Explicit Convergence Rates for Subgeometric Ergodic Markov Processes under Subordination

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Background and motivation

- Let X_t be a Markov process with state space E, transition function $P^t(x,\cdot)$, and stationary distribution π.
- Quantitative convergence rate:

$$\|P^t(x,\cdot)-\pi\|_{\mathrm{TV}} \le C(x)r(t), \quad x \in E, t \ge 0,$$

where $r:[0,\infty)\to(0,1]$ is the nondecreasing rate function.

- Typical examples for the rate r are $(\theta>0, \delta\in(0,1], \beta,\gamma>0)$

$$r(t)=\mathrm{e}^{-\theta t^\delta}, \quad r(t)=(1+t)^{-\beta}, \quad r(t)=[1+\log(1+t)]^{-\gamma}.$$

- Example: the following SDE admits such rates

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}Z_t,$$

where Z_t is an α-stable Lévy process and b satisfies some conditions. **Tool: Foster–Lyapunov criterion**

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- $$\|P^t(x,\cdot)-\pi\|_{\mathrm{TV}} \le C(x)r(t), \quad x\in E, t\ge 0.$$

- $X_t \rightsquigarrow X_{S_t}$: transition function $P^t_\phi(x,\cdot)$; S_t is an independent subordinator with Laplace exponent ϕ (introduced in detail later).

- Qualitative: $P^t_\phi(x,\cdot) \to \pi$ as $t\to\infty$

- **Aim:** describe r_ϕ via the rate function r of the original process:

$$\left\|P^t_\phi(x,\cdot)-\pi\right\|_{\mathrm{TV}} \le C(x)r_\phi(t), \quad x\in E,\ t>0.$$

- Explicit r_ϕ for

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Similar question: rates under discrete subordination

- X_n is a discrete time Markov chain with invariant measure π,

$$\|P^n(x,\cdot) - \pi\|_{\mathrm{TV}} \le C(x)r(n), \quad x \in E, n \in \mathbb{N}.$$

- $X_n \rightsquigarrow X_{T_n}$: transition kernel $P_\phi^n(x,\cdot)$; $\quad T_n$ is an independent discrete subordinator with Laplace exponent ϕ in in the sense of Bendikov and Saloff-Coste (Math. Nachr., 2012).

- **Question:** what can we say about r_ϕ?

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f-norm

- For generality, we replace the total variance norm by the so-called f-norm.
- Let $f : E \to [1, \infty)$. The f-norm of a signed measure μ is defined by

$$\|\mu\|_f := \sup_{|g| \le f} \left| \int_E g \, \mathrm{d}\mu \right|.$$

- Clearly, $\| \cdot \|_f \ge \| \cdot \|_{\mathrm{TV}}$.
- If f is bounded, $\| \cdot \|_f$ and $\| \cdot \|_{\mathrm{TV}}$ are even equivalent.

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Continuous time subordinator

- A subordinator S_t is an increasing Lévy process on $[0, \infty)$ with Laplace transform

$$\mathbb{E}\, \mathrm{e}^{-uS_t} = \mathrm{e}^{-t\phi(u)}, \quad u > 0,\ t \geq 0.$$

- $\phi : (0, \infty) \to (0, \infty)$ is a Bernstein function, i.e. $\phi \in C^\infty$ and $(-1)^{n-1}\phi^{(n)} \geq 0$ for all $n \in \mathbb{N}$.

- Every Bernstein function enjoys a unique (Lévy–Khintchine) representation

$$\phi(u) = bu + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-uy}\right) \nu(\mathrm{d}y), \quad u > 0.$$

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Bochner's subordination

- Assume that X_t and S_t are independent.
- If X_t is a Lévy process, then so does the subordinate process X_{S_t}.
- Example: α-stable process B_{S_t}, where B_t is a standard Brownian motion and S_t is an independent $\alpha/2$-stable subordinator.
- Generator: $A \rightsquigarrow -\phi(-A)$
- By independence, $P_\phi^t(x,\cdot)$ is given by

$$P_\phi^t(x,\cdot) = \int_{[0,\infty)} P^s(x,\cdot)\,\mathbb{P}(S_t \in \mathrm{d}s).$$

- A natural question: which (fine) properties can be preserved under subordination?

