Points of infinite multiplicity of a planar Brownian motion

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joint work with Elie Aïdékon, Zhan Shi (Paris 6)

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Let \((B_t)_{t \geq 0}\) be a planar Brownian motion starting from 0 and \(p \geq 2\). A point \(x\) is called of \(p\)-multiplicity if there exist \(0 < s_1 < s_2 < \ldots < s_p\) such that

\[
B_{s_1} = B_{s_2} = \ldots = B_{s_p} = x.
\]
• Dvoretzky, Erdös, Kakutani (1950) : Existence of \( p \)-multiple points for planar Brownian motion.

• Geman, Horowitz and Rosen (1984), Le Gall (1986) : The \( p \)-multiple self-intersection local time of \( B \) is the Radon measure \( \alpha_p \):

\[
\alpha_p(ds_1 \cdots ds_p) := \delta_0(B_{s_1} - B_{s_2}) \cdots \delta_0(B_{s_{p-1}} - B_{s_p})ds_1 \cdots ds_p.
\]

Consequently, almost surely for any \( p \geq 2 \), there exist points of multiplicity (exactly) \( p \).

• References : Le Gall, Rosen, Taylor, Yor..., Book by X. Chen (2009) “Random walk intersections.”

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Infinite multiplicity

Dvoretzky, Erdös and Kakutani (1958), Le Gall (1987)

There exist points of (exactly) countable multiplicity as well as points of uncountable multiplicity.

Consequence

Almost surely the planar Brownian motion has points of infinite multiplicity, and these points form a dense set on the range.
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Measuring the points of infinite multiplicity

**Problem**

Construct a measure supported by the points of infinite multiplicity, or *thick points*.

**Two natural ways to define the thick points**

1. By the occupation times;
2. By the number of crossings (i.e., number of excursions).
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Two natural ways to define the thick points

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2. By the number of crossings (i.e. number of excursions).
1st definition of thick points: by the occupation times

**Occupation times**

Let $0 < \alpha < 2$. A point $x$ is called a $\alpha$-thick point if

$$\lim_{r \to 0} \frac{1}{r^2 (\log r)^2} \int_0^T 1\{|B_s-x|<r\} \, ds = \alpha,$$

where $T$ denotes the hitting time of the unit circle, or any finite deterministic time.

**Typical behaviors (Ray 1963, Le Gall 1985, H. Shi 1995...)**

Almost surely,

$$\limsup_{r \to 0} \frac{1}{r^2 (\log 1/r) \log \log \log 1/r} \int_0^T 1\{|B_s|<r\} \, ds = 1.$$
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Thick points by the occupation times

Theorem: Dembo, Peres, Rosen and Zeitouni (2001)

For $0 \leq \alpha \leq 2$,

$$\dim_H \{\alpha\text{-thick points}\} = 2 - \alpha, \quad \text{a.s.},$$

with $\dim_H(A)$ the Hausdorff dimension of the set $A$.

Remark

The lower bound is the difficult part!

Dembo, Peres, Rosen and Zeitouni (2001)

As a consequence, they solved the Erdős-Taylor (1960) conjecture on the maximum of local times of a simple random walk on $\mathbb{Z}^2$. 
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**Erdös-Taylor (1960)’s conjecture**

Let $S$ be a simple symmetric random walk on $\mathbb{Z}^2$ and define $L_n^* := \max_x L_n^x$ with $L_n^x := \sum_{i=1}^n 1\{S_i = x\}$. Erdös-Taylor (1960) proved that almost surely,

$$\frac{1}{4\pi} \leq \liminf_{n \to \infty} \frac{L_n^*}{(\log n)^2} \leq \limsup_{n \to \infty} \frac{L_n^*}{(\log n)^2} \leq \frac{1}{\pi},$$

and conjectured that the upper bound is sharp.

