Limiting Behavior of Stationary Measures for Stochastic Evolution Systems

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Outline

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Motivation

"A major step in making the equation more relevant is to add a small stochastic term. ... It seems fair to say that all differential equations are better models of the world when a stochastic term is added and that their classical analysis is useful only if it is stable in an appropriate sense to such perturbations". (David Mumford, the Fields Medal in 1974 and the Wolf Prize in 2008)

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Problem: How to reveal the connections between deterministic systems and their stochastic perturbation systems ?

Some Works on Zero-Noise Limits

- Freidlin and Wentzell, 1979
  - For the non-degenerate diffusion case, via large-deviation technique to estimate the concentration
- P. Dupuis, E.s. Eills, 1997
  - a weak convergence approach to the theory of Large deviations
- Chii-Ruey Hwang, 1980, Ann. Probability
  - Gibbs measures for gradient system with additive noise
- Tu Sheng Zhang, etc
  - large-deviation on SPED with small perturbation

Some Works on Zero-Noise Limits-Continued

- Lai-Sang, Young 2002, 2005, J. Statis. Physics, Ergod. Th. & Dynam. Sys.
  - SRB measures can be realized
- Huang, Ji, Liu and Yi, 2015, 2016, Ann. Probability, Physica D
  - ODE + small white noise, including gradient systems
- Yao Li and Ying Fei Yi, 2016, Comm. Pure Appl. Math.
  - Study the systematic measures of biological network
- and so on ...

Heuristics

The weak limits of the stationary measures for small random perturbations systems represent idealizations of what we want to see.

Problem Statement

Objects

- ϵ — the noise intensity which is small;
- $X^{\epsilon}(t,x)$ — solution of stochastic differential equations driven by Lévy noise with the intensity ϵ;
- $\Phi(t,x) := X^{0}(t,x)$ — the corresponding deterministic dynamical system while $\epsilon = 0$;

• The map $\Phi : \mathbb{R}_+ \times M \longrightarrow M$ is called the *dynamical system* if the following properties hold:
(i) $\Phi_t(x)$ are continuous, for all $t \in \mathbb{R}_+, x \in M$,
(ii) $\Phi_0 = \mathrm{id}$, $\Phi_t \circ \Phi_s(x) = \Phi_{t+s}(x)$, for all $t, s \in \mathbb{R}_+$, $x \in M$. Here $\circ$ denotes composition of mappings.

• Probability measures μ^{ϵ} and μ on $\mathcal{B}(M)$ are called *stationary* or *invariant* with respect to $\{P_t^{\epsilon}\}_{t\geq 0}$ and $\{\Phi_t\}_{t\geq 0}$, respectively, if

$$P_t^{\epsilon}\mu^{\epsilon} = \mu^{\epsilon} \text{ for any } t \geq 0, \text{ and}$$

$$\mu \circ \Phi_t^{-1} = \mu \;\text{ for any } t \geq 0, \text{ respectively.}$$

Problem Statement-Continued

Objects

- $\mathcal{I}^\epsilon$ — stationary measures for $X^\epsilon(t,x)$. $\mathcal{I} := \bigcup_{0<\epsilon\leq\epsilon_0}\mathcal{I}^\epsilon$;
- $\mathcal{M}_L$ — weak limits: $\mu^{\epsilon_i} \xrightarrow{w} \mu$ as $\epsilon_i \to 0$;
- $\mathcal{S} := \overline{\{\mathrm{supp}(\mu) : \mu \in \mathcal{M}_{\mathrm{L}}\}}$.

Problems

• The most important questions are now what the constraction of $\mathcal{S}$ is ? and how the dynamics on $(\mathcal{S}, \Phi|_{\mathcal{S}})$ is ?

$\mathcal{S}$ is contained in the Birkhoff Center

$$B(\Phi) := \overline{\{x \in M : x \in \omega(x)\}}.$$

which responds to the complexity of dynamical system.

• What kind of invariant measures for dynamical system can be approximated by stationary measures for small perturbations systems ?

Strongly depends on the type of noise.

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Concrete Noise Perturbed Processes

- **SODEs Driven by a Lévy process**:

$$\begin{aligned} dX^{\epsilon,x}(t) =& b(X^{\epsilon,x}(t))dt + \epsilon\sigma(X^{\epsilon,x}(t))dW_t \\ &+ \epsilon \int_{|y|_{\mathbb{R}^l}<c} F(X^{\epsilon,x}(t-),y)\tilde{N}(dt,dy) \end{aligned} \tag{1.1}$$

with initial condition $X^{\epsilon,x}(0) = x \in \mathbb{R}^m$ and $\epsilon, c > 0$. The mappings $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to L_2(\mathbb{R}^k, \mathbb{R}^m)$, and $F : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ satisfy the some normal conditions.

