{R} | x < 0\}$.

Motivation-II

Theorem (Bretagnolle, 1971)
(i) For Case A, $C = \mathbf{R}$ and 0 is a regular point of $\{0\}$.
(ii) For Case B, either $C = \emptyset$ or $C = \mathbf{R}$, and if $C = \mathbf{R}$, then 0 is a regular point of $\{0\}$.
(iii) For Case C, suppose that X is not a compound Poisson process, then
(a) for Case C_1, $C = \emptyset$;
(b) for Case C_2, $C = \mathbf{R}^+$, 0 is not a regular point of $\{0\}$;
(c) for Case C_3, $C = \mathbf{R}$, 0 is not a regular point of $\{0\}$.

Remark For Case B, by one result of Kesten, we know that if $\int_0^\infty (x \wedge 1)\mu(dx) < \infty$ or $\int_{-\infty}^0 (|x| \wedge 1)\mu(dx) < \infty$, then $C = \mathbf{R}$, and thus any $x \in \mathbf{R}$ is a regular point of $\{x\}$ and hence (H) holds in this case.
$\Rightarrow$ our motivation

Main results-I

Denote by μ_+ and μ_- be the restriction of the Lévy measure μ on $(0,\infty)$ and $(-\infty,0)$, respectively. Define by $\bar{\mu}_-$ the image measure of μ_- under the map

$$x \mapsto -x, \ \forall x \in (-\infty, 0).$$

Theorem 1. Suppose that $Q = 0$ and $\int_0^\infty (1 \wedge x)\mu_+(dx) = \infty$. If there exist $\delta \in (0,1), k \in [0,1)$, and a measure ν on $\mathbf{R}^+$ satisfying $\int_{(0,\delta)} x\nu(dx) < \infty$, such that

$$\bar{\mu}_- \leq k\mu_+ + \nu. \tag{1}$$

Then X satisfies (H).

Main results-II

Theorem 2. If

$$\liminf_{\varepsilon \downarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\frac{\varepsilon}{|\log \varepsilon|}} > 0, \tag{2}$$

then X satisfies (H).

Main results-III

Remark 1. (i) The condition (2) does not require any controllability of $\mathrm{Im}(\psi)$ by $\mathrm{Re}(\psi)$.

(ii) For $\alpha > 0$, we define the measure ν_α on $(-1,1)$ by

$$\nu_\alpha(dx) := |x \log |x||^{1+\alpha} \mu(dx),\ x \in (-1,1).$$

Then, the condition (2) only requires slightly more than ν_α is an infinite measures on $(-1,1)$ for any $\alpha > 0$ in the sense as follows:
(a) Condition (2) implies that any ν_α is an infinite measure on $(-1,1)$.
(b) If for some $\beta > 2$,

$$\liminf_{\varepsilon \downarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\varepsilon / |\log \varepsilon|^\beta} = 0, \tag{3}$$

then ν_α is a finite measure on $(-1,1)$ for any $\alpha \in (0, \beta - 2)$.

Main results-IV

Proposition 4. If

$$\liminf_{|z|\to\infty} \frac{\mathrm{Re}\psi(z)}{\frac{|z|}{\log|z|}} > 0,$$

then X satisfies (H).

Proposition 5. If

$$\liminf_{\varepsilon\to 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2\mu(dx)}{\frac{\varepsilon}{|\log\varepsilon|[\log|\log\varepsilon|]}} > 0,$$

then X satisfies (H).

An example

Let X be a Lévy process on $\mathbf{R}$ with Lévy measure μ. Suppose that there exist positive constants c, δ, and a finite measure ν on $(0, \delta)$ such that

$$\mu(dx) + \nu(dx) \geq \frac{c}{x^2 |\log x|} dx \ \text{ on } (0, \delta).$$

Then X satisfies Condition (2), and thus (H) holds.

3.2 (H) for sum of Lévy processes without assumption on resolvent densities

◇ Main results

◇ Three lemmas for Theorem 6

◇ One lemma for Theorem 7

Main results-I

Theorem 6. Let X_1 and X_2 be two independent Lévy processes on $\mathbf{R}^n$. If X_1 satisfies (H) and X_2 is a compound Poisson process, then $X_1 + X_2$ satisfies (H).

Theorem 7. Let X_1 and X_2 be two independent Lévy processes on $\mathbf{R}^n$. If both X_1 and X_2 satisfy condition (S), then $X_1 + X_2$ satisfies (H).

Main results-II

By Theorem 6, we can strengthen the comparison Theorem 2.1 in Hu-Sun-Zhang (2015) as follows:

Proposition 8. Let X be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a, Q, μ). Suppose that μ_1 is a finite measure on $\mathbf{R}^n \backslash \{0\}$ such that $\mu_1 \leq \mu$. Denote $\mu_2 := \mu - \mu_1$ and let X' be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a', Q, μ_2), where
$a' := a + \int_{\{|x|<1\}} x\mu_1(dx)$. Then
(i) X and X' have same semipolar sets.
(ii) X and X' have same essentially polar sets.

