Time fractional equations and probabilistic representation

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Z.-Q. Chen, Time-fractional equations and probabilistic representation. *Chaos, Solitons and Fractals,* **102C** (2017) pp. 168-174.

Generator

Given a Markov process $(X, \mathbb{P}_x, x \in E)$ on E, its transition semigroup P_t is given by

$$P_t f(x) = \mathbb{E}_x[f(X_t)].$$

The infinitesimal generator $\mathcal{L}$ of X is

$$\mathcal{L}f(x) = \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}.$$

Hence $u(t,x) = P_t f(x)$ solves $\frac{\partial u}{\partial t} = \mathcal{L}u$ with $u(0,x) = f(x)$.

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Subordinate Markov Process

Suppose S_t is a subordinator independent of X with Laplace exponent ϕ:

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}.$$

There is a unique $\kappa \geq 0$ and a measure $\mu(dx)$ with $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$ so that

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})\mu(dx).$$

X_{S_t} is a Markov process, called subordinate Markov process. When X is symmetric, the infinitesimal generator of X_{S_t} is $\mathcal{L}_\phi := -\phi(-\mathcal{L})$.

Hence $u(t,x) := \mathbb{E}_x[f(X_{S_t})]$ solves $\frac{\partial u}{\partial t} = \mathcal{L}_\phi u$ with $u(0,x) = f(x)$.

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Inverse subordinator

Question: Let $E_t := \inf\{s > 0 : S_s > t\}$, inverse subordinator. Which equation does $v(t,x) := \mathbb{E}_x[f(X_{E_t})]$ solves?

Answer (the goal of this talk): Assume $\mu(0,\infty) = \infty$.

$$(\kappa\partial_t + \partial_t^w)\, v(t,x) = \mathcal{L}u(t,x) \quad \text{with } v(0,x) = f(x)$$

where

$$\partial_t^w g(t) := \frac{d}{dt}\int_0^t w(t-s)(g(s)-g(0))ds$$

with $w(r) := \mu([r,\infty))$.

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Caputo fractional derivative

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \left(f(s) - f(0) \right) ds,$$

where Γ is the Gamma function defined by
$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t} dt$.

A little physics: Let $u(t,x)$, $e(t,x)$ and $\vec{F}(t,x)$ denote the body temperature, internal energy and flux density, respectively. Then the relations

$$e(t,x) = \kappa u(t,x), \quad \vec{F}(t,x) = -\lambda \nabla u(t,x), \qquad \kappa, \lambda > 0,$$

$$\frac{\partial e}{\partial t}(t,x) = -\mathrm{div}\, \vec{F} \quad \text{(conservation law)}$$

yield the classical heat equation $\kappa \frac{\partial u}{\partial t} = \lambda \Delta u$.

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Material of thermal memory

However in real modeling, heat flow can be disrupted by the response of the material. It has been shown (e.g. in Lunardi and Sinestrari (1988), von Wolfersdorf (1994)) that in a material with thermal memory, the internal energy

$$e(t,x) = \kappa u(t,x) + \int_0^t n(t-s)u(s,x)ds.$$

Typically, $n(t)$ is a positive decreasing function that blows up near $t = 0$, indicating the nearer past affects the present more. When $n(t) = t^{-\beta}/\Gamma(1-\beta)$ for $\beta \in (0,1)$ and $u(0,x) = 0$, the heat equation becomes

$$\kappa \frac{\partial u}{\partial t} + \frac{\partial^\beta u}{\partial t^\beta} = -\mathrm{div}\vec{F} = \lambda \frac{\partial^2 u}{\partial x^2}.$$

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Subdiffusion

Fractional time equation $\frac{\partial^\beta u}{\partial t^\beta} = a\frac{\partial^2 u}{\partial x^2}$ has a close connection to subdiffusions.

Subdiffusion describes particle moves slower than Brownian motion, for example, due to particle sticking and trapping.

Example: (i) xerox machine, electrons in amorphous media tend to get trapped by local imperfections and then released due to thermal fluctuations.

(ii) hydrology: travel times of contaminants in groundwater are much longer than that of diffusion.

(iii) biology: proteins diffuse across cell membranes.

Time change of Brownian motion

A prototype of subdiffusion can be modeled by Brownian motion time-changed by an inverse stable subordinator.