Concrete Noise Perturbed Processes-Continued

- **Stochastic Reaction Diffusion Equation**

$$\begin{cases} dX(t,x) = \nu\Delta X(t,x)dt + g(x, X(t,x))dt \\ \qquad\qquad + \epsilon\sigma(x, X(t,x))dW(t), \\ X(t,x) = 0, \ x \in \partial\Lambda, \ t > 0, \\ X(0) = h \in L^2(\Lambda). \end{cases} \tag{1.2}$$

Here $\nu > 0$, $g : \Lambda \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \Lambda \times \mathbb{R} \to l^2$ are two measurable functions. $W(t) = (W_k(t))_{k\in\mathbb{N}}$ is a sequence of independent one dimensional standard Brownian motions.

Concrete Noise Perturbed Processes-Continued

- **2D Stochastic Navier-Stokes Equation Driven by Lévy Noise**

$$dX_t^{\epsilon,h} = \big(\nu A X_t^{\epsilon,h} + X_t^{\epsilon,h}\cdot\nabla X_t^{\epsilon,h} + h_0(X_t^{\epsilon,h})\big)dt \\ + \epsilon B(X_t^{\epsilon,h})dW_t + \epsilon\int_Z f(X_{t-}^{\epsilon,h},z)\tilde{N}(dt,dz), \qquad (1.3)$$

$$\mathrm{div} X_t^{\epsilon,h} = 0.$$

with a initial value $X_0^{\epsilon,h} = h$.

Concrete Noise Perturbed Processes-Continued

- **1D Stochastic Burgers Equation Driven by Lévy Noise**

$$\begin{aligned} dX_t^{\epsilon,h} =& \big(\Delta X_t^{\epsilon,h} + X_t^{\epsilon,h} \cdot \nabla X_t^{\epsilon,h}\big)dt \\ &+ \epsilon B(X_t^{\epsilon,h})dW_t + \epsilon \int_Z f(X_{t-}^{\epsilon,h}, z)\tilde{N}(dt, dz), \end{aligned} \tag{1.4}$$

with initial value $X_0^{\epsilon,h} = h$.

Concrete Noise Perturbed Processes-Continued

- **Stochastic Functional Differential Equations (SFDEs)**

$$\begin{aligned} & dX^{\epsilon,\phi}(t) = b(X_t^{\epsilon,\phi})dt + \epsilon\sigma(X_t^{\epsilon,\phi})dW(t), \\ & X_0^{\epsilon,\phi} = \phi \in \mathcal{C} := C([-\tau, 0], \mathbb{R}^m), \end{aligned} \tag{1.5}$$

where $W = \{W_t = (W_t^1, \cdots, W_t^k), t \geq 0\}$ is a k-dimensional Wiener process, $b(\cdot) : \mathcal{C} \to \mathbb{R}^m$ and $\sigma(\cdot) : \mathcal{C} \to \mathbb{R}^{m\times k}$ satisfy the regular conditions.

Basic Assumption

Let (M, ρ) be a Polish space, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Assume that $X^\epsilon(t, x)$ and $\Phi(t, x)$ are noise driven process and deterministic semi dynamical system on (M, ρ), respectively.

Hypothesis (PC)

For any $T > 0, \delta > 0$ and compact set $K \subset M$, one has

$$\lim_{\epsilon \to 0} \sup_{x \in K} \mathbb{P}\big\{\rho\big(X^\epsilon(T, x), \Phi(T, x)\big) \geq \delta\big\} = 0.$$

Main Result

The probability transition function

$$P_t^\epsilon(x, A) = \mathbb{P}\big(X^\epsilon(t, x) \in A\big), t \geq 0, x \in M, A \in \mathcal{B}(M).$$

Theorem 1 (Φ-Invariance)

Assume that (PC) holds. Let μ^{ϵ_i} be a sequence of invariant probability measure. If $\mu^{\epsilon_i} \xrightarrow{w} \mu$ as $\epsilon_i \to 0$, then μ is Φ-invariant.

