Points of infinite multiplicity of a planar Brownian motion

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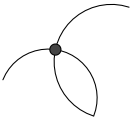
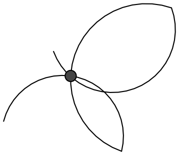
p-multiple point

Let $(B_t)_{t\geq 0}$ be a planar Brownian motion starting from 0 and $p \geq 2$. A point x is called of p-multiplicity if there exist $0 < s_1 < s_2 < ... < s_p$ such that

$$B_{s_1} = B_{s_2} = ... = B_{s_p} = x.$$



double point



triple point

- Dvoretzky, Erdös, Kakutani (1950) : Existence of p-multiple points for planar Brownian motion.
- Geman, Horowitz and Rosen (1984), Le Gall (1986) : The p-multiple self-intersection local time of B is the Radon measure α_p :

$$\alpha_p(ds_1 \cdots ds_p) := \delta_0(B_{s_1} - B_{s_2}) \cdots \delta_0(B_{s_{p-1}} - B_{s_p}) ds_1 \cdots ds_p.$$

  Consequently, almost surely for any $p \geq 2$, there exist points of multiplicity (exactly) p.
- References : Le Gall, Rosen, Taylor, Yor...,
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Infinite multiplicity

Dvoretzky, Erdös and Kakutani (1958), Le Gall (1987)

There exist points of (exactly) countable multiplicity as well as points of uncountable multiplicity.

Consequence

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Measuring the points of infinite multiplicity

Problem

Construct a measure supported by the points of infinite multiplicity, or *thick points*.

Two natural ways to define the thick points

1. By the occupation times ;
2. By the number of crossings (i.e. number of excursions).

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1st definition of thick points : by the occupation times

Occupation times

Let $0 < \alpha < 2$. A point x is called a *α-thick point* if

$$\lim_{r \to 0} \frac{1}{r^2 (\log r)^2} \int_0^T 1_{\{|B_s - x| < r\}} ds = \alpha,$$

where T denotes the hitting time of the unit circle, or any finite deterministic time.

Typical behaviors (Ray 1963, Le Gall 1985, H. Shi 1995...)

Almost surely,

$$\limsup_{r \to 0} \frac{1}{r^2 (\log 1/r) \log \log \log 1/r} \int_0^T 1_{\{|B_s| < r\}} ds = 1.$$

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Thick points by the occupation times

Theorem : Dembo, Peres, Rosen and Zeitouni (2001)

For $0 \leq \alpha \leq 2$,

$$\dim_{\mathrm{H}} \{\alpha\text{-thick points}\} = 2 - \alpha, \qquad a.s.,$$

with $\dim_{\mathrm{H}}(A)$ the Hausdorff dimension of the set A.

Remark

The lower bound is the difficult part !

Dembo, Peres, Rosen and Zeitouni (2001)

As a consequence, they solved the Erdös-Taylor (1960) conjecture on the maximum of local times of a simple random walk on $\mathbb{Z}^2$.

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Thick points by the occupation times

Erdös-Taylor (1960)'s conjecture

Let S be a simple symmetric random walk on $\mathbb{Z}^2$ and define $L_n^* := \max_x L_n^x$ with $L_n^x := \sum_{i=1}^n 1_{\{S_i = x\}}$. Erdös-Taylor (1960) proved that almost surely,

$$\frac{1}{4\pi} \le \liminf_{n\to\infty} \frac{L_n^*}{(\log n)^2} \le \limsup_{n\to\infty} \frac{L_n^*}{(\log n)^2} \le \frac{1}{\pi},$$

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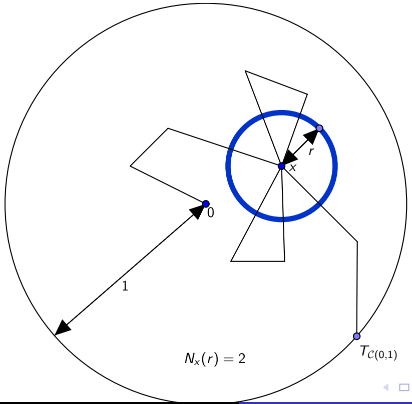
Thick points by the number of crossings

Let $N_x(r)$ be the number of crossings (i.e. number of excursions) from x to $\mathcal{C}(x,r)$ by the Brownian motion B till $T_{\mathcal{C}(0,1)}$, the hitting time of the unit circle $\mathcal{C}(0,1)$.