Let B is Brownian motion in $\mathbb{R}^d$ and S an β-stable subordinator. Define

$$E_t = \inf\{s > 0 : S_s > t\}.$$

Then one can see from random walk approximation with heavy tail holding times that B_{E_t} provides a model for anomalous sub-diffusion, where particles spread slower than Brownian particles.

Inverse subordinator

In general, given a Markov process X_t and an independent β-subordinator S, one can do time change to get a new process $Y_t = X_{E_t}$, where $E_t = \inf\{r \geq 0 : S_r > t\}$.

Question: What is the marginal distribution of Y_t?

Theorem (Baeumer-Meerschaert, 2001; Meerschaert-Scheffler, 2004): $u(t,x) = \mathbb{E}_x[f(X_{S_t})]$ solves

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \mathcal{L}_x u(t,x), \quad u(0,x) = f(x).$$

The self-similarity of the β-subordinator,

$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \quad \text{in distribution,}$$

played an important role in their derivation.

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General time-fractional derivative

In applications and numerical approximations, there is a need to consider more general fractional-time derivatives, for example where its value at time t may depend only on the finite range of the past from $t-\delta$ to t such as

$$\frac{d}{dt}\int_{(t-\delta)^+}^{t}(t-s)^{-\beta}\,(f(s)-f(0))\,ds.$$

Given a decreasing function w on $(0,\infty)$ with $\lim_{x\to\infty} w(x)=0$, define

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Questions

(i) Existence and uniqueness for solution of

$$(\kappa\partial_t + \partial_t^w)\, u = \mathcal{L}u \quad \text{with } u(0,x) = f(x),$$

and its probabilistic representation.

(ii) Given a strong Markov process X and subordinator S, what equation does $u(t,x) = \mathbb{E}_x\left[f(X_{E_t})\right]$ satisfy? Here

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Subordinator

Given a constant $\kappa \geq 0$ and an unbounded right continuous non-increasing function $w(x)$ on $(0, \infty)$ with $\lim_{x\to\infty} w(x) = 0$ and $\int_0^\infty (1 \wedge x)(-dw(x)) < \infty$, there is a unique subordinator $\{S_t; t \geq 0\}$ with Laplace exponent

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})(-dw(x)).$$

Laplace exponent: $\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}$.

Conversely, given a subordinator $\{S_t; t \geq 0\}$, there is a unique constant $\kappa \geq 0$ and a Lévy measure μ on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$ so that its Laplace exponent is given by above the display with $w(x) = \mu(x, \infty)$.

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Framework

From now on, we assume S_t is a subordinator with infinite Lévy measure μ and possible drift $\kappa \geq 0$. Let $\overline{S}_t = S_t - \kappa t$, which is a driftless subordinator with Lévy measure μ. Define $w(x) = \mu(x, \infty)$.

Facts: Since $\mu(0, \infty) = \infty$, $t \mapsto \overline{S}_t$ is strictly increasing. Hence the inverse subordinator E_t and $\overline{E}_t$ are continuous in t.

Suppose that $\{T_t; t \geq 0\}$ is a strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$ with the property that $\sup_{t>0} \|T_t\| < \infty$. Here $\|T_t\|$ denotes the operator norm of the linear map $T_t : \mathbb{B} \to \mathbb{B}$.

Note that by the uniform boundedness principle, $\sup_{t>0} \|T_t\| < \infty$ is equivalent to $\sup_{t>0} \|T_t f\| < \infty$ for every $f \in \mathbb{B}$.

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Main result

Theorem (C. 2017)

For every $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right]$ is the unique solution in $(\mathbb{B}, \|\cdot\|)$ to

$$\left(\kappa\partial_t + \partial_t^w\right) u(t,x) = \mathcal{L}u(t,x) \quad \text{with } u(0,x) = f(x)$$

in the following sense:

(i) $\sup_{t>0} \|u(t,\cdot)\| < \infty$, $x \mapsto u(t,x)$ *is in* $\mathcal{D}(\mathcal{L})$ *for each* $t \geq 0$ *with* $\sup_{t\geq 0} \|\mathcal{L}u(t,\cdot)\| < \infty$, *and both* $t \mapsto u(t,\cdot)$ *and* $t \mapsto \mathcal{L}u(t,\cdot)$ *are continuous in* $(\mathbb{B}, \|\cdot\|)$;

(ii) *for every* $t > 0$, $I_t^w(u) := \int_0^t w(t-s)(u(s,x) - f(x))ds$ *is absolutely convergent in* $(\mathbb{B}, \|\cdot\|)$ *and*

$$\lim_{\delta\to 0} \frac{1}{\delta}\left(\kappa u(t+\delta,\cdot) - \kappa u(t,\cdot) + I_{t+\delta}^w(u) - I_t^w(u)\right) = \mathcal{L}u(t,x)$$

in $(\mathbb{B}, \|\cdot\|)$.

