Asymptotic behavior for a long-range Domany-Kinzel model

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Outline

- Motivation and the Model
- Main Results
- Idea of the proof

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LETTER TO THE EDITOR

A long-range Domany–Kinzel model of directed percolation

T C Li and Z Q Zhang
Institute of Physics, Chinese Academy of Sciences, Beijing, China

Received 7 June 1983

Abstract. A long-range Domany–Kinzel model proposed by Wu and Stanley is solved using random-walk formulations. In this model, for every site (i, j) in a two-dimensional lattice there is a directed bond present from site (i, j) to $(i+1, j)$ with probability one. There are also $m+1$ directed bonds present from (i, j) to $(i-k, j+1)$, $k=-1, 0, 1, 2, 3 \ldots m-1$ with respective probabilities p_{k+1} where m is any positive integer. An exact expression is obtained to determine the critical percolation angle θ_c for any distribution of p_n. The system percolates in the region $\theta_c > \theta \geqslant 0$ with probability one and zero outside this region where θ is the angle measured from the x axis. If we let m go to infinity and p_n varies as $p_1 n^{-s}$, we find that when $s \leqslant 2$, $\theta_c = \pi$. A closed form expression of θ_c is obtained for $s \geqslant 2$. When m is large but finite, θ_c is also obtained for the following two distributions: (a) $p_n = a/(a+n)$ with $a > 0$, (b) $p_n = \beta/m$ and $\beta > 0$.

Directed percolation has been the focus of much attention in the past few years. This is not only because it forms a new universality class with anisotropic scaling but also because of its close relationship to the Reggeon field theory in high-energy physics and the Markov process with breaking, recombination and absorption that occur in chemistry and biology, etc. (For a review see Kinzel (1982).) In two dimensions, various methods, like series expansion (Blease 1977, Essam and De'Bell 1981, De'Bell and Essam 1983), Monte Carlo (Kertesz and Vicsek 1980, Dhar and Barma 1981), and finite-size scaling (Kinzel and Yeomans 1981, Domany and Kinzel 1981) have been performed and much progress has been achieved.

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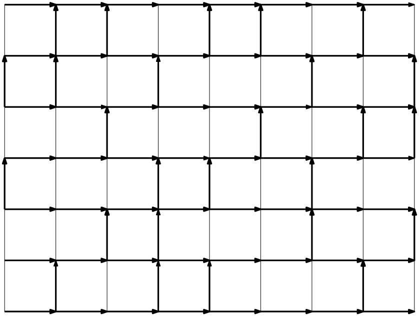
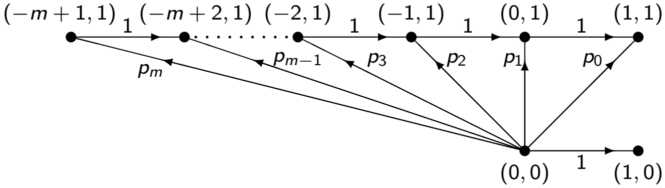
Domany-Kinzel model (1981)



In this talk we consider long-range Domany-Kinzel model as follows:



Given any $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, let
$N_\alpha = \lfloor \alpha N \rfloor = \sup\{t \in \mathbb{Z} : t \leq \alpha N\}$ with $N \in \mathbb{Z}_+$.

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Define the two point correlation function

$$\tau_m(N_\alpha, N) = \mathbb{P}_p\big((0,0) \to (N_\alpha, N)\big).$$

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Theorem (Domany and Kinzel (1981)) Let $m = 1$ and $p_0 = 0$ (square lattice). Given any $\alpha > 0$, there is $\alpha_{1,c} = (1 - p_1)/p_1$ such that

$$\lim_{N\to\infty} \tau_1(N_\alpha, N) = \begin{cases} 1 & \alpha > \alpha_{1,c}, \\ \frac{1}{2} & \alpha = \alpha_{1,c}, \\ 0 & \alpha < \alpha_{1,c}. \end{cases} \tag{1}$$

Theorem (Wu and Stanley (1982)) Let $m = 1$ (triangular lattice). Given any $\alpha > 0$, there is $\alpha_{1,c} = (1 - p_1)/(p_0 + p_1 - p_0 p_1)$ such that (1) holds.