Corollary (Birkhoff, Poincaré and Conley)

If μ is Φ-invariant, then $\mathrm{supp}(\mu)$ *is positively invariant for Φ, and*

$$\mathrm{supp}(\mu) \subset B(\Phi) = \overline{\{x \in M : x \in \omega(x)\}} - \text{Birkhoff Center.}$$

SODEs Driven by a Lévy Process

$$
\begin{aligned}
dX^{\epsilon,x}(t) =& b(X^{\epsilon,x}(t))dt + \epsilon\sigma(X^{\epsilon,x}(t))dW_t \\
&+ \epsilon\int_{|y|_{\mathbb{R}^l}<c} F(X^{\epsilon,x}(t-),y)\tilde{N}(dt,dy).
\end{aligned} \tag{3.1}
$$

For a C^2 scalar function V and $\epsilon \geq 0$, define an operator

$$
\begin{aligned}
\mathcal{L}^\epsilon V(x) :=& \langle \nabla V(x), b(x)\rangle + \frac{\epsilon^2}{2}\sum_{i,j=1}^m a_{ij}(x)\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \\
&+ \int_{|y|_{\mathbb{R}^l}<c} \big(V(x+\epsilon F(x,y)) - V(x) - \langle \nabla V(x), \epsilon F(x,y)\rangle\big)\nu(dy),
\end{aligned}
$$

where $A(x) = (a_{ij}(x)) = \sigma(x)\sigma^T(x)$ is the diffusion matrix.

SODEs Driven by a Lévy Process-Continued

Theorem 3 (Support on weak limits)

Let $b(x)$, $\sigma(x)$ and $F(x,y)$ in (3.1) be locally Lipschitz continuous. Suppose that there exists a nonnegative function $V(x) \in C^2(\mathbb{R}^m)$ such that

$$\inf_{|x|>R} V(x) \to +\infty, \quad \text{as } R \to \infty, \text{ and} \tag{3.2}$$

$$\sup_{|x|>R} \mathcal{L}^\epsilon V(x) \le -A_R \to -\infty \text{ as } R \to \infty. \tag{3.3}$$

Then
(1)(PC) holds.
(2) There exists an invariant probability measure μ_x^ϵ for $\epsilon > 0, x \in \mathbb{R}^m$.
(3) There exists $\mu_x^{\epsilon_i} \in \mathcal{I}^{\epsilon_i}$ such that $\mu_x^{\epsilon_i} \xrightarrow{w} \mu_x$ as $\epsilon_i \to 0$.

SODEs Driven by a Lévy Process-Continued

In fact,

Lemma (Sufficient conditions for (PC))

Let b, σ and F be locally Lipschitz continuous. If there exist a nonnegative function $V(x) \in C^2(\mathbb{R}^m)$, $\epsilon_0 > 0$ and a constant $c^* < +\infty$, such that (3.2) and

$$\mathcal{L}^{\epsilon}V(x) \leq c^*V(x), \quad \forall \epsilon \in [0, \epsilon_0] \tag{3.4}$$

hold. Then there exists a global unique solution $X^{\epsilon,x}(t)$ to (3.1) for all $x \in \mathbb{R}^m$, $\epsilon \in [0, \epsilon_0]$. Moreover the (PC) holds.

Sketch of the Proof

- The global existence and uniqueness of solution to (3.1) are similar as the proof in Khasminskii.

- Let $\tau_n^{\epsilon,x} = \inf\{t : |X^{\epsilon,x}(t)| > n\}$, $\tau_n^{0,x} = \inf\{t : |X^{0,x}(t)| > n\}$ & $S^n(r)$ be a nonincreasing C^∞ function with values in $[0,1]$ such that

$$S^n(r) = \begin{cases} 1 & \text{if } r \in [0,n], \\ \frac{n+\frac{1}{2}}{r} & \text{if } r \in [n+1,+\infty). \end{cases}$$

Construct functions

$$b_n(x) = b\big(xS^n(|x|)\big), \tag{3.5}$$

$\sigma_n(x)$ and $F_n(x,y)$ similarly. Then $b_n(x)$, $\sigma_n(x)$ and $F_n(x,y)$ satisfy global Lipschitz and linear growth conditions.