Main result (continued)

Theorem (C. 2017 (continued))

Moreover, when $\kappa > 0$, $t \mapsto u(t, \cdot)$ is globally Lipschitz continuous in $(\mathbb{B}, \|\cdot\|)$, and both $\partial_t u(t, \cdot)$ and $\partial_t^w u(t, \cdot) := \frac{d}{dt} I_t^w(u)$ exists as a continuous function taking values in $(\mathbb{B}, \|\cdot\|)$.

Remarks

(i) The assumption that $f \in \mathcal{D}(\mathcal{L})$ in the Theorem is to ensure that all the integrals involved in the proof are absolutely convergent in the Banach space $\mathbb{B}$. This condition can be relaxed if we formulate the time fractional equation in the weak sense when the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ is symmetric in a Hilbert space $L^2(E; m)$ and so its quadratic form can be used to formulate weak solutions.

(ii) Special cases or related work: Meerschaert and Scheffler (2008) and Kolokoltsov (2011), Toaldo (2015).

(iii) There are very limited results on uniqueness.

(iv) Warning: One needs to be very careful when dealing with time frictional equations. There are many literature that are careless about the absolute convergence of the integral.

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Uniqueness

For every $a > 0$, by Fubini theorem,

$$\int_0^a w(x)dx = \int_0^\infty \left(\int_0^{\xi \wedge a} dx \right) \mu(d\xi) = \int_a^\infty (\xi \wedge a)\mu(d\xi) < \infty.$$

The Laplace transform of w is

$$\int_0^\infty e^{-\lambda x} w(x)dx = \frac{1}{\lambda} \int_0^\infty \left(1 - e^{-\lambda \xi} \right) \mu(d\xi) = \frac{\phi_0(\lambda)}{\lambda}.$$

Suppose that $v(t, x)$ is a solution to the time fractional equation with $v(0, x) = 0$. Hence we have for every $t > 0$,

$$\kappa v(t, x) + \int_0^t w(t - r)v(r, x)dr = \int_0^t \mathcal{L}v(s, x)ds.$$

Uniqueness

Taking Laplace transform on both sides and denoting by $V(\lambda, x)$ the Laplace transform of $t \mapsto v(t, x)$, we have

$$\begin{aligned} V(\lambda, x)\left(\kappa + \int_0^\infty e^{-\lambda x} w(x)dx\right) &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\mathcal{L}v(t, x)dt \\ &= \frac{\mathcal{L}V(\lambda, x)}{\lambda}. \end{aligned}$$

It follows that $(\phi(\lambda) - \mathcal{L})V(\lambda, x) = 0$ for every $\lambda > 0$. Since $\mathcal{L}$ is the infinitesimal generator of a uniformly bounded strongly continuous semigroup $\{T_t, t \geq 0\}$, for every $\alpha > 0$, the resolvent $G_\alpha = \int_0^\infty e^{-\alpha t} T_t dt$ is well defined and is the inverse to $\alpha - \mathcal{L}$. Thus $V(\lambda, \cdot) = 0$ in $\mathbb{B}$ for every $\lambda > 0$. By the uniqueness of Laplace transform, we have $v(t, \cdot) = 0$ in $\mathbb{B}$ for every $t > 0$.

We first investigate some properties of subordinators.

Lemma

There is a Borel set $\mathcal{N} \subset (0, \infty)$ *having zero Lebesgue measure so that*

$$\mathbb{P}(\overline{S}_s \geq t) = \int_0^s \mathbb{E}\left[w(t - \overline{S}_r) 1_{\{t \geq \overline{S}_r\}} \right] dr$$

for every $s > 0$ *and* $t \in (0, \infty) \setminus \mathcal{N}$.

