The 13th Workshop on Markov Processes and Related Topics

Local Differential Geometry and Lévy's Occupation Time Arcsine Law

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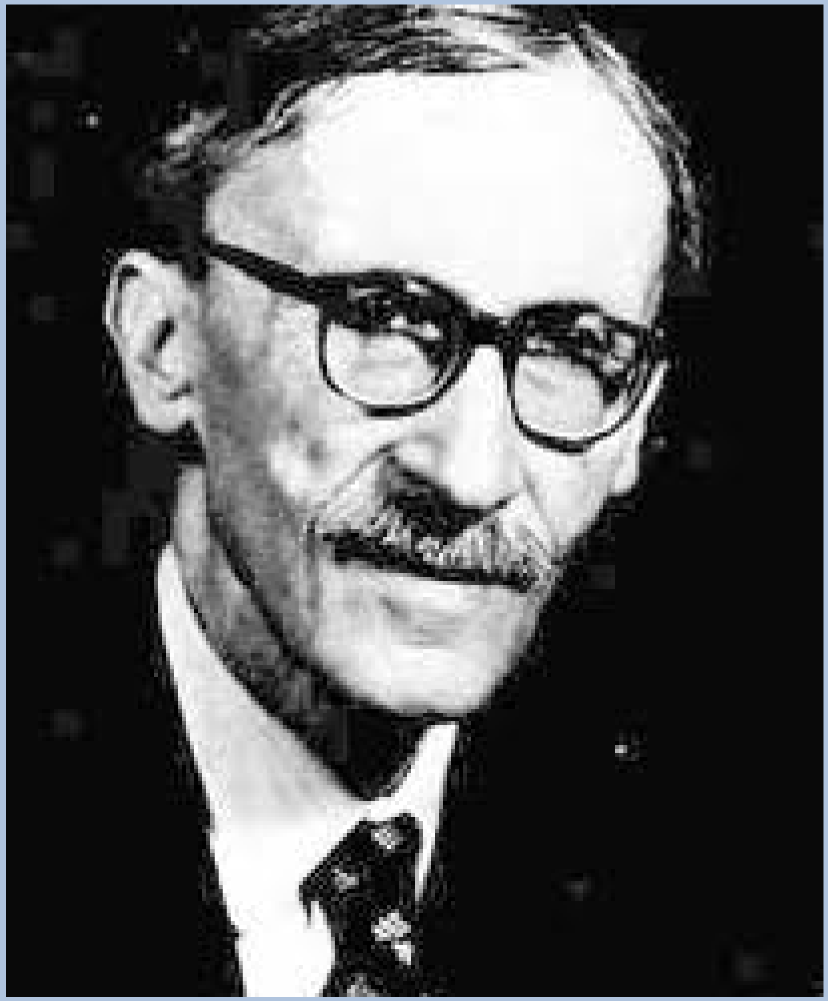
Paul Lévy (1886-1971)



Lévy’s Three Arcsine Laws

(1) Occupation of Brownian motion on the positive half line on [0,1];

(2) Last passage time of the origin before time 1;

(3) The time of the maximum of the path on [0,1].

We are concerned with the occupation time arcsine law.

Classical Occupation Time Arcsine Law

Let W be one dimensional Brownian motion.

$$\mathbb{P}\left[\int_0^1 1_{\mathbb{R}_+}(W_s)\,ds \le t\right] = \frac{2}{\pi}\arcsin\sqrt{t}.$$

The same hold for the half space defined by a hyperplane $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$.

The Problem to Be Investigated

Lower order correction (deviation) to the classical arcsine law near a hypersurface for a Brownian motion on $\mathbb{R}^n$ (or a general Riemannian manifold M).

If N is a hypersurface that divides the manifold M into two parts M_+ and M_-.

Consider the occupation time

$$T_t = \frac{1}{t}\int_0^t 1_{M_+}(X_s)\, ds.$$

We expect that the distribution of T_t is close to the arcsine law when t is small.

Question: What is the deviation of T_t from the arcsine law?

How to Describe the Deviation?

We will show that T_t is the sum of two random variables:

$$T_t = T_0 + Z_t,$$

where T_0 has exactly the arcsine law and Z_t is random variable going to zero as $t \to 0$.

What is the order of Z_t?

What is the joint law of T_0 and Z_t?

We will see that Z_t has the order of $\sqrt{t}$.

Intuition

* The problem should be local, depending only on the geometry near the starting point;

* The correction term depends on how the hypersurface deviates from the tangent hyperplane;

* In geometry, the mean curvature is a number that captures the deviation of a hypersurface from the tangent hyperplane.

What is the mean curvature of a hypersurface?