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Known results

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  Z.-Q. Chen-R. Song (2005, JFA), (2006, Math. Z.)
- Heat kernel estimates and potential theory for subordinate BM:
  Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček
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Our framework

- $$\left\|P^t(x,\cdot)-\pi\right\|_f \le C(x)r(t), \quad x\in E,\ t\ge 0.$$

- We focus on the following (sub-geometric) rates:

$$r(t)=\mathrm{e}^{-\theta t^{\delta}}, \quad r(t)=(1+t)^{-\beta}, \quad r(t)=[1+\log(1+t)]^{-\gamma},$$

where $\theta>0$, $\delta\in(0,1]$ and $\beta,\gamma>0$.

- **Our aim:** For such rates, determine r_ϕ such that

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- For simplicity, we only state our result for the special case $\phi(u)=u^\alpha$, $\alpha\in(0,1)$.

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Main result (in the special case $\phi(u) = u^\alpha$)

$$\left\|P^t(x,\cdot)-\pi\right\|_f \le C(x)r(t), \quad x\in E,\ t\ge 0. \qquad (\bigstar)$$

$$\left\|P^t_\phi(x,\cdot)-\pi\right\|_f \le C(x)r_\phi(t), \quad x\in E,\ t> 0. \qquad (\bigstar\bigstar)$$

Theorem (D.-Schilling-Song, Adv. Appl. Probab., 2017) (sub-exponential rate)

(1) If $(\bigstar)$ holds with rate $r(t) = \mathrm{e}^{-\theta t^\delta}$ for $\theta > 0$ and $\delta \in (0,1]$, then so does $(\bigstar\bigstar)$ with rate

$$r_\phi(t) = \exp\left[-C\, t^{\frac{\delta}{\alpha(1-\delta)+\delta}}\right],$$

where $C = C(\theta,\delta,\alpha) > 0$.

Main result (in the special case $\phi(u) = u^\alpha$)

$$\left\|P^t(x,\cdot) - \pi\right\|_f \le C(x) r(t), \quad x \in E,\ t \ge 0. \qquad (\bigstar)$$

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$$\boldsymbol{r_\phi(t) = \exp\left[-C\, t^{\frac{\delta}{\alpha(1-\delta)+\delta}}\right]},$$

where $C = C(\theta, \delta, \alpha) > 0$.

$$\left\| P^t(x,\cdot) - \pi \right\|_f \le C(x) r(t), \quad x \in E,\ t \ge 0. \qquad (\bigstar)$$

$$\left\| P^t_\phi(x,\cdot) - \pi \right\|_f \le C(x) r_\phi(t), \quad x \in E,\ t > 0. \qquad (\bigstar\bigstar)$$

Theorem (algebraic rate)

(2) If $(\bigstar)$ holds with rate $\boldsymbol{r(t) = (1+t)^{-\beta}}$ for $\beta > 0$, then so does $(\bigstar\bigstar)$ with rate

$$\boldsymbol{r_\phi(t) = (1+t)^{-\beta/\alpha}}.$$

Main result (cont.)

$$\left\|P^t(x,\cdot)-\pi\right\|_f \le C(x)r(t), \quad x\in E,\ t\ge 0. \qquad (\bigstar)$$

$$\left\|P^t_\phi(x,\cdot)-\pi\right\|_f \le C(x)r_\phi(t), \quad x\in E,\ t>0. \qquad (\bigstar\bigstar)$$

Theorem (logarithmic rate)

(3) If ($\bigstar$) holds with rate $r(t) = [1+\log(1+t)]^{-\gamma}$ for $\gamma>0$, then so does ($\bigstar\bigstar$) with rate

$$r_\phi(t) = [1+\log(1+t)]^{-\gamma}.$$

Compare the rates (for $t \gg 1$)

original process X_t	subordinate process X_{S_t}
e^{-t}	e^{-t}
$e^{-t^{\delta}}$	$e^{-t^{\frac{\delta}{\alpha(1-\delta)+\delta}}}$
$t^{-\beta}$	$t^{-\beta/\alpha}$
$\log^{-\gamma}(1+t)$	$\log^{-\gamma}(1+t)$