Proof: Show that both sides have the same Laplace transform.

More on subordinator

Define $G(0) = 0$ and $G(x) = \int_0^x w(y)dy$.

Corollary

Let $\mathcal{N} \subset (0, \infty)$ be the set in above Lemma, which has zero Lebesgue measure.

(i) $\displaystyle\int_0^\infty \mathbb{E}\left[w(t - \overline{S}_r)1_{\{t \geq \overline{S}_r\}}\right] dr = 1$ *for every* $t \in (0, \infty) \setminus \mathcal{N}$.

(ii) $\displaystyle\int_0^\infty \mathbb{E}\left[G(t - \overline{S}_r)1_{\{t \geq \overline{S}_r\}}\right] dr = t$ *for every* $t > 0$.

(iii) $\displaystyle\int_0^\infty \mathbb{E}\left[G(t - S_r)1_{\{t \geq S_r\}}\right] dr \leq t$ *for every* $t > 0$.

Proof: (i) follows from the lemma by taking $s \to \infty$.

Proof of Corollary

(ii) follows from (i) and Fubini theorem that

$$
\begin{aligned}
t &= \int_0^t \left(\int_0^\infty \mathbb{E}\left[w(s - \overline{S}_r) 1_{\{s \geq S_r\}} \right] dr \right) ds \\
&= \int_0^\infty \mathbb{E}\left[G(t - \overline{S}_r) 1_{\{t \geq \overline{S}_r\}} \right] dr.
\end{aligned}
$$

(iii) Since $G(x)$ is an increasing function in x, we have by (ii)

$$
\int_0^\infty \mathbb{E}\left[G(t - S_r) 1_{\{t \geq S_r\}} \right] dr \leq \int_0^\infty \mathbb{E}\left[G(t - \overline{S}_r) 1_{\{t \geq \overline{S}_r\}} \right] dr \leq t.
$$

Existence and probabilistic representation

(i) By the integration by parts formula, one can show that

$$\int_0^t w(t-r)\mathbb{P}(S_s > r)dr = G(t) - \mathbb{E}\left[G(t-S_s)1_{\{t\geq S_s\}}\right].$$

(ii) For $u(t,x) := \mathbb{E}_x\left[f(T_{E_t}f(x)\right]$, using above identity and an integration by parts,

$$\begin{aligned} & \int_0^t w(t-r)(u(r,x)-u(0,x))dr \\ = & \int_0^t w(t-r)\left(\int_0^\infty (T_sf(x)-f(x))d_s\mathbb{P}(S_s\geq r)\right)dr \\ = & \int_0^\infty (T_sf(x)-f(x))d_s\left(\int_0^t w(t-r)\mathbb{P}(S_s>r)dr\right) \\ = & -\int_0^\infty (T_sf(x)-f(x))d_s\mathbb{E}\left[G(t-S_s)1_{\{t\geq S_s\}}\right] \\ = & \int_0^\infty \mathbb{E}\left[G(t-S_s)1_{\{t\geq S_s\}}\right]\mathcal{L}T_sf(x)ds. \end{aligned}$$

(iii) we have by the Lemma 1 that

$$\mathbb{P}(S_r \geq s) = \mathbb{P}(\overline{S}_r \geq s-\kappa r) = \int_0^r \mathbb{E}\left[w(s-\kappa r-\overline{S}_y)1_{\{s-\kappa r>\overline{S}_y\}}\right] dy.$$

So for every $t > 0$,

$$\int_0^t \mathbb{P}(S_r \geq s)ds = (\kappa r)\wedge t + 1_{\{\kappa r<t\}}\mathbb{E}\int_0^r G(t-\kappa r-\overline{S}_y)1_{\{t-\kappa r>\overline{S}_y\}}dy.$$

Thus

$$\begin{aligned}\int_0^t \mathcal{L}u(s,x)ds &= \int_0^t \left(\int_0^\infty T_r\mathcal{L}f(x)d_r\mathbb{P}(S_r \geq s)\right) ds \\ &= \int_0^\infty T_r\mathcal{L}f(x)d_r\left(\int_0^t \mathbb{P}(S_r \geq s)ds\right).\end{aligned}$$