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Let $N_x(r)$ be the number of crossings (i.e. number of excursions) from $x$ to $C(x, r)$ by the Brownian motion $B$ till $T_{C(0,1)}$, the hitting time of the unit circle $C(0, 1)$. 

\[ N_x(r) = 2 \]
Fix $\alpha \in (0, 2)$. Bass, Burdzy and Koshnevisan (1994) studied the following set of the points with infinite multiplicity:

$$A_\alpha := \left\{ |x| \leq 1 : \lim_{r \to 0} \frac{N_x(r)}{\log 1/r} = \alpha \right\}.$$

**Carrying dimension**

For any finite measure $\beta$ on $B(\mathbb{R}^2)$, let $\text{Dim}(\beta)$ be the carrying dimension of $\beta$:

$$\text{Dim}(\beta) := \inf\{ r > 0 : \exists A \in B(\mathbb{R}^2), \beta(A^c) = 0, \dim_H(A) = r \}.$$
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Theorem (Bass, Burdzy and Koshnevisan (1994))

Let $\alpha \in (0, \frac{1}{2})$. Almost surely there exists a measure $\beta_\alpha$ carried by $A_\alpha$; [recalling $A_\alpha := \{|x| \leq 1 : \lim_{r \to 0} \frac{N_x(r)}{\log \frac{1}{r}} = \alpha \}$.] Moreover

$$\dim(\beta_\alpha) = 2 - \alpha, \quad \text{a.s.}$$
Questions

Question 1 (open) :
Are the two definitions of thick points (by the occupation times and by the number of crossings) are equivalent?

Question 2 (open) :
What is the Hausdorff dimension of $A_\alpha$?

Aim

- Extend Bass, Burdzy and Koshnevisan’s results to all $\alpha \in (0, 2)$;
- Establish a relationship between the two definitions of thick points.

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Theorem (Aïdékon, H., Shi (2017+))

For any $\alpha \in (0, 2)$. Almost surely there exists a random finite measure $\mathcal{M}_\infty^\alpha$ carried by $A_\alpha$ as well as by the set of $\alpha$-thick points. Moreover,

$$\text{Dim}(\mathcal{M}_\infty^\alpha) = 2 - \alpha, \quad \text{a.s.}$$

Consequence

Since $\mathcal{M}_\infty^\alpha$ is supported by the set of $\alpha$-thick points, the above Theorem yields that almost surely,

$$\dim_H \{\alpha\text{-thick points}\} \geq 2 - \alpha,$$

the lower bound in Dembo, Peres, Rosen and Zeitouni’s theorem.
Theorem (Aïdékon, H., Shi (2017+))

For any $\alpha \in (0, 2)$. Almost surely there exists a random finite measure $M_\alpha^\infty$ carried by $A_\alpha$ as well as by the set of $\alpha$-thick points. Moreover,

$$\dim(M_\alpha^\infty) = 2 - \alpha, \quad \text{a.s.}$$

Consequence

Since $M_\alpha^\infty$ is supported by the set of $\alpha$-thick points, the above Theorem yields that almost surely,

$$\dim_{\mathcal{H}} \{\alpha\text{-thick points}\} \geq 2 - \alpha,$$

the lower bound in Dembo, Peres, Rosen and Zeitouni’s theorem.
Some properties of $M^{\alpha}_{\infty}$

Nice domain

- By a domain in $\mathbb{R}^2$ we mean an open, connected and bounded subset of $\mathbb{R}^2$.
- A domain $D$ is said a nice domain if its boundary $\partial D$ is a finite union of analytic curves.
- We say that a boundary point is nice if the boundary is an analytic curve locally around the point.
Some properties of $\mathcal{M}^\alpha_\infty$

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Two Probabilities

Probability $\mathbb{P}_D^{z,z'}$

Let $D$ be a nice domain. Let $z \neq z'$ two nice points of $\partial D$. We denote by $\mathbb{P}_D^{z,z'}$ the law of the Brownian motion starting from $z$ and conditioned to exit $D$ at $z'$. 

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Two Probabilities

**Probability** $Q_{x,D}^{z,z',\alpha}$

Let $D$ be a nice domain, and let $x \in D$ and $z, z'$ distinct nice points in $\partial D$. Under $Q_{x,D}^{z,z',\alpha}$, $(B_t)_{t \geq 0}$ can be split into three parts:

(i) Until time $T_x$, $B$ is a Brownian motion starting from $z$ and conditioned at touching $x$ before $\partial D$;
(ii) : After touching $x$, it is a concatenation of Brownian loops generated by PPP with intensity $1_{[0,\alpha]}dt$ in time and $\mu_D(x,x)$ in space, where $\mu_D(x,x)$ denotes the law of Brownian loops in $D$ at $x$;
(iii) : the last part is a standard Brownian motion starting at $x$, conditioned to exit $D$ at $z'$. 

