Dirichlet fractional heat kernel estimates on a horn-shaped domain

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Outline

1. **Introduction**
   - framework
   - Known results
2. **Main results**

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1. Introduction
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2. Main results

Symmetric jump processes

- Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a symmetric Dirichlet forms on $L^2(\mathbb{R}^d; dx)$ as follows

$$\mathscr{E}(f,f) := \int\int_{\mathbb{R}^d \times \mathbb{R}^d / \Delta} \big(f(x) - f(y)\big)^2 J(x,y)dxdy,$$

$$\mathscr{D}(\mathscr{E}) := \overline{C_c^\infty(\mathbb{R}^d)}^{\mathscr{E}_1}.$$

- $J(x,y) : \mathbb{R}^d \times \mathbb{R}^d / \Delta \to \mathbb{R}_+$ is symmetric.
- There exists a symmetric Hunt process $(X_t)_{t \geqslant 0}$ associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

Symmetric jump processes killed on domains

- Given an open set $D \subseteq \mathbb{R}^d$.
- Let $(\mathscr{E}_D, \mathscr{D}(\mathscr{E}_D))$ be a Dirichlet form on $L^2(D; dx)$ as follows

$$\mathscr{E}_D(f,f) := \int\int_{\mathbb{R}^d \times \mathbb{R}^d / \Delta} \big(f(x) - f(y)\big)^2 J(x,y) dxdy,$$

$$\mathscr{D}(\mathscr{E}_D) := \{f \in \mathscr{D}(\mathscr{E}); f(z) = 0 \text{ for } q.e. z \notin D\}.$$

- There exists a symmetric Hunt process $(X_t^D)_{t \geqslant 0}$ associated with $(\mathscr{E}_D, \mathscr{D}(\mathscr{E}_D))$.

Symmetric jump processes killed on domains

- When $D = \mathbb{R}^d$, $(\mathscr{E}_D, \mathscr{D}(\mathscr{E}_D)) = (\mathscr{E}, \mathscr{D}(\mathscr{E}))$.
- Let $\tau_D := \inf\{t \geqslant 0; X_t \notin D\}$, then

$$X_t^D = \begin{cases} X_t, & t < \tau_D; \\ \partial, & t \geqslant \tau_D; \end{cases}$$

Dirichlet heat kernel

- Suppose the heat kernel $p(t,x,y)$ exists for $(X_t)_{t\geqslant 0}$,

$$\mathbb{P}_x(X_t \in dy) = p(t,x,y)dy = p(t,y,x)dy.$$

- (Dirichlet heat kernel) There exists a $p^D : \mathbb{R}_+ \times D \times D \to \mathbb{R}_+$ such that

$$Pp_x(X_t^D \in dy) = p^D(t,x,y)dy = p^D(t,y,x)dy,\ x,y \in D.$$

Dirichlet heat kernel

- $$p^D(t,x,y) = p(t,x,y) - \mathbb{E}_x\left[p(t-\tau_D, X_{\tau_D}, y)1_{\{t \geqslant \tau_D\}}\right], \ x,y \in D$$
  $$p^D(t,x,y) = 0, \quad x \notin D \text{ or } y \notin D$$

- Green function $G^D(x,y)$ (transient case),
  $$G^D(x,y) = \int_0^\infty p^D(t,x,y)dy, \ \ x,y \in D.$$

- $$\mathbb{P}_x(\tau_D > t) = \int_D p^D(t,x,y)dy, \ \ x \in D,$$
  $$\mathbb{E}_x[\tau_D] = \int_D G^D(x,y)dy, \ \ x \in D.$$

Regularity condition on the boundary

- We say that $U \subseteq \mathbb{R}^d$ is (κ, r)-fat if there exist $\kappa > 0$, $R_0 > 0$ such that for every $x \in \overline{U}$ and $r \in (0, R_0)$ we can find a $\xi_x \in U$ satisfying that $B(\xi_x, \kappa r) \subseteq U \cap B(x, r)$.
- We say that U is (uniformly) $C^{1,1}$ if there exist $r > 0$, $\lambda > 0$ such that for every $x \in \partial U$, we can find a $C^{1,1}$-function $\varphi_x : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi_x(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi_x\|_\infty \leqslant \lambda$, $|\nabla\varphi_x(y) - \nabla\varphi_x(z)| \leqslant \lambda|y - z|$, and an orthonormal coordinate system CS_x with its origin at x such that

$$B(x, r_x) \cap U = \{y = (y_1, \tilde{y}) \in B(0, r) \text{ in } CS_x : \ y_1 > \varphi_x(\tilde{y})\},$$

where $\tilde{y} := (y_2, \dots, y_d)$.

Regularity condition on the boundary

- $\delta_D(x) := dist(x, \partial D), x \in D.$
- We say that $U \subseteq \mathbb{R}^d$ satisfies that uniformly interior ball condition when there exists a $r_0 > 0$ so that for every $x \in D$ with $\delta_D(x) < r_0$, we could find a $z_x \in \partial D$ satisfying that $B(x_0, r_0) \subseteq D$, where $x_0 := z_x + r_0(x - z_x)/|x - z_x|$.
- $C^{1,1} \Longrightarrow$ uniformly interior ball condition $\Longrightarrow \kappa$-fat.