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Theorem (Li and Zhang (1983))
Given any $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, there is

$$\alpha_{m,c} = \frac{\sum_{k=1}^{m} q_k q_{k+1} \cdots q_m - (m-1)}{1 - q_0 q_1 \cdots q_m}$$

such that

$$\lim_{N \to \infty} \tau_m(N_\alpha, N) = \begin{cases} 1 & \alpha > \alpha_{m,c}, \\ \frac{1}{2} & \alpha = \alpha_{m,c}, \\ 0 & \alpha < \alpha_{m,c}, \end{cases}$$

where $q_i = 1 - p_i,\ i = 0, 1, 2, ..., m$.

Theorem (Wu and Stanley (1982)) For triangular lattice ($m = 1$) and $\alpha < \alpha_{1,c}$ and α close to $\alpha_{1,c}$,

$$\tau_1(N_\alpha, N) \le e^{-NI(\alpha)}, \quad I(\alpha) \approx (\alpha_{1,c} - \alpha)^2.$$

We say that $f(n) \approx g(n)$ means $\lim_{n\to\infty} f(n)/g(n) \in (0, \infty)$.

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Theorem (Chang and Chen (2013)) For triangular lattice ($m = 1$) , there is $\alpha_{1,c}$ such that

$$\begin{cases} \tau_1(N_\alpha, N) \leq e^{-NI(\alpha)}, & \alpha < \alpha_{1,c}, \\ \tau_1(N_\alpha, N) = \frac{1}{2} + \frac{O(1)}{\sqrt{N}}, & \alpha = \alpha_{1,c}, \\ 1 - \tau_1(N_\alpha, N) \leq e^{-NI(\alpha)}, & \alpha > \alpha_{1,c}, \end{cases}$$

where $I(\alpha) \approx (\alpha - \alpha_{1,c})^2$.

Main result 1

Given $m \in \mathbb{Z}_+$ and $p_k \in [0,1)$, $k = 0, 1, 2, ..., m$ with $p_0 \vee p_1 \vee \cdots \vee p_m > 0$, there is a critical aspect ratio

$$\alpha_{m,c} = \frac{\sum_{k=1}^{m} q_k q_{k+1}^2 \cdots q_m^{m-k+1} - (m-1)}{1 - q_0 q_1 \cdots q_m}$$

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such that

$$\tau_m(N_{\alpha_{m,c}}, N) = \frac{1}{2} + O(\frac{1}{\sqrt{N}}),$$

and

$$\begin{aligned} \tau_m(N_\alpha, N) &\leq e^{-NI(\alpha)} \quad \text{for} \quad \alpha < \alpha_{m,c}, \\ 1 - \tau_m(N_\alpha, N) &\leq e^{-NI(\alpha)} \quad \text{for} \quad \alpha > \alpha_{m,c} \end{aligned}$$

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Moreover given any positive regularly varying sequence $\{\mathcal{L}_N\}$ where $\mathcal{L}_N \to \infty$ as $N \to \infty$, we have

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Main result 2

When $|\alpha_c - \alpha| < 1$, we have

$$\begin{aligned} \tau(N_\alpha, N) &\leq e^{-NI(\alpha)}, \quad \text{for } \alpha < \alpha_c, \\ 1 - \tau(N_\alpha, N) &\leq e^{-NI(\alpha)}, \quad \text{for } \alpha > \alpha_c, \end{aligned}$$

where

$$I(\alpha) \approx \begin{cases} \left(|\alpha - \alpha_c|\right)^{\frac{1+\epsilon}{\epsilon}}, & \text{for } s \in (3,4], \\ |\alpha - \alpha_c|^2, & \text{for } s > 4 \end{cases}$$

for any $\epsilon \in (0, s-3)$.