Existence and probabilistic representation

$$
\begin{aligned}
&= \mathbb{E}\int_0^{t/\kappa} T_r\mathcal{L}f(x)\Big(\kappa + G(t-\kappa r-\overline{S}_r)1_{\{t-\kappa r>\overline{S}_r\}} \\
&\qquad -\kappa\int_0^r w(t-\kappa r-\overline{S}_y)1_{\{t-\kappa r>\overline{S}_y\}}dy\Big)dr \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr \\
&\qquad +\kappa\int_0^{t/\kappa} T_r\mathcal{L}f(x)\left(1-\mathbb{P}(S_r\ge t)\right)dr \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr \\
&\qquad +\kappa\int_0^{t/\kappa} \mathbb{P}(E_t>r)d_r\left(T_rf(x)-f(x)\right)
\end{aligned}
$$

Existence and probabilistic representation

$$
\begin{aligned}
&= \int_0^\infty T_r \mathcal{L} f(x) \mathbb{E}\left[G(t-S_r) 1_{\{t \geq S_r\}}\right] dr \\
&\quad + \kappa \int_0^\infty \left(T_r f(x) - f(x)\right) d_r \mathbb{P}(E_t \leq r) \\
&= \int_0^\infty T_r \mathcal{L} f(x) \mathbb{E}\left[G(t-S_r) 1_{\{t \geq S_r\}}\right] dr + \kappa \mathbb{E}\left[T_{E_t} f(x) - f(x)\right] \\
&= \int_0^\infty T_r \mathcal{L} f(x) \mathbb{E}\left[G(t-S_r) 1_{\{t \geq S_r\}}\right] dr + \kappa (u(t,x) - u(0,x)).
\end{aligned}
$$

Thus we have for every $t > 0$,

$$
\kappa(u(t,x) - u(0,x)) + \int_0^t w(t-r)(u(r,x) - u(0,x)) dr = \int_0^t \mathcal{L} u(s,x) ds.
$$

Consequently, $(\kappa \partial_t + \partial_t^w) u(t,x) = \mathcal{L} u(t,x)$ in $\mathbb{B}$ as $t \mapsto \mathcal{L} u(t, \cdot)$ is continuous in $(\mathbb{B}, \| \cdot \|)$.

Fundamental solution

When the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ has an integral kernel $p(t, x, y)$ with respect to some measure $m(dx)$, then there is a kernel $q(t, x, y)$ so that

$$u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right] = \int_E q(t,x,y) f(y) m(dy);$$

in other words,

$$q(t,x,y) := \mathbb{E}\left[p(E_t,x,y)\right] = \int_0^\infty p(s,x,y) d_s \mathbb{P}(E_t \leq s)$$

is the fundamental solution to the time fractional equation $\left(\kappa \partial_t + \partial_t^w\right) u = \mathcal{L} u$.

In a ongoing work with Kim, Kumagai and Wang, two-sided estimates on $q(t, x, y)$ are obtained when $\kappa = 0$ and $\{T_t; t \geq 0\}$ is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

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Examples

(i) When $\{S_t; t \geq 0\}$ is a β-subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, Then S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx$. Hence

$$w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Thus the time fractional derivative $\partial_t^w f$ is exactly the Caputo derivative of order β. In this case, our Theorem recovers the main result of Baeumer-Meerschaert (2001) and Meerschaert-Scheffler (2004).

Truncated stable-subordinator

(ii) A truncated β-stable subordinator $\{S_t; t \geq 0\}$ is driftless and has Lévy measure

$$\mu_\delta(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx$$

for some $\delta > 0$. In this case,

$$\begin{aligned} w_\delta(x) &:= \mu_\delta(x,\infty) = 1_{\{0<x\leq\delta\}} \int_x^\delta \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy \\ &= \frac{1}{\Gamma(1-\beta)} \left(x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x). \end{aligned}$$

The corresponding the fractional derivative is

$$\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t \left((t-s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.$$

Stability

Clearly, as $\lim_{\delta\to\infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta}$. Consequently, $\partial_t^{w_\delta} f(t) \to \partial_t^{w} f(t)$, the Caputo derivative of f of order β, in the distributional sense as $\delta \to 0$. Using the probabilistic representation in the main Theorem, one can deduce that as $\delta \to \infty$, the solution to the equation $\partial_t^{w_\delta} u = \mathcal{L} u$ with $u(0,x) = f(x)$ converges to the solution of $\partial_t^{\beta} u = \mathcal{L} u$ with $u(0,x) = f(x)$.