Sketch of the Proof-Continued

Let $X_n^{\epsilon,x}(t)$ be the solution associated with functions $b_n(x)$, $\sigma_n(x)$ and $F_n(x,y)$. Then $X^{\epsilon,x}(t) = X_n^{\epsilon,x}(t)$ for $t \leq \tau_n^{\epsilon,x}$. For large n, $\sup_{x\in K} \mathbb{P}(\tau_n^{\epsilon,x} \leq T)$ is sufficiently small & $\inf_{x\in K} \tau_n^{0,x} > T$. Therefore,

$$
\begin{aligned}
&\sup_{x\in K} \mathbb{P}\{|X^{\epsilon,x}(T) - X^{0,x}(T)| \geq \delta\} \\
\leq &\sup_{x\in K} \mathbb{P}\{|X_n^{\epsilon,x}(T) - X_n^{0,x}(T)| \geq \delta, T < \tau_n^{\epsilon,x} \wedge \tau_n^{0,x}\} \\
&+ \sup_{x\in K} \mathbb{P}(\tau_n^{\epsilon,x} \wedge \tau_n^{0,x} \leq T) \\
\leq &\frac{1}{\delta^2} \sup_{x\in K} \mathbb{E}|X_n^{\epsilon,x}(T) - X_n^{0,x}(T)|^2 + \sup_{x\in K} \mathbb{P}(\tau_n^{\epsilon,x} \leq T) \to 0, \text{ as } \epsilon \to 0.
\end{aligned}
$$

SODEs Driven by a Lévy Process

Lemma (The existence and tightness of stationary measure)

Suppose that $b(x)$, $\sigma(x)$ and $F(x,y)$ in (3.1) are locally Lipschitz continuous, and that there exists a scalar function $V(x) \in C^2(\mathbb{R}^m, \mathbb{R}_+)$ such that (3.2) and (3.3) hold. Then $\mathcal{I}^\epsilon$ is nonempty, and the set $\mathcal{I} = \bigcup_{0<\epsilon\leq\epsilon_0} \mathcal{I}^\epsilon$ of stationary measures is tight.

The idea of the proof is borrowed from Khasminskii to the prove the existence of stationary measures. Using Portmanteau Theorem to prove the tightness.

Nonlinear Case: Polynomial Growth

Corollary

Suppose that $b(x)$, $\sigma(x)$ and $F(x,y)$ in (3.1) are locally Lipschitz continuous. If there are positive constants c_1, c_2 and $q \geq 2$ such that for $|x|$ sufficiently large, one has

$$\langle b(x), x\rangle \leq -c_1|x|^q,$$

$$\frac{1}{2}\|\sigma(x)\|_2^2 + \int_{|y|_{\mathbb{R}^l}<c} |F(x,y))|^2\nu(dy) \leq c_2|x|^q,$$

then the conclusions of **Theorem 3** *hold.*

Proof Note that $V(x) = \frac{1}{2}|x|^2$.

Nonlinear Case: Monotone Cyclic Feedback Systems

Consider a typical monotone cyclic feedback system driven by Lévy process

$$
\begin{aligned}
dx^i(t) =& \big[- b_i x^i(t) + f^i(x^{i+1}(t)) \big] dt \\
&+ \epsilon \sum_{j=1}^{k} \sigma_{ij}(x(t)) dW_j(t) + \epsilon \int_{|y|_{\mathbb{R}^l} < c} F^i(x(t-), y) \tilde{N}(dt, dy),
\end{aligned} \tag{3.6}
$$

where b_i is positive constant, f^i is bounded and continuously differentiable with bounded derivative for $i = 0, 1, \cdots, N$, $(N+1) \times k$-dispersion matrix $\sigma(x) = (\sigma_{ij}(x))$ and F have global Lipschitz continuous and linear growth properties.

Nonlinear Case: Monotone Cyclic Feedback Systems

Note Let $V(x) = \frac{1}{2}|x|^2$, it is easy to see that the assumptions of **Theorem 3** hold, therefore $\mathcal{S} \subset B(\Phi)$. Furthermore,

Theorem 7 (Mallet-Paret and Smith, 2002)

Let $\Phi(t)$ be a solution of unperturbed system (3.6) on $[0, \infty)$. Then $B(\Phi) = \mathcal{E} \cup \mathcal{P}$. More precisely, either
(1) $\omega(x)$ is a single non-constant periodic orbit; or
(2) for solutions with $u(t) \in \omega(x)$ for all $t \in \mathbb{R}$, we have that

$$\alpha(u) \cup \omega(u) \subset \mathcal{E}.$$

Here

- $\mathcal{E}$ and $\mathcal{P}$ denote the set of equilibria and nontrivial periodic orbits, respectively.
- $\alpha(x)$ and $\omega(x)$ denote the α- and ω-limit sets of solution in the phase space $\mathbb{R}^{N+1}$, respectively.

