Long Brownian bridges in hyperbolic spaces converge to Brownian trees

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# Table of Contents

1. Hyperbolic Brownian motion
2. Convergence to CRT
Hyperbolic space $H^d$
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- Poincaré coordinates with $d_H s = \sqrt{|dx|^2 + dy^2}$:
  
  $$H^d = \mathbb{R}^{d-1} \times \mathbb{R}^* = \{(x, y) : x \in \mathbb{R}^{d-1}, y > 0\}$$
Hyperbolic space $H^d$

- Poincaré coordinates with $d_{H^d} = \sqrt{\frac{|dx|^2 + dy^2}{y}}$:
  $$H^d = \mathbb{R}^{d-1} \times \mathbb{R}_+ = \{(x, y) : x \in \mathbb{R}^{d-1}, y > 0\}$$

- Polar coordinates $H^d = \mathbb{R}_+ \times S^{d-1} = \{(\rho, \phi)\}$. For example, Poincaré disk $H^2$
Hyperbolic space $H^d$

- Poincaré coordinates with $\text{d}_{H}s = \sqrt{\text{d}x^2 + \text{d}y^2}$:
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- Polar coordinates $H^d = \mathbb{R}_+ \times S^{d-1} = \{(\rho, \phi)\}$. For example, Poincaré disk $H^2$

\[
d_{H}s = \frac{2dr}{1 - r^2}
\]
Brownian motion on $H^d$

Hyperbolic Laplacian $\Delta_H = y^2\left(\frac{\partial^2}{\partial y^2} + \Delta_{x}^{d-1}\right) + (2 - d)y\frac{\partial}{\partial y}$

$$\Delta_H = \frac{\partial^2}{\partial \rho^2} + (d - 1) \coth \rho \frac{\partial}{\partial \rho} + (\sinh \rho)^{-2} \Delta_{\phi}^{S^{d-1}}$$
Brownian motion on \( H^d \)

- Hyperbolic Laplacian \( \Delta_H = y^2 \left( \frac{\partial^2}{\partial y^2} + \Delta_x^{d-1} \right) + (2 - d)y \frac{\partial}{\partial y} \)

\[
\Delta_H = \frac{\partial^2}{\partial \rho^2} + (d - 1) \coth \rho \frac{\partial}{\partial \rho} + (\sinh \rho)^{-2} \Delta_{\phi}^{S^{d-1}} > 0
\]

- Brownian motion: \((B(t))_{t \geq 0}\) Markov process in \( H^d \) with generator \( \frac{1}{2} \Delta_H \).

\[
\frac{d_H(o, B_t)}{t} \xrightarrow{a.s.} \ell_d > 0.
\]
Brownian motion on Poincaré disk
Brownian bridge: \((b_T(t); 0 \leq t \leq T)\) is \((B(t); 0 \leq t \leq T)\) conditionally on \(B(0) = B(T) = o_H\)
Hyperbolic Brownian bridges

1. Brownian bridge: \((b_T(t); 0 \leq t \leq T)\) is \((B(t); 0 \leq t \leq T)\) conditionally on \(B(0) = B(T) = o_H\)

2. Heat kernel [Grigor’yan-Noguchi’97]

\[
p_t(x, y) = p_d(d_H(x, y), t) \xrightarrow{t \to \infty} C_d(x, y)e^{-\lambda_d t}t^{-3/2}
\]

For example, \(p_3(\rho, t) = \frac{1}{(4\pi t)^{3/2}} \frac{\rho}{\sinh(\rho)} e^{-t-\frac{\rho^2}{4t}}\)
Hyperbolic Brownian bridges

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3. Transition density of Brownian bridge with respect to \(m(dz) = \sinh(\rho)d\rho d\phi\)

\[
p_{s, t}^b(x, y) = \frac{p_s(o_H, x)p_{t-s}(x, y)p_{T-t}(y, o_H)}{p_T(o_H, o_H)} C_d(o_H, x) C_d(x, y) C_d(y, o_H) s^{-3/2} (t - s)^{-3/2} (T - t)^{-3/2}
\]

\[
\sim \frac{C_d(o_H, x) C_d(x, y) C_d(y, o_H) s^{-3/2} (t - s)^{-3/2} (T - t)^{-3/2}}{C_d(o_H, o_H) T^{-3/2}}
\]
1 Hyperbolic Brownian motion

2 Convergence to CRT
Main result: convergence to Brownian continuum random tree (CRT)

Let $\rho_T(t) = d_H(o_H, b_T(t))$.

**Theorem [Bougerol-Jeulin'99]**

$\left( \frac{\rho_T(tT)}{\sqrt{T}}, t \in [0, T] \right) \xrightarrow{(law)} \left( e_t, t \in [0, 1] \right)$

where $(e_t, t \in [0, 1])$ is normalised Brownian excursion.
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Let \( \rho_T(t) = d_H(o_H, b_T(t)) \).