A general lemma

$$\left\|P^t(x,\cdot)-\pi\right\|_f \le C(x)r(t), \quad x\in E,\ t\ge 0. \qquad (\bigstar)$$

$$\left\|P^t_\phi(x,\cdot)-\pi\right\|_f \le C(x)r_\phi(t), \quad x\in E,\ t>0. \qquad (\bigstar\bigstar)$$

Lemma

If $(\bigstar)$ holds with some rate function r, then so does $(\bigstar\bigstar)$ with rate function $r_\phi(t)=\mathbb{E}\,r(S_t)$.

Proof:

$$\left\|P^t_\phi(x,\cdot)-\pi\right\|_f = \left\|\int_{[0,\infty)}\left(P^s(x,\cdot)-\pi\right)\mu_t(\mathrm{d}s)\right\|_f$$

$$\le \int_{[0,\infty)}\left\|P^s(x,\cdot)-\pi\right\|_f\,\mu_t(\mathrm{d}s) \le C(x)\int_{[0,\infty)} r(s)\,\mu_t(\mathrm{d}s) = C(x)\mathbb{E}\,r(S_t).$$

A general lemma

$$\left\|P^t(x,\cdot)-\pi\right\|_f \le C(x)r(t), \quad x\in E,\ t\ge 0. \qquad (\bigstar)$$

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Our task

- Recall typical examples for the rate r of the original process are

$$r(t) = \mathrm{e}^{-\theta t^{\delta}},\ \ r(t) = (1+t)^{-\beta},\ \ r(t) = [1+\log(1+t)]^{-\gamma}$$

for $\theta > 0$, $\delta \in (0,1]$ and $\beta, \gamma > 0$.

- To get explicit rates for the subordinate process, the crucial point is to bound

$$\mathbb{E}\,\mathrm{e}^{-\theta S_t^{\delta}}, \qquad \mathbb{E} S_t^{-\beta}, \qquad \mathbb{E}\log^{-\gamma}(1+S_t)$$

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- Byproduct: moment estimates for general subordinator

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Theorem

If $\nu(\mathrm{d}y) \ge c\, y^{-1-\alpha}\, \mathrm{d}y$ for $c > 0, \alpha \in (0,1)$, then for some $C = C(\theta, \delta, c, \alpha) > 0$

$$\mathbb{E}\, \mathrm{e}^{-\theta S_t^\delta} \le \exp\left[-C\, t^{\frac{\delta}{\alpha(1-\delta)+\delta}}\right] \quad \text{for all } t \gg 1.$$

Algebraic moment: $\mathbb{E}S_t^{-\beta}$, where $\beta > 0$

Theorem

(1) We always have

$$\mathbb{E}S_t^{-\beta} \geq \frac{1}{\mathrm{e}\beta\Gamma(\beta)}\left[\phi^{-1}\left(\frac{1}{t}\right)\right]^{\beta} \quad \text{for all } t > 0.$$

(2) If

$$\liminf_{u\to\infty} \frac{\phi(\lambda u)}{\phi(u)} > 1 \quad \text{for some (hence, all) } \lambda > 1,$$

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then for some $C = C(\beta) > 0$

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Typical examples for Bernstein function ϕ satisfying $(\spadesuit)$ are

- $\phi(u) = \log(1+u)$;
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Discrete time subordinator (Bendikov/Saloff-Coste, 2012)

- ϕ is a Bernstein function of the form

$$\phi(u) = \int_{(0,\infty)} \left(1 - \mathrm{e}^{-uy}\right) \nu(\mathrm{d}y), \quad u > 0$$

  such that $\phi(1) = 1$.