FDE: Hopfield Neural Network Models

Consider the stochastic delayed Hopfield equations

$$dX^\epsilon(t) = [-BX^\epsilon(t) + Ag(X^\epsilon(t-\tau))]dt + \epsilon\sigma(X_t^\epsilon)dW(t) \qquad (3.7)$$

where $B = \mathrm{diag}(b_1, \cdots, b_m)$, $g(x) = (g_1(x_1), \cdots, g_m(x_m))^T$, $A = (a_{ij})_{m\times m}$, and $\sigma(\phi) = (\sigma_{ij}(\phi))$ is an $m \times m$ matrix defined on $\mathcal{C} = C([-\tau, 0], \mathbb{R}^m)$.

Assumptions on g and σ:

(A$_1$) There exists a positive constant $\widetilde{L}, M$ such that for all $x,\ y \in \mathbb{R}^m$

$$|g(x) - g(y)| \leq \widetilde{L}|x-y|,\ |g(x)| \leq M.$$

(A$_2$) There exists a positive constant L such that for all $\phi, \psi \in \mathcal{C}$

$$\|\sigma(\phi) - \sigma(\psi)\|_2 \leq L\|\phi - \psi\|.$$

(A$_3$) The diffusion matrix $\sigma\sigma^T$ is *uniformly elliptic* in $\mathcal{C}$

FDE: Hopfield Neural Network Models

Theorem 8

Assume $(\mathrm{A}_1)-(\mathrm{A}_3)$. *Let* $\gamma = 9[\frac{2\sqrt{\kappa}-1}{(\sqrt{\kappa}-1)^2}+1]$. *If*

$$\min_{1\le i\le m} b_i > \frac{\gamma^2 e^{6\tau}\left(16\widetilde{L}^3|A|^3\right)^2}{(1-\kappa e^{-3\tau})^2}, \kappa \in (1, e^{3\tau}), \tag{3.8}$$

then
(1) for each $\epsilon \in (0,1]$, *the system* (3.7) *has a unique invariant measure* μ^ϵ *for the segment process* $\{X_t^\epsilon\}_{t\ge 0}$.
(2) μ^ϵ *weakly converges to* δ_p *as* $\epsilon \to 0$, *where* p *is a globally asymptotically stable equilibrium for differential equations* (3.7) *with* $\epsilon = 0$.

Stochastic Reaction Diffusion Equation with a Polynomial Nonlinearity

Let $\Lambda \subset \mathbb{R}$ be a bounded domain with smooth boundary $\partial\Lambda$,

$$g(u) = \sum_{i=0}^{2k-1} a_i u^i, \quad a_{2k-1} < 0. \tag{3.9}$$

Stochastic reaction diffusion equation with Dirichlet boundary conditions:

$$\begin{cases} dX(t,x) = \Delta X(t,x)dt + g(X(t,x))dt + \epsilon\sigma(x, X(t,x))dW(t), \\ X(t,x) = 0, \ x \in \partial\Lambda, \ t > 0, \\ X(0) = h \in L^2(\Lambda). \end{cases} \tag{3.10}$$

Here $\sigma : \Lambda \times \mathbb{R} \to l^2$ is measurable function.

Stochastic Reaction Diffusion Equation with a Polynomial Nonlinearity-Continued

(C) There exist $c_1, c_2 > 0$ and $h \in L^1(\Lambda)$ such that for all $u, u' \in \mathbb{R}$ and $x \in \Lambda$,

$$\begin{aligned} \|\sigma(x,u) - \sigma(x,u')\|_{l^2}^2 &\leq c_1 |u - u'|^2, \\ \|\sigma(x,u)\|_{l^2}^2 &\leq c_2 |u|^2 + h(x). \end{aligned}$$

Theorem 9

Assume that g is given in (3.9) and σ satisfies condition (C), then any limiting measures of stationary measures for (3.10) are supported in the set of equilibrium points $\mathcal{E}$ of (3.10) with $\epsilon = 0$.

Important facts: Deeply Depends on the Type of Noise

Suppose that $X^{0,x_0}(t)$ is a bounded solution of $\dot{x} = b(x)$.

Denote by $\mathcal{I}_{x_0}$ the set of invariant measures generated by the family of probability measures

$$P^{0,t}(x_0, B) = \frac{1}{t}\int_0^t \delta_{X^{0,x_0}(s)}(B)ds$$

via Krylov-Bogoliubov procedure.