**Theorem [Bougerol-Jeulin’99]**

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\left( \frac{\rho_T(tT)}{\sqrt{T}}, t \in [0, T] \right) \xrightarrow{\text{(law)}} (e_t, t \in [0, 1])
\]

where \((e_t, t \in [0, 1])\) is normalised Brownian excursion.

**Theorem [C.-Miermont’16]**

For the range \( \mathcal{R}_T = \{b_T(t); t \in [0, T]\} \),

\[
\left\{ \left( \frac{\rho_T(tT)}{\sqrt{T}}, t \in [0, T] \right), (\mathcal{R}_T, \frac{d_H}{\sqrt{T}}, o_H) \right\} \xrightarrow{T \to \infty} (e, \mathcal{T}_e)_{\text{CRT}}
\]
Related results

Theorem [C.-Miermont’16, Stewart’16]

For simple RW on regular tree,

\[
\left( \{ S_i; 0 \leq i \leq 2n \}, \frac{d_{T_k}}{\sqrt{2n}}, \circ \bigg| S_0 = S_{2n} = o \right) \xrightarrow{n \to \infty} T_e
\]

Theorem [Anker-Bougerol-Jeulin’02]

Hyperbolic Brownian bridge \( b_T \xrightarrow{T \to \infty} b_\infty \), which is called the infinite Brownian loop.

Theorem [C.-Miermont’16]

\[
(\{ b_\infty(t); t \in \mathbb{R} \}, ad_H, o_H) \xrightarrow{a \downarrow 0} T_X \text{ where } X \text{ is two-ended 3-dimensional Bessel processes.}
\]
Convergence to CRT

- Convergence of radial part (Done by Bougerol and Jeulin)
- Invariance by re-rooting (same as CRT $\mathcal{T}_e$)
- $\delta$-hyperbolicity (tree structure of $\mathcal{R}_T$)
Invariance by re-rooting

Let $\psi_x : H \to H$ be the unique hyperbolic isometry sending $x$ to $o_H$ such that $x \mapsto \psi_x(o_H)$ preserves the measure $m(dx)$. For example,

$$\psi_x(z) = \frac{z - x}{1 - \bar{x}z} : H^2 \to H^2$$
Invariance by re-rooting

Let $\psi_x : H \to H$ be the unique hyperbolic isometry sending $x$ to $o_H$ such that $x \mapsto \psi_x(o_H)$ preserves the measure $m(dx)$. For example,

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Lemma

For any $t \in [0, T]$, $b_T$ and $\psi_{b_T(t)}(b_T(\cdot + t \mod T))$ have the same distribution.
Hyperbolic Brownian bridge

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Hyperbolic Brownian motion

Convergence to CRT

\[ b_T \overset{\text{law}}{=} \psi_{b_T}(t)(b_T(\cdot + t \mod T)) \]
Tightness in the sense of Gromov-Hausdorff from invariance by re-rooting

1. Tightness of radial part.
2. Finite number of $\eta$-balls cover $\mathcal{R}_T := \{ b_T(t); 0 \leq t \leq T \}$, w.h.p. In fact,

$$
\mathbb{P}(\mathcal{R}_T \notin \bigcup_{i=0}^{N-1} B_H(b_T(\frac{i}{N} T), \eta\sqrt{T})) \\
\leq \sum_{i=0}^{N-1} \mathbb{P} \left( \exists t \in \left[ \frac{i}{N} T, \frac{i+1}{N} T \right], d_H(b_T(t), b_T(\frac{i}{N} T)) > \eta \sqrt{T} \right) \\
= N \mathbb{P} \left( \sup_{t \leq 1/N} \frac{d_H(b_T(tT), o_H)}{\sqrt{T}} > \eta \right) \\
\approx N \mathbb{P} \left( \sup_{t \leq 1/N} e_t > \eta \right) \leq C_\eta \sqrt{N} e^{-c_\eta N}
$$
\(\delta\)-hyperbolicity

\[\begin{align*}
 \text{Figure: } [AB] & \subset B_\delta([AC]) \cup B_\delta([BC]), \quad [AC] \subset B_\delta([AB]) \cup B_\delta([BC]) \\
 \text{and } [BC] & \subset B_\delta([AC]) \cup B_\delta([AB])
\end{align*}\]

For \(H^2\) with \(d_H/\sqrt{T}\), \(\delta = \log(\sqrt{2} + 1)/\sqrt{T}\). For trees, \(\delta = 0\).
Tree structure from $\delta$-hyperbolicity and tightness

Figure: Brownian bridge moves along the geodesics
Conclusion:

- Tightness
- "tree-like structure"
- \( \Rightarrow \) radial convergence to normalized Brownian excursion

\( \Rightarrow \) Brownian tree is the unique choice for the limit.