Fix $\alpha \in (0,2)$. Bass, Burdzy and Koshnevisan (1994) studied the following set of the points with infinite multiplicity :

$$\mathcal{A}_\alpha := \Big\{ |x| \le 1 : \lim_{r \to 0} \frac{N_x(r)}{\log 1/r} = \alpha \Big\}.$$

Carrying dimension

For any finite measure β on $\mathscr{B}(\mathbb{R}^2)$, let $\mathrm{Dim}(\beta)$ be the carrying dimension of β :

$$\mathrm{Dim}(\beta) := \inf\{ r > 0 : \exists A \in \mathscr{B}(\mathbb{R}^2), \, \beta(A^c) = 0, \, \dim_{\mathrm{H}}(A) = r \}.$$

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Theorem (Bass, Burdzy and Koshnevisan (1994))

Let $\alpha \in (0, \frac{1}{2})$. Almost surely there exists a measure β_α carried by $\mathcal{A}_\alpha$; [recalling $\mathcal{A}_\alpha := \{|x| \le 1 : \lim_{r\to 0} \frac{N_x(r)}{\log 1/r} = \alpha\}$.] Moreover

$$\mathrm{Dim}(\beta_\alpha) = 2 - \alpha, \qquad \text{a.s.}$$

Questions

Question 1 (open) :

Are the two definitions of thick points (by the occupation times and by the number of crossings) are equivalent ?

Question 2 (open) :

What is the Hausdorff dimension of $\mathcal{A}_\alpha$?

Aim

- Extend Bass, Burdzy and Koshnevisan's results to all $\alpha \in (0, 2)$;
- Establish a relationship between the two definitions of thick points.

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Theorem (Aidékon, H., Shi (2017+))

For any $\alpha \in (0,2)$. Almost surely there exists a random finite measure $\mathcal{M}_\infty^\alpha$ carried by $\mathcal{A}_\alpha$ as well as by the set of α-thick points. Moreover,

$$\mathrm{Dim}(\mathcal{M}_\infty^\alpha) = 2 - \alpha, \qquad \textit{a.s.}$$

Consequence

Since $\mathcal{M}_\infty^\alpha$ is supported by the set of α-thick points, the above Theorem yields that almost surely,

$$\dim_{\mathrm{H}} \{\alpha\text{-thick points}\} \geq 2 - \alpha,$$

the lower bound in Dembo, Peres, Rosen and Zeitouni's theorem.

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Some properties of $\mathcal{M}_\infty^\alpha$

Nice domain

- By a domain in $\mathbb{R}^2$ we mean an open, connected and bounded subset of $\mathbb{R}^2$.
- A domain D is said a nice domain if its boundary ∂D is a finite union of analytic curves.
- We say that a boundary point is nice if the boundary is an analytic curve locally around the point.

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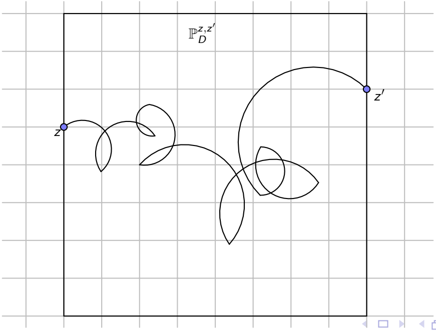
Two Probabilities

Probability $\mathbb{P}_D^{z,z'}$

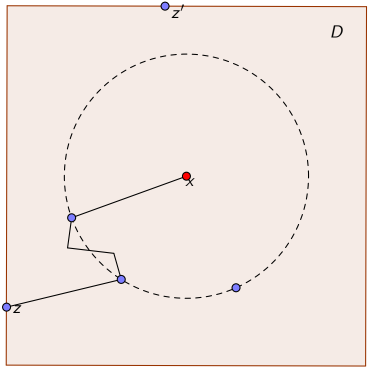
Let D be a nice domain. Let $z \neq z'$ two nice points of ∂D. We denote by $\mathbb{P}_D^{z,z'}$ the law of the Brownian motion starting from z and conditioned to exit D at z'.