Choose $r = r(x_0) > 0$ such that $\overline{\gamma^+(x_0)} = \overline{\{X^{0,x_0}(t) : t \geq 0\}} \subset B_r(O)$. Then construct a C^∞ diffusion term σ:

$$\sigma(x) = \begin{cases} \text{zero matrix } 0 & \text{if } x \in B_r(O), \\ \text{constant matrix } J & \text{if } x \in (\bar{B}_{r+1}(O))^c. \end{cases}$$

Important facts: Deeply Depends on the Type of Noise-Continued

Then for SDEs

$$dX^{\epsilon,x_0}(t) = b(X^{\epsilon,x_0}(t))dt + \epsilon\sigma(X^{\epsilon,x_0}(t))dW_t, \quad X^{\epsilon,x_0}(0) = x_0 \qquad (4.1)$$

has a unique solution $X^{\epsilon,x_0} = X^{0,x_0}$.

Furthermore, we have

Facts

$X^{\epsilon,x_0}(t) = X^{0,x_0}(t)$ for all $t \geq 0$ and $\mathcal{I}_{x_0} \subset \mathcal{I}$ for all ϵ. In particular, for any $\mu^{\epsilon,x_0} = \mu \in \mathcal{I}_{x_0}$, one has $\mu^{\epsilon,x_0} \xrightarrow{w} \mu$ as $\epsilon \to 0$.

Important facts: Deeply Depends on the Type of Noise-Continued

Theorem 10 (Uniquely ergodic)

Suppose the assumptions of Theorem 3 are satisfied. If the matrix

$$a(x) := \sigma(x)\sigma^T(x) \quad \text{is invertible,} \tag{4.2}$$

then the semigroup P_t is irreducible. Furthermore, if

(a) *for any $n > 1$, there exists a nonnegative function $c_n \in L^2(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l), \nu)$ such that $\sup_{|x|\le n} |F(x,y)| \le c_n(y), \quad y \in \mathbb{R}^l$,*

(b) *there exist positive constants C and C_r for any $r > 0$ such that*

$$\int_{|y|_{\mathbb{R}^l}<c} \|D_x F(0,y)\|_2^2 \nu(dy) \le C,$$

$$\int_{|y|_{\mathbb{R}^l}<c} \|D_x F(x_1,y) - D_x F(x_2,y)\|_2^2 \nu(dy) \le C_r |x_1 - x_2|^2, \ |x_1| \vee |x_2| \le r,$$

then the semigroup P_t has the strong Feller property.

Example 1: The Bernoulli Lemniscate System

Example (The Bernoulli Lemniscate System)

Let $I(x,y) = (x^2+y^2)^2 - 4(x^2-y^2)$. Define

$$V(I) := \frac{I^2}{2(1+I^2)^{\frac{3}{4}}}, \quad H(I) := \frac{I}{(1+I^2)^{\frac{3}{8}}}.$$

Consider the vector field

$$b(x,y) := -\left[\nabla V(I) + \left(\frac{\partial H(I)}{\partial y}, -\frac{\partial H(I)}{\partial x}\right)^T\right].$$

For the deterministic system

$$\begin{cases} \dfrac{dx}{dt} = b_1(x,y), \\ \dfrac{dy}{dt} = b_2(x,y). \end{cases} \tag{4.3}$$

The global behavior of (4.3) are sketched in following Figure.

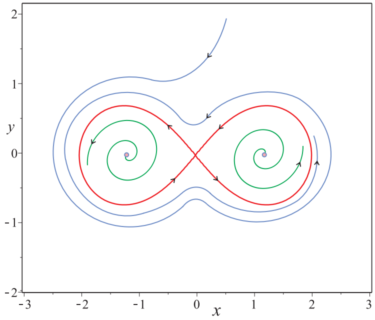


Figure: The phase portrait of (4.3) with $b(x, y) = -\nabla V(x, y) - \Theta(x, y)$.

The Birkhoff center $B(\Phi) = \{O(0, 0), P^+(\sqrt{2}, 0), P^-(-\sqrt{2}, 0)\}$, where O is a saddle point, P^+, P^- are unstable spirals.

Proposition

The system (4.3) *has the equilibria* $O(0,0)$, $P^+(\sqrt{2},0)$ *and* $P^-(-\sqrt{2},0)$. $V(I)$ *is its Lyapunov function.*

(1) The Birkhoff center $B(\Phi) = \{O, P^+, P^-\}$.

(2) When the initial point p *locates outside of the Bernoulli Lemniscate:*

$$L: \quad (x^2+y^2)^2 = 4(x^2-y^2), \tag{4.4}$$

its ω*-limit set* $\omega(p) = L$, *which is a red curve in Figure;*

(3) When the initial point $p \neq P^-$ *(resp.* $p \neq P^+$ *) locates left (resp. right) inside of the Bernoulli Lemniscate, its* ω*-limit set the left (resp. right) branch of* L.