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2. **Main results**

Killed Brownian motion

- [E.B. Davies 87], [Q.S. Zhang 02] Suppose X_t is a Brownnian motion, D is a $C^{1,1}$ bounded set,

$$p^D(t,x,y) \asymp t^{-d/2} e^{-\frac{|x-y|^2}{t}} \left(1 \wedge \frac{\delta_D(x)^{1/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{1/2}}{\sqrt{t}}\right)$$

$$x, y \in D, \ t \in (0,1].$$

α-stable process

- [Z.Q. Chen, R.M. Song 98], [T. Kulczycki 98] Suppose $J(x,y) = \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}$, D is $C^{1,1}$ bounded and $d \geqslant 2$,

$$G^D(x,y) \asymp |x-y|^{-d+\alpha}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right), \ x,y \in D.$$

- [Z.Q. Chen, T. Kumagai 03,08] Suppose $J(x,y) = \frac{\kappa(x,y)}{|x-y|^{d+\alpha}}$, $D = \mathbb{R}^d$,

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, \ \ x,y \in \mathbb{R}^d, \ t > 0.$$

α-stable process

- [Z.Q. Chen, P.Kim, R.M. Song 10] Suppose $J(x,y) = \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}$, D is (uniformly) $C^{1,1}$,

$$p^D(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)$$

$x, y \in D,\ t \in (0,1]$.

- [Z.Q. Chen, P.Kim, R.M. Song 10] Moreover, if D is bounded,

$$p^D(t,x,y) \asymp e^{-\lambda t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},\ x,y \in D,\ t \in (1,+\infty).$$

α-stable process

- [Z.Q. Chen, J. Tokle 11] Suppose $J(x,y)=\frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}$, D is half-space like $C^{1,1}$ domain,

$$p^D(t,x,y)\asymp\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)$$

$$x,y\in D,\ t\in(1,+\infty).$$

- [Z.Q. Chen, J. Tokle 11] Moreover, if D is exterior bounded $C^{1,1}$ domain,

$$p^D(t,x,y)\asymp\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)$$

$$x,y\in D,\ t\in(1,+\infty).$$

α-stable process

- [K. Bogdan, T. Grzywny, M. Ryznar 10] Suppose $J(x,y)=\frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}$, D is a κ-fat domain,

$$p^D(t,x,y)\asymp\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\mathbb{P}_x\big(\tau_D>t\big)\mathbb{P}_y\big(\tau_D>t\big)$$

$$x,y\in D,\ \ t\in(0,1].$$

- Key ingredients: boundary Harnack inequality ([R.M. Song, J. Wu 98], [K. Bogdan, T. Kulczycki, M. Kwasnicki 08]), estimates for exit probability, Lévy system, intrinsic ultracontractivity.

Other jump processes

- Truncated stable process $J(x, y) = \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}} 1_{\{|x-y| \leqslant 1\}}$.
[Z.Q. Chen, P. Kim, T. Kumagai 08], [P. Kim, K. Kim 14], [J.M. Wang 2017+]

- Relativistic stable process [Z.Q. Chen, P. Kim, R.M. Song 11,12,12]

- Subordinated Brownian motion [Z.Q. Chen, P. Kim, R.M. Song 16], [P. Kim, R.M. Song and Z. Vondracek 12]

- Non-symmetric setting [L. Xie, X. Zhang 14], [Z.Q. Chen, X. Zhang 16]

General symmetric jump process

- General symmetric jump process

$$J(x,y) \asymp \frac{\kappa(x,y)}{|x-y|^d \Phi(|x-y|)\chi(|x-y|)}, \ x,y \in \mathbb{R}^d$$

- $$L^{-1} \leqslant \kappa(x,y) \leqslant L, \ |\kappa(x,x+h) - \kappa(x,x)| \leqslant L|h|^\eta.$$

- For some $0 < \alpha_1 \leqslant \alpha_2 < 2$,

$$C_1(\frac{R}{r})^{\alpha_1} \leqslant \frac{\Phi(R)}{\Phi(r)} \leqslant C_2(\frac{R}{r})^{\alpha_2}, \ 0 < r \leqslant R.$$

- $r \mapsto -(\Phi(r)^{-1} r^{-d})'/r$ is decreasing on $(0,\infty)$.

General symmetric jump process

- χ is a nondecreasing function on $(0, \infty)$ with $\chi(r) \equiv \chi(0)$ for $r \in (0, 1]$ and

$$L_1 \exp(c_1 r^\gamma) \leqslant \chi(r) \leqslant L_2 \exp(c_2 r^\gamma), \quad r > 1.$$

- Related results [Z.Q. Chen, P. Kim, T. Kumagai 11], [Z.Q. Chen, P. Kim, R.M. Song 14], [Z.Q. Chen, P. Kim 16], [T. Grzywny, K. Kim, P. Kim 16]

horn-shaped domain

- Suppose that $f : (0, \infty) \to (0, \infty)$ is a bounded and continuous function such that $\lim_{u\to\infty} f(u) = 0$. The open set $D_f = \{x \in \mathbb{R}^d : x_1 > 0, |\tilde{x}| < f(x_1)\}$ is called the horn-shaped domain.
- D_f is not (uniformly) $C^{1,1}$ nor κ-fat.
- [M. Kwasnicki 09], [X. Chen, P. Kim, J. Wang 17+] If $J(x, y) = \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}$, $D = D_f$, $f(s) = \log^{-\theta}(1 + s)$ with $\theta > \alpha^{-1}$, then $(T_t^D)_{t\geqslant 0}$ is intrinsically ultracontractive.