Stability

(iii) If we define

$$\eta_\delta(r) = \frac{\Gamma(2-\beta)\,\delta^{\beta-1}}{\beta} w_\delta(r) = (1-\beta)\delta^{\beta-1}\left(x^{-\beta} - \delta^{-\beta}\right) 1_{(0,\delta]}(x),$$

then $\eta_\delta(r)$ converges weakly to the Dirac measure concentrated at 0 as $\delta \to 0$. So $\partial_t^{\eta_\delta} f(t)$ converges to $f'(t)$ for every differentiable f. It can be shown that the subordinator corresponding to η_δ, that is, subordinator with Lévy measure

$$\nu_\delta(dx) := \frac{(1-\beta)\,\delta^{\beta-1}}{\beta} x^{-(1+\beta)} 1_{(0,\delta]}(x)dx,$$

converges as $\delta \to 0$ to deterministic motion t moving at constant speed 1. Using the main Theorem, one can show that the solution to the equation $\partial_t^{\eta_\delta} u(t,x) = \mathcal{L}u(t,x)$ with $u(0,x) = f(x)$ converges to the solution of the heat equation $\partial_t u = \mathcal{L}u$ with $u(0,x) = f(x)$.

Occupation measure

- $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in E\}$: strong Markov process on E with infinitesimal generator $\mathcal{L}$.
- $S = \{S_t; t \geq 0\}$: subordinator independent of X with infinite Lévy measure μ.
- ϕ: Laplace exponent of S; that is, $\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}$.

$$\mathbb{E}\left[S_t\right] = t\phi'(0).$$

- $E_t := \inf\{s > 0 : S_s > t\}$, and $X_t^* := X_{E_t}$.

Suppose D is an open subset of E and define $\tau_D := \{t > 0 : X_t \notin D\}$ and first exit time from D by

$$\tau_D^* := \inf\left\{t > 0 : X_t^* \notin D\right\}.$$

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Occupation measure

Theorem

For every measurable function $f \geq 0$ on D and $x \in D$,

$$\mathbb{E}_x\left[\int_0^{\tau_D^*} f(X_t^*)dt\right] = \phi'(0)\,\mathbb{E}_x\left[\int_0^{\tau_D} f(X_t)dt\right] = \phi'(0)G_D f(x).$$

In other words, $\nu_x^{*,D} = \phi'(0)\nu_x^D$ for every open set $D \subset E$ and every $x \in D$.

Remarks

(i) Taking $f = 1$ yields

$$\mathbb{E}_x\left[\tau_D^*\right] = \phi'(0)\,\mathbb{E}_x\left[\tau_D\right] \quad \text{for every } x \in D.$$

When X is either a diffusion process determined by a stochastic differential equation driven by Brownian motion or a rotationally symmetric α-stable process on $\mathbb{R}^d$, and $\{S_t; t \geq 0\}$ is a tempered β-stable subordinator having Laplace exponent $\phi(\lambda) = (\lambda + m)^\beta - m^\beta$ for some $m > 0$ and $0 < \beta < 1$, the above display recovers the main result of W. Deng, X. Wu and W. Wang (*Europhysics Letters* 2017), derived there using a PDE method.

Remarks

(ii)

$$u(t,x) := \mathbb{E}_x\left[f(X^D_{E_t})\right] = \mathbb{E}_x\left[f(X^*_t); t < \tau^*_D\right]$$

is the strong solution to

$$(\kappa\partial_t + \partial^w_t)\, u(t,x) = \mathcal{L}^D u(t,x) \quad \text{with } u(0,x) = f(x) \text{ in } D.$$

On the other hand, $G_D f(x)$ is the solution to the Poisson equation $\mathcal{L}v = -f$ in D with $v = 0$ on D^c. Hence by the above theorem, for $f \in \mathcal{D}(\mathcal{L}^D)$,

$$G^*_D f(x) = \int_0^\infty \mathbb{E}_x\left[f(X_t); t < \tau^*_D\right] dt = \int_0^\infty u(t,x)dt$$

is the solution to the Poisson equation

$$\mathcal{L}^D v = -\phi'(0) f \quad \text{in } D \quad \text{with } v = 0 \text{ on } D^c.$$

Thank you!