Two Probabilities

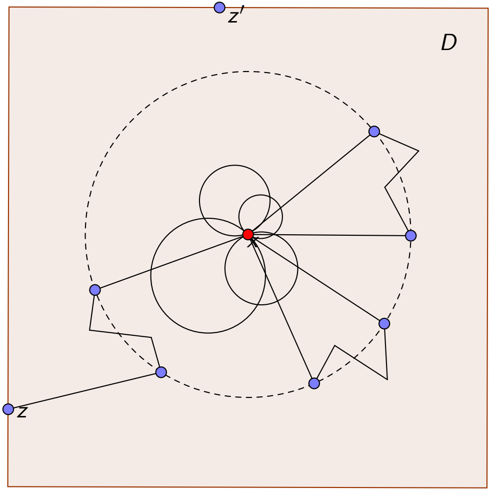
Probability $\mathbb{Q}_{x,D}^{z,z',\alpha}$

Let D be a nice domain, and let $x \in D$ and z, z' distinct nice points in ∂D. Under $\mathbb{Q}_{x,D}^{z,z',\alpha}$, $(B_t)_{t\geq 0}$ can be split into three parts :



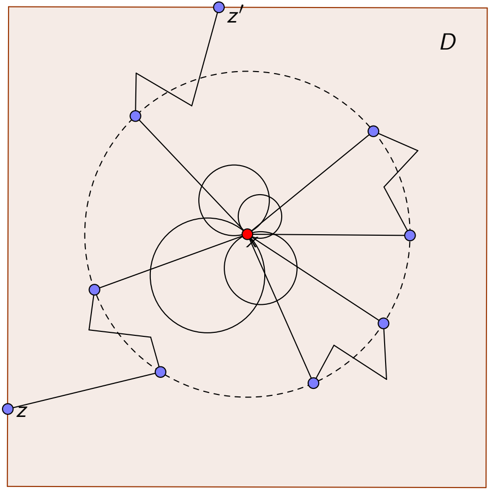
(i) Until time T_x, B is a Brownian motion starting from z and conditioned at touching x before ∂D;

Probability $\mathbb{Q}_{x,D}^{z,z',\alpha}$



(ii) : After touching x, it is a concatenation of Brownian loops generated by PPP with intensity $\mathbf{1}_{[0,\alpha]}dt$ in time and $\mu_D(x,x)$ in space, where $\mu_D(x,x)$ denotes the law of Brownian loops in D at x ;

Probability $\mathbb{Q}^{z,z',\alpha}_{x,D}$



(iii) : the last part is a standard Brownian motion starting at x, conditioned to exit D at z'.

Some properties of $\mathcal{M}_\infty^\alpha$

Spine decomposition

Let $0 < \alpha < 2$. Let $z \neq z'$ be two nice points of ∂D. For any nonnegative measurable function f, we have

$$\mathbb{E}_D^{z,z'} \int_D f(x, B_\cdot)\mathcal{M}_\infty^\alpha(dx) = \int_D \mathbb{E}_{\mathbb{Q}_{x,D}^{z,z',\alpha}}\Big(f(x, B_\cdot)\Big) M_D(x,\alpha)dx,$$

where $M_D(x,\alpha)$ is a deterministic positive function.

Remark

The above result is an extension of Bass, Burdzy and Koshnevisan's Theorem 5.2 (where they limited $\alpha \in (0, \frac{1}{2})$ and considered a Brownian motion instead of an excursion).