Main result 3

Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\ du, \quad \Psi(x) = 1 - \Phi(x) \approx_{x>0} \frac{e^{-x^2/2}}{x}.$$

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Given $p_n \approx pn^{-s}$ for $s \in (0, \infty]$ and $p \in (0, 1)$. Let $\rho \in (0, \infty)$ and a positive regularly varying sequence $\{\ell_n\}_{n=1}^{\infty}$ and let $\alpha_N^- = \alpha_c - N^{-\frac{\rho(s\wedge 4-3)}{(s\wedge 4-2)}} \ell_N$ and $\alpha_N^+ = \alpha_c + N^{-\frac{\rho(s\wedge 4-3)}{(s\wedge 4-2)}} \ell_N$. Let $\mathcal{L}_N$ is any positive regularly varying sequence where $\mathcal{L}_N \to \infty$ as $N \to \infty$.

Then for $s \in (3,4)$ as $N \to \infty$ both

$$\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)$$

$$\begin{cases} \leq \exp\left(-cN^{1-\rho}\ell_N^{\frac{s-2}{s-3}}\right), \quad \rho \in (0,1), \\ \end{cases}$$

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For $s = 4$, as $N \to \infty$ both

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Main result 3

For $s \in (4, 5)$ as $N \to \infty$ both

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For $s \in (4,5)$ as $N \to \infty$ both

$$\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)$$

$$\begin{cases} \leq \exp\left(-cN^{1-\rho}\ell_N^2\right), & \rho \in (0,1), \\ = O(1)\max\{\frac{1}{\ell_N}e^{-2\ell_N}, \frac{1}{N^{\frac{s-4}{2}}}\}, & \rho = 1, \\ & \lim_{N\to\infty} \ell_N = \infty, \\ = \Psi(L) + O(1)\max\{\ell_N - L, \frac{1}{N^{\frac{s-4}{2}}}\}, & \rho = 1, \\ & \lim_{N\to\infty} \ell_N := L \in [0,\infty), \\ = \frac{1}{2} + O\left(\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}\right), & \rho \in (1, s-3], \end{cases}$$

Main result 3

For $s \in (4,5)$ as $N \to \infty$ both

$$\tau(N_{\alpha_N^-}, N), 1-\tau(N_{\alpha_N^+}, N)$$

$$\begin{cases} \leq \exp\left(-cN^{1-\rho}\ell_N^2\right), & \rho \in (0,1), \\ = O(1)\max\{\frac{1}{\ell_N}e^{-2\ell_N}, \frac{1}{N^{\frac{s-4}{2}}}\}, & \rho = 1, \\ & \lim_{N\to\infty} \ell_N = \infty, \\ = \Psi(L) + O(1)\max\{\ell_N - L, \frac{1}{N^{\frac{s-4}{2}}}\}, & \rho = 1, \\ & \lim_{N\to\infty} \ell_N := L \in [0,\infty), \\ = \frac{1}{2} + O\left(\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}\right), & \rho \in (1, s-3], \\ = \frac{1}{2} + O\left(\frac{1}{N^{\frac{s-4}{2}}}\right), & \rho > s-3. \end{cases}$$

For $s = 5$ as $N \to \infty$ both

$$\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)$$

$$\begin{cases} \leq \exp\left(-cN^{1-\rho}\ell_N^2\right), & \rho \in (0,1), \\ \\ \\ \\ \end{cases}$$

For $s = 5$ as $N \to \infty$ both

$$\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)$$
$$\begin{cases} \leq \exp\big(-cN^{1-\rho}\ell_N^2\big), & \rho \in (0,1), \\ = O(1)\max\{\frac{1}{\ell_N}e^{-2\ell_N}, \frac{\mathcal{L}_N}{\sqrt{N}}\}, & \rho = 1, \lim_{N\to\infty}\ell_N = \infty, \end{cases}$$

