On a time-dependent Eggenberger-Pólya urn model

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Outline

Pólya-Eggenberger Urn (1923)

- Suppose an urn initially contains w white and r red balls.
- One ball is drawn at random and then replaced together with c balls of the same color. Repeat the procedure *ad infinitum*.
- Denote the added balls situation by the replacement matrix

$$M = \begin{matrix} \text{the drawn ball is white} \\ \text{the drawn ball is red} \end{matrix} \overset{\begin{matrix}\text{white} & \text{red}\end{matrix}}{\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}}.$$

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- After the nth action, let W_n be the number of white balls and T_n be the number of total balls. Also let $X_n = W_n/T_n$ and $\mathcal{F}_n = \sigma\{W_1, \ldots, W_n\}$.

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- Then $T_{n+1} = T_n + c$ and $W_{n+1} \overset{(d)}{=} W_n + c\xi_{n+1}$, where $\xi_{n+1} \mid_{W_n} \sim^d \text{Ber}(X_n)$.

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- Thus

$$
\begin{aligned}
X_{n+1} &\overset{(d)}{=} \frac{W_n + c\xi_{n+1}}{T_{n+1}} \\
&= \frac{T_n}{T_{n+1}} X_n + \frac{c\xi_{n+1}}{T_{n+1}} \\
&= X_n + \frac{c}{T_{n+1}}(\xi_{n+1} - X_n).
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- Hence $\{X_n\}$ is a bounded martingale.
- By martingale convergence theorem, $\{X_n\}$ converges almost surely. Furthermore, the distribution of $\lim_{n\to\infty} X_n$ follows a beta distribution with parameters b/c and r/c.

Pemantle's urn (1989)

- Suppose an urn initially contains w white and r red balls.
- At time n, one ball is drawn at random and then replaced together with c_n balls of the same color, where c_n is a positive integer. Repeat the procedure *ad infinitum*.
- Then after the nth drawn, the replacement matrix

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$$X_{n+1} \stackrel{(d)}{=} \frac{T_n}{T_{n+1}} X_n + \frac{c_{n+1}\xi_{n+1}}{T_{n+1}}$$
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If $\sum_{n=1}^{\infty}\left(\frac{c_n}{T_{n-1}}\right)^2 < \infty$, what is the distribution of X?