Near the starting point, let $\tilde{x} = (x^2, \ldots, x^n)$ be euclidean coordinates on the tangent hyperplane, then the surface is given by an equation $z = f(\tilde{x})$. We can choose $\tilde{x}$ such that

$$f(x) = \sum_{i=2}^{n} K_i x_i^2 + O(|x|^3).$$

Then

$$H = \sum_{i=2}^{n} K_i.$$

Technical Setup

In order to obtain a probabilistic description of the deviation from the arcsine law, we need to put time scaled Brownian motion on the same probability space. In $\mathbb{R}^n$, this means that we consider the scaled Brownian motion

$$X^t = \{B_{ts},\ s \geq 0\}.$$

Now the occupation time we are interested in is just

$$L_t = \int_0^1 I_{M_+}(X^t_s)\, ds.$$

This setup will help us describe the joint law of the leading arcsine term and the deviation term.

Statement of the Main Result

Let W be a one dimensional Brownian motion and L its local time at $x = 0$.

$$T_t = \int_0^1 1_{\mathbb{R}_+}(W_u)du - \frac{\sqrt{t}}{2} H \int_0^1 u\, dL_u + o(\sqrt{t}).$$

Observations:

(1) The leading order of correction is $\sqrt{t}$;

(2) The correction is proportional to the mean curvature;

(3) The correction depends on the local time of W through the integral

$$\int_0^1 u\, dL_u.$$

We will see why the local time at $x = 0$ appears in the deviation term.

Semi-geodesic coordinates

Straighten the hypersurface into a hyperplane:

$x^1 =$ the signed distance to the hypersurface;

$\tilde{x} = \left\{x^2, \ldots, x^n\right\}$, the usual euclidean coordinates on the tangent hyperplane.

The metric (length element) $ds^2 = g_{ij}dx^i dx^j$ is given by

$$g_{ij}(x) = \begin{cases} \delta_{1j}, & i = 1, 1 \le j \le n; \\ \delta_{ij} + 2\Pi_{ij}x_1 + O(|x|^2), & 2 \le i, j \le n. \end{cases}$$

Here $\Pi = \{\Pi_{ij}\}$ is called the second fundamental form. Its relation with the mean curvature H is

$$H = \mathrm{Tr}\Pi = \sum_{i=2}^{n} \Pi_{ii}.$$

Change of Coordinates for the Laplace Operator

The general formula for the Laplace operator in curvilinear coordinates is

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

In the semi-geodesic coordinates $\left\{ x^i \right\}$, the Laplace operator becomes

$$\Delta = \left(\frac{\partial}{\partial x^1} \right)^2 + b^1(x) \frac{\partial}{\partial x^1} + \text{partial derivatives with respect to } \tilde{x},$$

where

$$b^1(x) = H + O(|x|).$$

Brownian Motion in Curvilinear Coordinates

Once we have the generator (the Laplace operator) in the curvilinear coordinates we can write down the stochastic differential equation for the Brownian motion $X^t = \{B_{ts}\}$. Its first component is given by

$$X_s^{t,1} = \sqrt{t} W_s + \frac{t}{2} \int_0^s b^1(X_u^t)\, du.$$

Recall that $b^1(x) = H + O(|x|)$, we see that

$$\frac{X_s^{t,1}}{\sqrt{t}} \sim W_s + \frac{\sqrt{t}}{2} H s.$$

Approxmate Occupation Time

By the choice of the coordinates, the occupation time for the half space M_+ is simply

$$T_t = \int_0^1 1_{\mathbb{R}_+}(X_s^{t,1})\, ds.$$

From the approximation for the first component $X^{t,1}$, we have

$$T_t \sim \int_0^1 I_{\mathbb{R}_+}\left(W_s + \frac{\sqrt{t}}{2} H s\right)\, ds.$$

It is clear now that

$$\lim_{t \downarrow 0} T_t = \int_0^1 1_{\mathbb{R}+}(W_s)\, ds.$$

This is the exactly the term with the arcsine law.

In order to find out the deviation from the arcsine term, we need the local time for Brownian motion.

Brownian Local Time and Occupation Time Formula

Our discovery is that the leading deviation term depends also on the local time L_t^x of a one dimensional Brownian motion W. It is defined as the unique continuous increasing process such that

$$|W_t - x| = |x| + \int_0^t \mathrm{sgn}(W_s - x)\, dW_s + \frac{1}{2} L_t^x.$$

It is a classical result that L_t^x is jointly continuous in (t, x).

The occupation time formula is

$$\int_0^t \Phi(W_s, s)\, ds = \int_{\mathbb{R}} \left[\int_0^t \Phi(x, s) dL_s^x \right] dx.$$

The Final Step

From the occupation time formula we have immediately

$$
\begin{aligned}
T_t &\sim \int_0^1 I_{\mathbb{R}_+}\left(W_s + \frac{\sqrt{t}}{2}Hs\right)\,ds \\
&\sim \int_0^1 I_{\mathbb{R}_+}(W_s)\,ds - \int_0^1 I_{[0,\sqrt{t}Hs/2]}(W_s)\,ds \\
&\sim \int_0^1 I_{\mathbb{R}_+}(W_s)\,ds - \frac{\sqrt{t}}{2}H\int_0^1 s\,dL_s.
\end{aligned}
$$

We have omitted some technical details. But what we described is basically the whole proof.

Thank You!