(iii) if X' satisfies (H), then X satisfies (H).
(iv) if X satisfies (H) and X' has resolvent densities w.r.t. the Lebesgue measure, then X' satisfies (H).

Three lemmas for Theorem 6

Lemma 1. Let X be a Lévy process on $\mathbf{R}^n$ $(n > 1)$ satisfying (H). Then, for any nonempty proper subspace S of $\mathbf{R}^n$, the projection process Y of X on S satisfies (H).

Lemma 2. Let X be a Lévy process on $\mathbf{R}^n$ $(n > 1)$ with Lévy-Khintchine exponent (a, Q, μ). Suppose that for some proper subspace S of $\mathbf{R}^n$, the projection process X_S of X on S satisfies (H) and $\mu(\mathbf{R}^n \backslash S) < \infty$. Then X satisfies (H).

Lemma 3. Let X_1 and X_2 be two independent Lévy processes on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. If X_1 satisfies (H) and X_2 is a compound Poisson process, then $X = (X_1, X_2)$ satisfies (H).

One lemma for Theorem 7

Lemma 4. Let M be a symmetric nonnegative definite $n \times n$ matrix. Then,

$$x \in \sqrt{M}\mathbf{R}^n \quad \Leftrightarrow \quad \exists c > 0 \; s.t. \; |\langle x, z\rangle| \leq c\sqrt{\langle z, Mz\rangle}, \; \forall z \in \mathbf{R}^n.$$

3.3 (H) for sum of Lévy processes under assumption that resolvent densities exist

Throughout this part, we assume that X_1 and X_2 are two independent Lévy processes on $\mathbf{R}^n$ such that $X_1 + X_2$ has resolvent densities w.r.t. the Lebesgue measure. We denote by ψ_1 and ψ_2 the Lévy-Khintchine exponents of X_1 and X_2, respectively.

Main results-I

Theorem 9. Suppose that
(i) X_1 has resolvent densities w.r.t. the Lebesgue measure and satisfies (H).
(ii) Any finite measure ν of finite 1-energy w.r.t. $X_1 + X_2$ has finite 1-energy w.r.t. X_1.
(iii) There exists a constant $c > 0$ such that

$$|\mathrm{Im}\psi_2| \leq c(1 + \mathrm{Re}\psi_1 + \mathrm{Re}\psi_2).$$

Then $X_1 + X_2$ satisfies (H).

Main results-II

Proposition 10. If one of the following conditions is fulfilled, then any finite measure ν of finite 1-energy w.r.t. $X_1 + X_2$ has finite 1-energy w.r.t. X_1.

(i) There exists a constant $c > 0$ such that

$$|\psi_2| \leq c(1 + \mathrm{Re}(\psi_1)).$$

(ii) There exists a constant $c > 0$ such that

$$\begin{cases} \mathrm{Re}\psi_2 \leq c\left(1 + \mathrm{Re}\psi_1 + \frac{(\mathrm{Im}\psi_1)^2}{1+\mathrm{Re}\psi_1}\right), \\ |\mathrm{Im}\psi_2| \leq c(1 + \mathrm{Re}\psi_1 + \mathrm{Re}\psi_2). \end{cases}$$

(iii) There exists a constant $c > 0$ such that

$$\begin{cases} \mathrm{Re}\psi_2 \leq c\left(1 + \mathrm{Re}\psi_1 + \frac{(\mathrm{Im}\psi_1)^2}{1+\mathrm{Re}\psi_1)}\right), \\ (\mathrm{Im}\psi_2)^2 \leq c(1 + \mathrm{Re}\psi_1 + \mathrm{Re}\psi_2)\left(1 + \mathrm{Re}\psi_1 + \frac{(\mathrm{Im}\psi_1)^2}{1+\mathrm{Re}\psi_1}\right). \end{cases}$$

Main results-III

Corollary 11. Suppose that
(i) X_1 has bounded resolvent densities w.r.t. the Lebesgue measure and satisfies (H).
(ii) There exists a constant $c > 0$ such that

$$|\mathrm{Im}\psi_2| \leq c(1 + \mathrm{Re}\psi_1 + \mathrm{Re}\psi_2).$$

Then $X_1 + X_2$ satisfies (H).

Main results-III

Remark 2.

(i) One-dimensional Brownian motion has bounded resolvent densities.

(ii) Any spectrally one sided one-dimensional Lévy process with unbounded variation has bounded resolvent densities.

(iii) Any one-dimensional Lévy process satisfying the condition (2) of Theorem 2 has bounded resolvent densities.

Outline

Some open problems

Question 1. Does any subordinator with zero drift satisfy (H) ?

Question 2. Does any one-dimensional Lévy processes in Case C_1 satisfy (H)?

Question 3. Does any pure jump Lévy process satisfy (H)?

Question 4. Does any one-dimensional Lévy process in Case B satisfy (H)?

Question 5. Suppose that X_1 and X_2 are two independent Lévy processes, and both of them satisfy (H). Does $X_1 + X_2$ satisfy (H)?

Question 6. Suppose that X_1 and X_2 are two independent Lévy processes, and both of them satisfy (H). Does $X = (X_1, X_2)$ satisfy (H)?

Thank you for your attention!

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