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Change of measure

Let $D_1 \subset D$ be two nice domains, and $x \in D_1$. Let $z \neq z'$ distinct nice points of ∂D.

$$\frac{\mathrm{d}\mathbb{Q}_{x,D}^{z,z',\alpha}}{\mathrm{d}\mathbb{P}_{D}^{z,z'}}\Big|_{\mathscr{F}_{D_1}^{+}} = \frac{M_{D_1}(x,\alpha)}{e^{-\alpha C_D(x)}\xi_D(x,z,z')},$$

where

$$M_{D_1}(x,\alpha) := e^{-\alpha C_{D_1}(x)} \sum_{k\geq 1} \frac{\alpha^{k-1}}{(k-1)!} \sum_{(\underline{g}_k,\underline{d}_k)_{D_1}} \prod_{i=1}^{k} \xi_{D_1}(x, e_{g_i}, e_{d_i}),$$

where $C_{D_1}(x) := -\int_{\partial D_1} \log(|x-y|) H_{D_1}(x,y)dy$ and $\xi_{D_1}(x,y,z) := \frac{2\pi H_{D_1}(x,y)H_{D_1}(x,z)}{H_{D_1}(y,z)}$, $d_0 := 0$ and the sum runs over all $\underline{g}_k = (g_1,\ldots,g_k)$ and $\underline{d}_k = (d_1,\ldots,d_k)$ such that (g_i,d_i) are the starting and ending times of distinct excursions inside D_1.

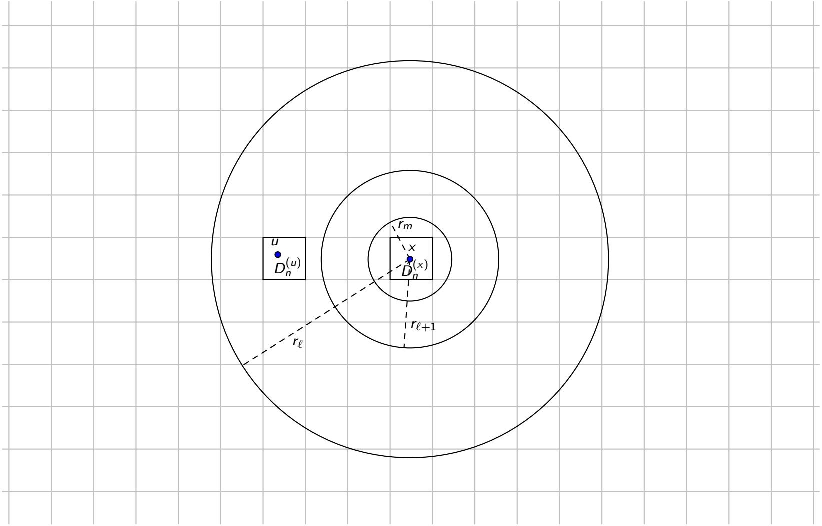
Martingale

For simplicity, let D be the square $(0,1)^2$ and $z \neq z'$ nice points of ∂D. Let $\mathscr{D}_n$ be the set of open squares in D generated by the grid of mesh size 2^{-n}. We define for any Borel set A,

$$\mathcal{M}^{\alpha}_{\mathscr{D}_n}(A) := \int_A M_{D_n^{(x)}}(x,\alpha)dx.$$

The process $(\mathcal{M}^{\alpha}_{\mathscr{D}_n}(A))_{n\geq 0}$ is a martingale under $\mathbb{P}_D^{z,z'}$.

The martingale measure $\mathcal{M}^{\alpha}_{\mathscr{D}_n}(A) := \int_A M_{D_n^{(x)}}(x,\alpha)dx$



Definition of $\mathcal{M}_\infty^\alpha$

Since $\mathcal{M}_{\mathscr{D}_n}^\alpha(A)$ is a nonnegative martingale, we can define

$$\mathcal{M}_\infty^\alpha(A) := \lim_{n\to+\infty} \mathcal{M}_{\mathscr{D}_n}^\alpha(A), \qquad \mathbb{P}_D^{z,z'}\text{-a.s.}$$

Uniform convergence

Let $0 < \alpha < 2$. Let A be a Borel set in D. Under $\mathbb{P}_D^{z,z'}$, the martingale $(\mathcal{M}_{\mathscr{D}_n}^\alpha(A))_n$ converges in L^1 to $\mathcal{M}_\infty^\alpha(A)$.

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THANK YOU