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$$
\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)
\begin{cases}
\leq \exp\left(-cN^{1-\rho}\ell_N^2\right), & \rho \in (0,1), \\
= O(1) \max\{\frac{1}{\ell_N} e^{-2\ell_N}, \frac{\mathcal{L}_N}{\sqrt{N}}\}, & \rho = 1, \lim_{N\to\infty} \ell_N = \infty, \\
= \Psi(L) + O(1) \max\{\ell_N - L, \frac{\mathcal{L}_N}{\sqrt{N}}\}, & \rho = 1, \\
 & \lim_{N\to\infty} \ell_N := L \in [0, \infty), \\
= \frac{1}{2} + \frac{\ell_N}{N^{\frac{\rho}{2} - \frac{1}{2}}}, & \rho \in (1, 2], \\
= \frac{1}{2} + O\left(\frac{\mathcal{L}_N}{\sqrt{N}}\right), & \rho > 2.
\end{cases}
$$

For $s > 5$ as $N \to \infty$ both

$$\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)$$
$$\begin{cases} \leq \exp\left(-cN^{1-\rho}\ell_N^2\right), & \rho \in (0,1), \\ & \\ & \end{cases}$$

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Idea of the proof (I)

$$\tau_m(N_\alpha, N) = \sum_{k \le N_\alpha} C_{m,N}(k)$$

for $N \in \mathbb{N}$.
By the definition of our model, for any $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, we have

$$C_{m,n+1}(k) = \sum_{j \in \mathbb{Z}} C_{m,n}(k-j) D_m(j)$$

If $m = 2$, consider $D_2(0) = q_2(p_1 + p_2 - p_1 p_2)$:

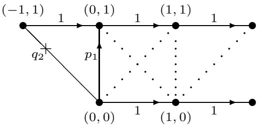

FIG. 2: $P(\{(0,0) \to (0,1)\}) = p_1 q_2$.

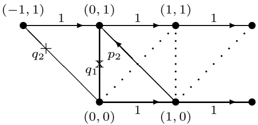

FIG. 3: $P(\{(1,0) \to (0,1)\}) = p_2(q_1 q_2)$.

consider $D_2(1) = q_1 q_2^2(1 - q_0 q_1 q_2)$

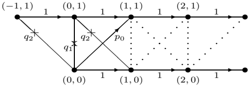

FIG. 4: $P(\{(0,0) \to (1,1)\}) = (p_0 q_1 q_2) q_2$.

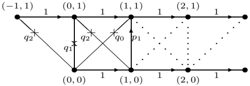

FIG. 5: $P(\{(1,0) \to (1,1)\}) = (q_0 q_1 q_2)(p_1 q_2)$.

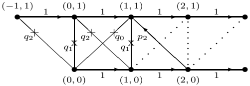

FIG. 6: $P(\{(2,0) \to (1,1)\}) = (q_0 q_1 q_2)(q_1 q_2) p_2$.

In general, we can obtain that

$$D_m(j) = \begin{cases} 0, & \text{if} \quad j \le -m, \\ 1 - q_m & \text{if } j = -m+1, \\ U_m(j) & \text{if } j \in \{-m+2, ..., 0\}, \\ (\bar{q}_m)^{j-1} (\prod_{l=1}^{m} q_l^l)(1 - \bar{q}_m) & \text{if} \quad j \ge 1, \end{cases}$$

and if $j = \{-m+2, ..., 0\}$

$$U_m(j) = q_{-j+2} q_{-j+3}^2 \cdots q_m^{m+j-1} (1 - q_{-j+1} \cdots q_m),$$

where $\bar{q}_m := q_0 q_1 \cdots q_m$.

Idea of the proof (II)

Since $\sigma^2 = \infty$ for $s \in (3, 4]$, we can't use Berry-Esseen theorem directly.

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Given any $N \in \mathbb{N}$, let

$$N_t(s) = \begin{cases} (-\alpha_c \vee 1)\left(\frac{N}{(\mathcal{L}_N)^2}\right)^{\frac{1}{s-2}}, & s \in (3, 4), \\ (-\alpha_c \vee 1)\sqrt{N}, & s \geq 4, \end{cases}$$

where $\{\mathcal{L}_N\}$ is any slowly very sequence where $\mathcal{L}_N \to \infty$ as $N \to \infty$.