For the perturbed system driven by Brownian motion:

$$\begin{cases} dx = b_1(x,y)dt + \epsilon[\sigma_{11}(x,y)dW_t^1 + \sigma_{12}(x,y)dW_t^2], \\ dy = b_2(x,y)dt + \epsilon[\sigma_{21}(x,y)dW_t^1 + \sigma_{22}(x,y)dW_t^2]. \end{cases} \tag{4.5}$$

Theorem 12

Suppose that $\sigma_{ij}(i,j \in \{1,2\})$ *satisfies global Lipschtiz condition, thus there exist nonnegative constants* C_1, C_2 *such that*

$$|a_{ij}(x,y)| = |\left(\sigma(x,y)\sigma^T(x,y)\right)_{ij}| \leq C_1|(x,y)|^2 + C_2, \ \textit{for } i,j = 1,2,$$

(i) *if* $C_1 = 0$, *then for any* ϵ, *the system* (4.5) *admits at least one stationary measure* μ^ϵ;

(ii) *if* $C_1 > 0$, *then the system* (4.5) *possesses at least one stationary measure* μ^ϵ *for* $0 < \epsilon < \frac{1}{8\sqrt{26C_1}}$.

If, in addition, the diffusion matrix $a(x,y)$ *is positively definite everywhere, then for a given* ϵ *as above, the stationary measure* μ^ϵ *is unique, and* $\mu^\epsilon \xrightarrow{w} \delta_O(\cdot)$ *as* $\epsilon \to 0$, *where* O *is a saddle point.*

Sketch of the Proof

1. The conclusions of Theorem 3 hold because of the following fact

$$\mathcal{L}^\epsilon V(x,y) \leq -[\frac{1}{4\sqrt{2}} - 208\sqrt{2}C_1\epsilon^2]r^2 + 208\sqrt{2}C_2\epsilon^2 \to -\infty \text{ as } r \to \infty.$$

   This implies (i), (ii) and $\mathrm{supp}(\mu) \subset \{O, P^+, P^-\}$, where μ is the weak limit measure.

2. Also the uniqueness of μ^ϵ follows from Theorem 10.

3. To show $\mu(\{P^+, P^-\}) = 0$.
   By the measure estimate theorem [Huang-Ji-Liu-Yi], it is easy to see that

$$\mu^\epsilon(\Omega_{\rho_0}) \leq \mu^\epsilon(\Omega_\rho) e^{-\frac{\widetilde{m}}{\widetilde{M}}\int_{\rho_0}^{\rho}\frac{1}{t}dt} \leq e^{-\frac{\widetilde{m}}{\widetilde{M}}\int_{\rho_0}^{\rho}\frac{1}{t}dt}, \quad \rho \in (\rho_0, \rho_M).$$

   Then, using the fact of $\mu^\epsilon \xrightarrow{w} \mu$ and the openness of Ω_{ρ_0}. Finally, letting $\rho_0 \to 0$, one has $\mu(\{P^+\}) = 0$.

Example 2: The May-Leonard System

Example (The May-Leonard system)

Consider the May-Leonard system with a white noise perturbation:

$$\begin{cases} dy_1 = y_1(1 - y_1 - \beta y_2 - \gamma y_3)dt + \epsilon y_1 \circ dW_t, \\ dy_2 = y_2(1 - y_2 - \beta y_3 - \gamma y_1)dt + \epsilon y_2 \circ dW_t, \\ dy_3 = y_3(1 - y_3 - \beta y_1 - \gamma y_2)dt + \epsilon y_3 \circ dW_t, \end{cases} \tag{4.6}$$

where $\circ$ denotes the Stratonovich stochastic integral, $\beta,\ \gamma > 0$ and ϵ denotes noise intensity.

Stochastic Decomposition Formula

Theorem 14 (Chen, Dong, Jiang, Niu and Zhai)

Let Φ^ϵ, Φ^0 be the solutions of (4.6) and the corresponding deterministic system without noise, respectively. Then

$$\Phi^\epsilon(t,\omega,y) = g^\epsilon(t,\omega,g_0)\Phi^0\Big(\int_0^t g^\epsilon(s,\omega,g_0)ds, \frac{y}{g_0}\Big), \tag{4.7}$$

where g^ϵ is the solution of stochastic logistic equation

$$dg = g(1-g)dt + \epsilon g \circ dW_t. \tag{4.8}$$

Asymptotic Properties for Φ^0

- M. Hirsch (1988) pointed out that the flow Φ^0 admits an invariant surface Σ (called *carrying simplex*), such that every trajectory in $\mathbb{R}^3_+ \setminus \{O\}$ is asymptotic to one in Σ. So we draw phase portraits on Σ. Here carrying simplex Σ is homeomorphic to $\{y \in \mathbb{R}^3_+ : \sum_i y_i = 1\}$.