Main results

- [M. Kwasnicki 09], [X. Chen, P. Kim, J. Wang 17+] If $J(x,y)=\frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}$ and $(T_t^D)_{t\geqslant 0}$ is intrinsically ultracontractive, then

$$p^D(1,x,0)\asymp\phi_1(x)\asymp\delta_D(x)^{\alpha/2}f(x_1)^{\alpha/2}(1+|x|)^{-d-\alpha},\ x\in D.$$

- For simplicity, suppose $f:\mathbb{R}_+\to\mathbb{R}_+$ is non-increasing, define

$$t_0(y):=\inf\Big\{t>0;\exp\Big(-\frac{c_1t}{f(y_1)^\alpha}\Big)\leqslant\frac{c_2t}{(1+|y|)^{d+\alpha-1}}\Big\},\ y\in D.$$

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$$\Psi(t,y):=\Bigg(1\wedge\frac{\delta_D(y)^{\alpha/2}\big(f(y_1)^{\alpha/2}\wedge\sqrt{t}\big)}{t}\Bigg),\ y\in D,t>0.$$

Main results

- $$\Gamma(t,y) := \int_{\{z\in D, |z|\leqslant \frac{|y|}{2}\wedge z_0(t)\}} e^{-\frac{ct}{f(z_1)^\alpha}} dz,\ y\in D.$$
  $$z_0(t) := \inf\{s>0; f(s)^\alpha \log(1+s)\leqslant t\}.$$

Theorem

Suppose $\lim_{s\to\infty} f(s)(1+s)^k = \infty$ *for some* $k>0$, *then for every* $x,y\in D$ *with* $x_1>y_1$,

$$p^D(t,x,y) \asymp \Psi(t,x)\Psi(t,y)\cdot \begin{cases} \exp\Big(-\frac{c_1 t}{f(y_1)^\alpha}\Big)\Big(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\Big) \\ t\in(0,t_0(y)\wedge 1], \\ \frac{t}{(1+|x|)^{d+\alpha}}\frac{t}{(1+|y|)^{d+\alpha}}\Gamma(t,y) \\ t\in(t_0(y)\wedge 1, 1], \end{cases}$$

Example

Theorem

Suppose $f(s) = \log^{-\theta}(1+s)$ *with some* $\theta > \frac{1}{\alpha}$, *then*

$$p^D(t,x,y) \asymp \Psi(1\wedge t,x)\Psi(1\wedge t,y)\cdot \begin{cases} \exp\left(-\frac{c_1 t}{f(y_1)^\alpha}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ \quad t \in (0, c_0 \log^{-(\theta\alpha-1)}(1+|y|)], \\ \frac{t}{(1+|x|)^{d+\alpha}} \frac{t}{(1+|y|)^{d+\alpha}} \exp(ct^{-1/(\theta\alpha-1)}) \\ \quad t \in (c_0 \log^{-(\theta\alpha-1)}(1+|y|), 1] \\ e^{-\lambda t}, \quad t \in (1, +\infty). \end{cases}$$

Examples

Theorem

Suppose $f(s) = \log^{-\theta}(1+s)$ *with some* $\theta \leqslant \frac{1}{\alpha}$, *then*

$$p^D(t,x,y) \asymp \Psi(t,x)\Psi(t,y) \cdot \begin{cases} \exp\left(-\frac{c_1 t}{f(y_1)^{\alpha}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ t \in (0,1]. \end{cases}$$

Theorem

Suppose $f(s) = exp(-c_1 s)$, *then*

$$p^D(t,x,y) \asymp \Psi(t,x)\Psi(t,y) \cdot \begin{cases} \exp\left(-\frac{c_1 t}{f(y_1)^{\alpha}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ t \in (0, e^{-c_0(1+|y|)}], \\ t^2 \exp(-c_1(1+|x|)) \exp(-c_2(1+|y|)), \\ t \in (e^{-c_0(1+|y|)}, 1] \end{cases}$$

Idea of proof

- $$\mathbb{E}_x\left[\tau_{B(z_x,t^{1/\alpha})}\right] \leqslant c_1\delta_D(x)^{\alpha/2}\left(\sqrt{t}\wedge f(x_1)^{\alpha/2}\right), \quad x\in D.$$

- $$\mathbb{P}_x(\tau_D>t)\leqslant c_1\Psi(t,x)\left(e^{-\frac{ct}{f(x_1)^\alpha}}+\frac{t}{(1+|x|)^{d+\alpha-1}}\right),\ x\in D.$$

Thank you for your attention!