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where $\{\mathcal{L}_N\}$ is any slowly very sequence where $\mathcal{L}_N \to \infty$ as $N \to \infty$.
Define $\tilde{S}_N = \sum_{k=1}^N \tilde{Y}_k$ where $\{\tilde{Y}_k\}$ is i.i.d. random variable with distribution

$$\mathrm{Prob.}(\tilde{Y}_k = j) = \tilde{D}(j) = \begin{cases} cD(j), & \text{if } j \geq -N_t(s), \\ 0, & \text{if } j < -N_t(s), \end{cases}$$

where c is normalization constant. Let $\tilde{\alpha}_{c,N} = Exp.(\tilde{Y}_1)$.

Then we obtain

$$\tilde{\alpha}_{c,N} - \alpha_c \approx \begin{cases} (\mathcal{L}_N)^{\frac{2(s-3)}{s-2}} N^{-\frac{s-3}{s-2}}, & s \in (3,4), \\ N^{-\frac{s-3}{2}}, & s \geq 4, \end{cases}$$

$$\tilde{\sigma}_N^2 := \mathrm{var}(\tilde{Y}_1) \approx \begin{cases} N^{\frac{4-s}{s-2}} (\mathcal{L}_N)^{-\frac{2(4-s)}{(s-2)}}, & s \in (3,4), \\ \mathcal{L}_N, & s = 4, \\ 1, & s > 4. \end{cases}$$

Then we obtain

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and

$$\tilde{\eta}_N = \mathrm{Exp.}(\tilde{Y}_1)^3 \approx \begin{cases} N^{\frac{5-s}{s-2}} (\mathcal{L}_N)^{-\frac{2(5-s)}{s-2}}, & s \in (3,4), \\ N^{\frac{5-s}{2}}, & s \in [4,5), \\ \mathcal{L}_N, & s = 5, \\ 1, & s > 5. \end{cases}$$

Use Berry-Esseen theorem we obtain

$$
\begin{aligned}
&\text{Prob.}(\tilde{S}_N \le \alpha_c N) \\
= \;& \int_{-\infty}^{\frac{N(\alpha_c - \tilde{\alpha}_{c,N})}{\sqrt{N\tilde{\sigma}_N^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du + \frac{O(\tilde{\eta}_N)}{\sqrt{N}\tilde{\sigma}_N^3} \\
= \;& \begin{cases} O\left(\frac{1}{\mathcal{L}_N}\right), & s \in (3,4), \\ & \\ & \end{cases}
\end{aligned}
$$

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= & \begin{cases} O\left(\frac{1}{\mathcal{L}_N}\right), & s \in (3,4), \\ \frac{1}{2} + O\left(\frac{1}{\sqrt{\mathcal{L}_N}}\right), & s = 4, \\ & \\ & \end{cases}
\end{aligned}$$

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= & \begin{cases} O\left(\frac{1}{\mathcal{L}_N}\right), & s \in (3,4), \\ \frac{1}{2} + O\left(\frac{1}{\sqrt{\mathcal{L}_N}}\right), & s = 4, \\ \frac{1}{2} + O\left(\frac{1}{N^{\frac{s-4}{2}}}\right), & s \in (4,5), \end{cases}
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\end{aligned}
$$

Use Berry-Esseen theorem we obtain

$$\begin{aligned}
&\mathrm{Prob.}(\tilde{S}_N \leq \alpha_c N) \\
&= \int_{-\infty}^{\frac{N(\alpha_c - \tilde{\alpha}_{c,N})}{\sqrt{N\tilde{\sigma}_N^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + \frac{O(\tilde{\eta}_N)}{\sqrt{N}\tilde{\sigma}_N^3} \\
&= \begin{cases} O\big(\frac{1}{\mathcal{L}_N}\big), & s \in (3,4), \\ \frac{1}{2} + O\big(\frac{1}{\sqrt{\mathcal{L}_N}}\big), & s = 4, \\ \frac{1}{2} + O\big(\frac{1}{N^{\frac{s-4}{2}}}\big), & s \in (4,5), \\ \frac{1}{2} + O\big(\frac{\mathcal{L}_N}{\sqrt{N}}\big), & s = 5, \\ \frac{1}{2} + O\big(\frac{1}{\sqrt{N}}\big), & s > 5. \end{cases}
\end{aligned}$$