- Φ^0 always possesses

(1) *equilibria*: $O(0,0,0)$;

(2) *three axial equilibria*: $R_1(1,0,0)$, $R_2(0,1,0)$, $R_3(0,0,1)$;

(3) *the unique positive equilibrium*: $P = \frac{1}{1+\beta+\gamma}(1,1,1)$;

(4) *possible planar equilibria*: $R_{12} = \frac{1}{1-\beta\gamma}(1-\beta, 1-\gamma, 0)$, $R_{23} = \frac{1}{1-\beta\gamma}(0, 1-\beta, 1-\gamma)$, $R_{31} = \frac{1}{1-\beta\gamma}(1-\gamma, 0, 1-\beta)$.

Table: The classification for the flow Φ^0 on Σ

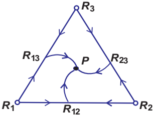
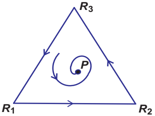
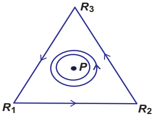
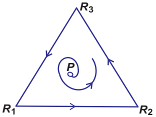
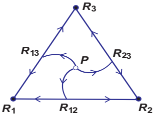
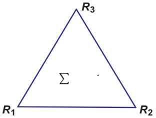
Parameter conditions	Equilibria	Phase Portrait
a: $0 < \beta, \gamma < 1$	$O, R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, P$	
b: (i) $\beta + \gamma < 2$ (ii) $\beta \geq 1, \gamma < 1$ or $\gamma \geq 1, \beta < 1$	O, R_1, R_2, R_3, P	
c: (i) $\beta + \gamma = 2$ (ii) $\beta, \gamma \neq 1$	O, R_1, R_2, R_3, P	

Table: continued

Parameter conditions	Equilibria	Phase Portrait
d: (i) $\beta+\gamma>2$ (ii) $\gamma>1, \beta\leq 1$ or $\gamma\leq 1, \beta>1$	O, R_1, R_2, R_3, P	
e: $\beta, \gamma > 1$	$O, R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, P$	
f: $\beta=\gamma=1$	$\forall x \in \Sigma \cup \{O\}$	

Complete Depiction

Theorem 15 (Chen, Dong, Jiang, Niu and Zhai)

(i) *For any equilibrium* $Q \in \mathcal{E}$, $\mu^{\epsilon}_{Q}(\cdot) \xrightarrow{w} \delta_{Q}(\cdot)$ *as* $\epsilon \to 0$, *which is valid to the cases* a, b, e, f.

(ii) *For the case* c, ν^{ϵ}_{h} *converges weakly to the Haar measure on the closed orbit* $\Gamma(h)$ *as* $\epsilon \to 0, 0 < h \le \frac{1}{27}$.

(iii) *For the case* d, *if* $\mu^{i} := \nu^{\epsilon^{i}}_{y} \in \mathcal{M}_{S}(\epsilon_0)$, $i = 1, 2, \cdots$, *satisfying* $\epsilon^{i} \to 0$ *and* $\mu^{i} \xrightarrow{w} \mu$ *as* $i \to \infty$, *then* $\mu(\{R_1, R_2, R_3\}) = 1$.

Conlusions

♠ The Bernoulli Lemniscate Example shows that the Birkhoff center consists of the origin O (saddle point) and strongly unstable spirals $\{P^+, P^-\}$. Relatively, the O is more stable than $\{P^+, P^-\}$.

All weak limits support at the O under the diffusion matrix is nondegenerate.

• The most unstable positions for the deterministic system may be recognised by the added small stochastic term.

♠ The May-Leonard Example in case d indicates that the Birkhoff center of Φ^0 on $\Sigma = \{y \in \mathbb{R}^3_+ : \sum_i y_i = 1\}$ is composed of P (which is strongly repelling on Σ) and three saddles $\{R_1, R_2, R_3\}$(which are relatively more stable than P).

The weak limits are more complex, but keep the original deterministic classification.

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Thank you!