$$\sum_{n=1}^{\infty}\left(\frac{c_n}{T_{n-1}}\right)^2 < \infty$$

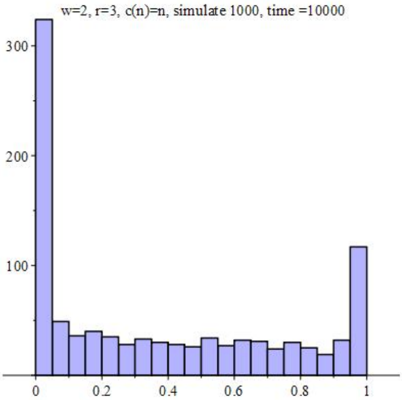

Figure 1. $c_n = n$

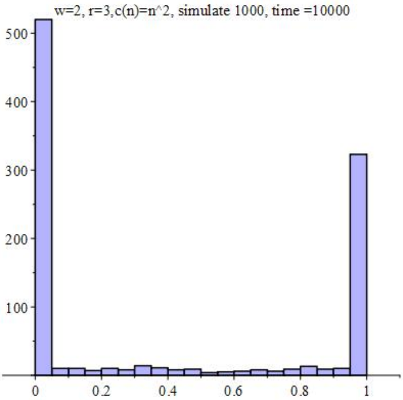

Figure 2. $c_n = n^2$

$$\sum_{n=1}^{\infty} \left(\frac{c_n}{T_{n-1}} \right)^2 < \infty$$

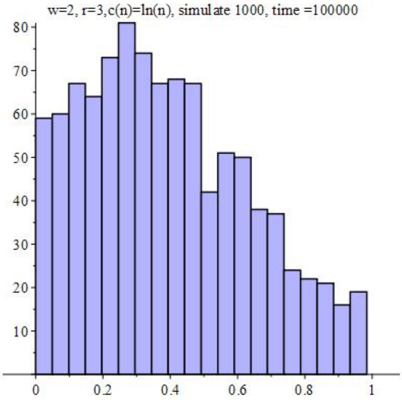

Figure 3. $c_n = \ln(n)$

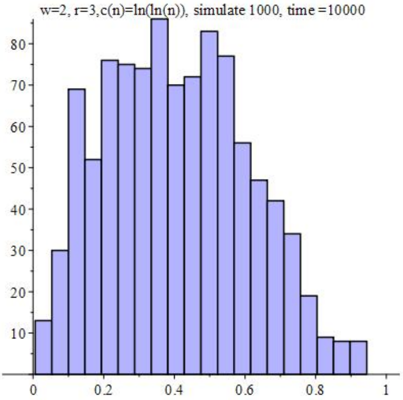

Figure 4. $c_n = \ln(\ln(n))$

Example

For Pemantle's urn, if $w = r = 1$ and $c_n = n$, then the probability that all drawn are of the same color is $\frac{2}{3} \times \frac{6}{7} \times \cdots > 0$. Thus the probability of $X \in \{0, 1\}$ is positive.

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In 1989, Pemantle showed that if $\{c_n\}_{n \geq 1}$ is a bounded sequence, then $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 0$, that is , X has no atoms on $[0, 1]$.

The urn of Johnson, Kotz, and Mahmoud (2004)

- Johnson *et al.* (2004) proposed a general Pólya urn models with multiple drawn.
- In their model, the drawn, say $m \geq 1$, can be with or without replacement and the replacement matrix is

$$M = \begin{array}{c} \#\text{white balls drawn} \\ \text{m} \\ \text{m-1} \\ \vdots \\ 1 \\ 0 \end{array} \begin{pmatrix} -(m-1) & m \\ -(m-2) & m-1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- They gave an recursion formula for the distribution of white balls.
- They also gave the expectation and the variance of the number of white balls.

Chen-Wei Urn (2005)

- Suppose an urn initially contains w white and r red balls.
- Chen-Wei considered that at each step, $m \geq 1$ balls are randomly drawn and then note their colors, say k white and $m-k$ red balls. Replace the drawn balls together with ck white and $c(m-k)$ red balls. Repeat the procedure *ad infinitum*.

The replacement matrix is $M = \begin{pmatrix} cm & 0 \\ c(m-1) & c \\ \vdots & \vdots \\ c & c(m-1) \\ 0 & cm \end{pmatrix}.$

- After the nth action, let W_n be the number of white balls and T_n be the number of total balls in the urn. Also let $X_n = W_n / T_n$.

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- Then $T_{n+1} = T_n + cm$ and $W_{n+1} \stackrel{(d)}{=} W_n + c\xi_{n+1}$, where $\xi_{n+1}\,|_{W_n} \sim^d$ Hypgeo$(W_n, T_n - W_n, m)$, that is,

$$\mathbb{P}\{\xi_{n+1} = k | W_n\} = \frac{\binom{W_n}{k}\binom{T_n - W_n}{m-k}}{\binom{T_n}{m}} = \frac{\binom{T_n X_n}{k}\binom{T_n(1-X_n)}{m-k}}{\binom{T_n}{m}},$$

where $0 \leq k \leq m$.

- Note that $E[\xi_{n+1} \mid \mathcal{F}_n] = \sum_{k=0}^{m} \frac{k\binom{T_n X_n}{k}\binom{T_n(1-X_n)}{m-k}}{\binom{T_n}{m}} = mX_n$.

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- Then

$$X_{n+1} = \frac{T_n}{T_{n+1}} X_n + \frac{c}{T_{n+1}} \xi_{n+1}$$
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- Hence $\{X_n\}$ is a bounded martingale.
- Furthermore, as $n \to \infty$, X_n converges almost surely to an absolutely continuous random variable.