Since

$$\mathrm{Prob.}(S_N \leq \alpha_c N) = \mathrm{Prob.}(\tilde{S}_N \leq \alpha_c N)$$
$$+\mathrm{Prob.}(S_N \leq \alpha_c N, \exists j \in \{1, 2., , , N\} \text{ such that } Y_j < (\alpha_c \wedge -1)N),$$

and

$$\begin{aligned} & \mathrm{Prob.}(S_N \leq \alpha_c N : \exists Y_j < -N_t(s)) \\ \leq \; & \mathrm{Prob.}(\exists j \in \{1, 2., , , N\} \text{ such that } Y_j < -N_t(s)) \\ \leq \; & N \max_{j > N_t(s)} D(-j) \\ \approx \; & \begin{cases} \dfrac{(\mathcal{L}_N)^{\frac{2(s-1)}{s-2}}}{N^{\frac{1}{s-2}}}, & s \in (3,4), \\ N^{-\frac{s-3}{2}}, & s \geq 4. \end{cases} \end{aligned}$$

With the definition of N_α given in the introduction, we have

$$\tau(N_{\alpha_c}, N) \approx \mathrm{Prob.}(S_N \leq \alpha_c N) \approx \mathrm{Prob.}\left(\tilde{S}_N \leq \alpha_c N\right).$$

Idea of the proof (III)

Using Chernov's inequality, we obtain that when $\alpha \neq \alpha_c$,

$$\mathrm{Prob.}(S_N \leq N_\alpha) < e^{-NI(\alpha)}, \quad \mathrm{Prob.}(S_N > N_\alpha) \leq e^{-NI(\alpha)}$$

where

$$I(\alpha) = \sup_{t \in (0,1)} \left\{ \alpha \log t - \log \hat{D}(t) \right\} := \alpha \log t_\alpha - \log \hat{D}(t_\alpha).$$

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Hence

$$\frac{\alpha}{t_\alpha} = \frac{\frac{d\hat{D}(t_\alpha)}{d\,t}}{\hat{D}(t_\alpha)}.$$

Note that $\frac{d^2\,\hat{D}(t_\alpha)}{d\alpha^2}|_{\alpha=\alpha_c} = \infty$ for $s \in (3,4]$, So we use the fractional derivative.

It is easy to check that for $s \in (3,4]$

$$\frac{d^{1+\epsilon}\,\hat{D}(t_\alpha)}{d\alpha^{1+\epsilon}}|_{\alpha=\alpha_c} = \frac{d^{\epsilon}\,\hat{D}'(t_\alpha)}{d\alpha^{\epsilon}}|_{\alpha=\alpha_c} = O(1)\sum_{n\in\mathbb{Z}} n^{1+\epsilon}D(n)$$

exists, so, $\frac{d^\epsilon\, t_\alpha}{d\alpha^\epsilon}|_{\alpha=\alpha_c}$ also exists, for $\epsilon \in (0, s-3)$. By Taylor formula, for $\epsilon \in (0, s-3)$ and $|\alpha - \alpha_c| \in (0,1)$, we have

$$I(\alpha) = (\alpha - \alpha_c)(t_\alpha - t_{\alpha_c}) + O(1)(t_\alpha - t_{\alpha_c})^{1+\epsilon},$$

and

$$t_\alpha - t_{\alpha_c} = O(1)(\alpha - \alpha_c)^{\frac{1}{\epsilon}}.$$

Then

$$I(\alpha) \approx |\alpha - \alpha_c|^{\frac{1+\epsilon}{\epsilon}}, \quad \epsilon \in (0, s-3).$$

Thank You