- Set

$$c(\phi, m) = \frac{1}{m!} \int_{(0,\infty)} y^m \mathrm{e}^{-y} \nu(\mathrm{d}y), \quad m \in \mathbb{N}.$$

- Since

$$\sum_{m=1}^{\infty} c(\phi, m) = \int_{(0,\infty)} \left(1 - \mathrm{e}^{-y}\right) \nu(\mathrm{d}y) = \phi(1) = 1,$$

  we know that $\{c(\phi, m) : m \in \mathbb{N}\}$ is a probab. measure on $\mathbb{N}$.

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$$T_n := \sum_{k=1}^{n} R_k,$$

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rates under discrete subordination

- X_n is a discrete time Markov chain with invariant measure π,

$$\|P^n(x,\cdot)-\pi\|_f \le C(x)r(n), \quad x\in E, n\in \mathbb{N}.$$

- $X_n \rightsquigarrow X_{T_n}$: transition kernel $P_\phi^n(x,\cdot)$; $\quad T_n$ is an independent discrete subordinator with Laplace exponent ϕ.

- Generator: $I-P \rightsquigarrow \phi(I-P)$.

- **Aim:** If the rate functions are

$$r(n)=\mathrm{e}^{-\theta n^\delta}, \quad r(n)=n^{-\beta}, \quad r(n)=\log^{-\gamma}(2+n),$$

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Main result (in the special case $\phi(u) = u^\alpha$)

Theorem (D., arXiv:170605533)

Main results are collected in the following table:

original chain X_n	subordinate chain X_{T_n}
$\mathrm{e}^{-n^\delta} \quad (0 < \delta \le 1)$	$\mathrm{e}^{-n^{\frac{\delta}{\alpha(1-\delta)+\delta}}}$
$n^{-\beta} \quad (\beta > 0)$	$n^{-\beta/\alpha}$
$\log^{-\gamma}(2+n) \quad (\gamma > 0)$	$\log^{-\gamma}(2+n)$

Sketch of the proof

- As in the time-continuous case, we need to bound

$$\mathbb{E}\, \mathrm{e}^{-\theta T_n^\delta}, \qquad \mathbb{E}\, T_n^{-\beta}, \qquad \mathbb{E} \log^{-\gamma}(1 + T_n)$$

  as $n \to \infty$.

- To this aim, we need the technique from the theory of completely monotone functions.
- A function $g : (0, \infty) \to \mathbb{R}$ is called a completely monotone function if $g \in C^\infty$ and $(-1)^n g^{(n)} \geq 0$ for all $n = 0, 1, 2, \cdots$
- **Bernstein theorem:** g is a completely monotone function iff there exists a unique measure μ on $[0, \infty)$ s.t.

$$g(x) = \int_{[0,\infty)} \mathrm{e}^{-xt} \mu(\mathrm{d}t).$$

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A crucial lemma

Lemma (D., arXiv:170605533)

Let T_n be a discrete time subordinator with Bernstein function ϕ, and S_t be a continuous time subordinator with the same Bernstein function ϕ. If g is a completely monotone function, then

$$\mathbb{E}g(T_n) \le \mathbb{E}g(S_n).$$

Since the functions

$$x \mapsto \mathrm{e}^{-x^\delta}, \quad x \mapsto x^{-\beta}, \quad x \mapsto \log^{-\gamma}(1+x)$$

are completely monotone functions, this allows us to bound

$$\mathbb{E}\,\mathbf{e}^{-T_n^\delta}, \qquad \mathbb{E}\,T_n^{-\beta}, \qquad \mathbb{E}\log^{-\gamma}(1+T_n)$$

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$$x \mapsto \mathrm{e}^{-x^\delta}, \quad x \mapsto x^{-\beta}, \quad x \mapsto \log^{-\gamma}(1+x)$$

are completely monotone functions, this allows us to bound

$$\mathbb{E}\,\mathrm{e}^{-T_n^\delta}, \qquad \mathbb{E}\,T_n^{-\beta}, \qquad \mathbb{E}\log^{-\gamma}(1+T_n)$$

by the corresponding estimates for continuous time subordinator S_t.

Thanks for Your Attention!