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Some properties of $\mathcal{M}_{\alpha}^\infty$

**Spine decomposition**

Let $0 < \alpha < 2$. Let $z \neq z'$ be two nice points of $\partial D$. For any nonnegative measurable function $f$, we have

$$\mathbb{E}_{D}^{z,z'} \int_{D} f(x, B.) \mathcal{M}_{\alpha}^\infty(dx) = \int_{D} \mathbb{E}_{Q_{x,D}^{z,z'},\alpha}^{\alpha} \left( f(x, B.) \right) M_{D}(x, \alpha) dx,$$

where $M_{D}(x, \alpha)$ is a deterministic positive function.

**Remark**

The above result is an extension of Bass, Burdzy and Koshnevisan’s Theorem 5.2 (where they limited $\alpha \in (0, \frac{1}{2})$ and considered a Brownian motion instead of an excursion).
Some properties of $M^\infty_\alpha$

**Spine decomposition**

Let $0 < \alpha < 2$. Let $z \neq z'$ be two nice points of $\partial D$. For any nonnegative measurable function $f$, we have

$$E_{\partial D}^{z,z'} \int_D f(x, B.) \mathcal{M}^\alpha_\infty (dx) = \int_D E_{Q_{x,D}^{z,z',\alpha}} \left( f(x, B.) \right) M_D(x, \alpha) dx,$$

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Change of measure

Let $D_1 \subset D$ be two nice domains, and $x \in D_1$. Let $z \neq z'$ distinct nice points of $\partial D$.

$$
\frac{dQ^{z,z',\alpha}_{x,D}}{dP^{z,z'}_{D}} \bigg|_{\mathcal{F}_D^+} = \frac{M_{D_1}(x, \alpha)}{e^{-\alpha C_D(x)} \xi_D(x, z, z')} ,
$$

where

$$
M_{D_1}(x, \alpha) := e^{-\alpha C_{D_1}(x)} \sum_{k \geq 1} \frac{\alpha^{k-1}}{(k-1)!} \sum (g_k, d_k)_{D_1} \prod_{i=1}^k \xi_{D_1}(x, e_{g_i}, e_{d_i}),
$$

where $C_{D_1}(x) := -\int_{\partial D_1} \log(|x - y|) H_{D_1}(x, y) dy$ and

$$
\xi_{D_1}(x, y, z) := \frac{2\pi H_{D_1}(x,y) H_{D_1}(x,z)}{H_{D_1}(y,z)} ,
$$

$d_0 := 0$ and the sum runs over all $g_k = (g_1, \ldots, g_k)$ and $d_k = (d_1, \ldots, d_k)$ such that $(g_i, d_i)$ are the starting and ending times of distinct excursions inside $D_1$. 

Yueyun Hu (Paris 13) joint work with Elie Aïdékon, Zhan Shi (Paris 6) Points of infinite multiplicity of a planar Brownian motion
For simplicity, let $D$ be the square $(0, 1)^2$ and $z \neq z'$ nice points of $\partial D$. Let $\mathcal{D}_n$ be the set of open squares in $D$ generated by the grid of mesh size $2^{-n}$. We define for any Borel set $A$,

$$M^\alpha_{\mathcal{D}_n}(A) := \int_A M_{D_n(x)}(x, \alpha) dx.$$ 

The process $(M^\alpha_{\mathcal{D}_n}(A))_{n \geq 0}$ is a martingale under $\mathbb{P}^{z, z'}_D$. 

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The martingale measure $\mathcal{M}_n^\alpha (A) := \int_A M_{D_n(x)}(x, \alpha) dx$
Definition of $\mathcal{M}_\infty^\alpha$

Since $\mathcal{M}_{\mathcal{D}_n}^\alpha(A)$ is a nonnegative martingale, we can define

$$\mathcal{M}_\infty^\alpha(A) := \lim_{n \to +\infty} \mathcal{M}_{\mathcal{D}_n}^\alpha(A), \quad \mathbb{P}_D^{z,z'} - \text{a.s.}$$

Uniform convergence

Let $0 < \alpha < 2$. Let $A$ be a Borel set in $D$. Under $\mathbb{P}_D^{z,z'}$, the martingale $(\mathcal{M}_{\mathcal{D}_n}^\alpha(A))_n$ converges in $L^1$ to $\mathcal{M}_\infty^\alpha(A)$. 

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THANK YOU