The urn of Aoudia and Perron (2012)

- Aoudia and Perron (2012) proposed a new model which at time n, M_n balls are sampled and a multiple of C_n of the drawn balls are added, where M_n and C_n are random variables.
- They showed that $\{X_n\}$ is a bounded martingale and converges almost surely.
- They also showed that $X \sim^d$ Bernoulli$(\frac{w}{w+r})$ if and only if $\sum_{n=1}^{\infty} E\left[\frac{C_{n+1}^2 M_{n+1} X_n(1-X_n)(T_n-M_{n+1})}{T_{n+1}^2(T_n-1)}\right] = \frac{wr}{(w+r)^2}$.

- Assume an urn initially contains w white and r red balls.
- After the nth adding balls, suppose $m \geq 1$ balls are randomly drawn and then note their colors, say k white and $m - k$ red balls.
- Replace the drawn balls together with $c_{n+1}k$ white and $c_{n+1}(m-k)$ red balls. Repeat the procedure *ad infinitum*.
- If $m = 1$, then the above model is Pemantle's urn.
- If $c_1 = c_2 = \cdots = c$, then the above model is Chen-Wei urn.

The replacement matrix at time n is

$$M_n = \begin{pmatrix} mc_n & 0 \\ (m-1)c_n & c_n \\ \vdots & \vdots \\ c_n & (m-1)c_n \\ 0 & mc_n \end{pmatrix}.$$

- After the nth action, let W_n, T_n and X_n defined as before.
- Then $\{X_n\}$ is a bounded martingale and so X_n converges almost surely to a random variable, say X.
- Let $\rho_n = c_n/T_{n-1}$, $n \in \mathbb{N}$.

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Theorem 1.

(i) If $\sum_{j=1}^{\infty} \rho_{j+1}^2 = \infty$, then X follows a Bernoulli distribution with parameter $w/(w+r)$.

(ii) If $\{c_n\}_{n\geq 1}$ is a bounded sequence by c, then X is absolutely continuous.

Proposition 1. (Chen and Wei(2005))

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space and let $(\Omega_n)_{n\geq 1}$ be a sequence of increasing events such that $\mathbb{P}\{\cup_{n=1}^{\infty}\Omega_n\} = 1$. If there exist nonnegative Borel measurable functions $(f_n)_{n\geq 1}$ such that $\mathbb{P}(\Omega_n \cap X^{-1}(B)) = \int_B f_n(x)dx$ for all Borel sets B, then $f = \lim_{n\to\infty} f_n$ exists almost everywhere, and f is the density of X.

Proposition 2.

For $n \geq 1$, let

$$\Omega_n = \{\omega : cm \leq W_n(\omega) \leq T_n - cm\}.$$

Then $\Omega_{n+1} \supset \Omega_n$ and $P(\cup_{n=1}^{\infty} \Omega_n) = 1$.

The proof of Theorem 1.(ii) with $m = 1$

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- For any given $\epsilon > 0$, choose $\delta = \epsilon / (T_{\ell-1} \exp\left\{-\sum_{j=\ell}^{\infty} \rho_j^2\right\}) > 0$.

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- For any given $\epsilon > 0$, choose $\delta = \epsilon / (T_{\ell-1} \exp\left\{-\sum_{j=\ell}^{\infty} \rho_j^2\right\}) > 0$.
- Let $x_1 < x_1^{'} \leq x_2 < x_2' \leq \cdots < x_s < x_s^{'}$ and $\sum_{i=1}^{s} (x_i^{'} - x_i) < \delta$.

Then by Fatou's lemma,

$$\begin{aligned}\sum_{i=1}^{s} \Pr\left(\{x_i < X < x_i^{'}\} \mid \Omega_\ell\right) &= \sum_{i=1}^{s} E[1_{\{x_i < X < x_i'\}} \mid \Omega_\ell] \\ &\leq \sum_{i=1}^{s} \liminf_{n\to\infty} E[1_{\{x_i < X_n < x_i'\}} \mid \Omega_\ell] \\ &= \sum_{i=1}^{s} \liminf_{n\to\infty} P(x_i < X_n < x_i^{'} \mid \Omega_\ell).\end{aligned}$$

Since $X_n = W_n / T_n$,

$$\sum_{i=1}^{s} \liminf_{n\to\infty} P(x_i < X_n < x_i' \mid \Omega_\ell)$$

$$= \sum_{i=1}^{s} \liminf_{n\to\infty} P(T_n x_i < W_n < T_n x_i' \mid \Omega_\ell)$$

$$= \sum_{i=1}^{s} \liminf_{n\to\infty} \left[\sum_{T_n x_i < k < T_n x_i'} \Pr(W_n = k \mid \Omega_\ell) \right]$$

$$\le \left[\sum_{i=1}^{s} (x_i' - x_i) \right] \left[\liminf_{n\to\infty} T_n \left(\max_k \Pr(W_n = k \mid \Omega_\ell) \right) \right]$$

$$\le \delta \liminf_{n\to\infty} T_n \left(\max_k \Pr(W_n = k \mid \Omega_\ell) \right).$$

Since $\rho_n = c_n / T_{n-1}$,

$$T_n = T_{n-1} + c_n = T_{n-1}(1 + \rho_n) = \cdots = T_{\ell-1} \prod_{s=\ell}^{n} (1 + \rho_s).$$

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Thus

$$\sum_{i=1}^{s} \Pr\Big(\{x_i < X < x_i'\} \mid \Omega_\ell\Big)$$

$$\leq \delta T_{\ell-1} \liminf_{n\to\infty} \left[\prod_{j=1}^{n}(1+\rho_j)\right] \left[\max_k \Pr(W_n = k \mid \Omega_\ell)\right].$$

Observe that

$$\max_k \Pr(W_n = k \mid \Omega_\ell)$$

$$= \max_k \{\Pr(W_n = k \mid W_{n-1} = k)\Pr(W_{n-1} = k \mid \Omega_\ell)$$

$$+ \Pr(W_n = k \mid W_{n-1} = k - c_n)\Pr(W_{n-1} = k - c_n \mid \Omega_\ell)\}$$

$$\leq \left(1 - \frac{k}{T_{n-1}} + \frac{k - c_n}{T_{n-1}}\right)\left(\max_k \Pr(W_{n-1} = k \mid \Omega_\ell)\right)$$

$$= (1 - \rho_n)\max_k \Pr(W_{n-1} = k \mid \Omega_\ell)$$

$$\vdots$$

$$\leq \prod_{j=\ell}^{n}(1 - \rho_j).$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{s} \Pr(x_i < X < x_i^{'} \mid \Omega_\ell) \\
\leq& \delta T_{\ell-1} \liminf_{n\to\infty} \left[\prod_{j=\ell}^{n}(1+\rho_j)\right]\left[\prod_{j=\ell}^{n}(1-\rho_j)\right] \\
\leq& \delta T_{\ell-1} \exp\left\{-\sum_{j=\ell}^{\infty}\rho_j^2\right\} \text{ (since } 1-x \leq e^{-x}) \\
<& \epsilon.
\end{aligned}$$

Hence by Theorem 31.7 of Billingsley (1995), the restriction of X to Ω_ℓ has a density.

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Thanks for Your Attention

$$
\begin{aligned}
&\max_k \Pr(W_n = k \mid \Omega_\ell) \\
&= \max_k \{ \sum_{i=0}^{m} \Pr(W_n = k \mid W_{n-1} = k - ic_n) \Pr(W_{n-1} = k - ic_n \mid \Omega_\ell \\
&\le \left(\max_k \Pr(W_{n-1} = k \mid \Omega_\ell) \right) \max_k \left\{ \sum_{i=0}^{m} \frac{\binom{k-ic_n}{i}\binom{T_{n-1}-k+ic_n}{m-i}}{\binom{T_{n-1}}{m}} \right\} \\
&\le \max_k \Pr(W_{n-1} = k \mid \Omega_\ell) \left(1 - \rho_n + \frac{\rho_n c_n (m-1)}{T_{n-1} - 1} \right) \\
&\quad \vdots \\
&\le \prod_{j=\ell}^{n} \left(1 - \rho_j + \frac{\rho_j c_j (m-1)}{T_{j-1} - 1} \right).
\end{aligned}
$$