Cutoff for 1-dim variable speed random walks

Guan-Yu Chen
(Joint with Takashi Kumagai)

Chiao Tung Uniersity, Taiwan

The 13th Workshop
on Markov Processes and Related Topics

Outline

Total variation and its mixing time

Consider an irreducible Markov chain on a finite set $\mathcal{S}$ with transition matrix K and stationary distribution π.

- The total variation of $(\mathcal{S}, K, \pi)$ is defined by

$$d_{\text{TV}}(t) := \max_{x\in\mathcal{S}} \|K^t(x,\cdot) - \pi\|_{\text{TV}} = \max_{x\in\mathcal{S},\, A\subset\mathcal{S}} \{K^t(x,A) - \pi(A)\}.$$

- The mixing time of the total variation is defined by

$$T_{\text{TV}}(\epsilon) = \min\{t \geq 0 | d_{\text{TV}}(t) \leq \epsilon\}, \quad \forall \epsilon \in (0,1).$$

Simple random walks with reflecting boundaries

Let $\mathcal{S}=\{0,1,...,n\}$ and K,π be given by

$$\begin{cases} K(i,i\pm 1)=1/2, \ \forall 0<i<n, \\ K(0,1)=K(n,n-1)=1, \end{cases}, \quad \begin{cases} \pi(i)=1/n, \ \forall 0<i<n, \\ \pi(0)=\pi(n)=1/(2n). \end{cases}$$

Let $\mathcal{S}'=\mathbb{Z}_{2n}$ and K',π' be given by

$$K'(i,i\pm 1)=1/2, \quad \pi'(i)=1/(2n).$$

Through the mapping $\{i,2n-i\}\subset\mathcal{S}'\mapsto i\in\mathcal{S}$, one has

$$d_{\text{TV}}(t):=\max_{x\in\mathcal{S}}\|K^t(x,\cdot)-\pi\|_{\text{TV}}=d'_{\text{TV}}(t):=\max_{y\in\mathcal{S}'}\|(K')^t(y,\cdot)-\pi'\|_{\text{TV}}.$$

Bounding mixing times using spectral information

Observe that K' has the following eigenvalues and normalized eigenvectors

$$\beta_i = \cos\frac{\pi i}{n}, \quad \phi_i(j) = \sqrt{2}\cos\frac{\pi ij}{n}, \quad i,j \in \mathcal{S}'.$$

To see an upper bound, set $d_2'(t) = \max_x \|(K')^t(x,\cdot)/\pi' - 1\|_{\ell^2(\pi')}$. Then,

$$d_{\mathrm{TV}}'(t) \le \frac{1}{2}d_2'(t) = \frac{1}{2}\left(\sum_{i=1}^n \beta_i^{2t}\right)^{1/2} = \frac{1}{2}\left(\sum_{i=1}^n \left(\cos\frac{\pi i}{n}\right)^{2t}\right)^{1/2}.$$

For a lower bound, taking ϕ_1 as a testing function yields

$$d_{\mathrm{TV}}'(t) = \frac{1}{2}\|(K')^t - \pi'\|_{\ell^\infty(\pi')\to\ell^\infty(\pi')} \ge \frac{(K')^t(0,\phi_1/\sqrt{2})}{2} = \frac{1}{2}\left(\cos\frac{\pi}{n}\right)^t.$$

Constant speed random walks

As the random walk $(\mathcal{S}', K', \pi')$ does not converge (in distribution), let's consider one continuous time variant, called the constant speed random walk, $(\mathcal{S}', Q, \pi')$ with infinitesimal generator

$$Q(i,j) = K'(i,j) \quad \forall i \neq j, \quad Q(i,i) = -1 \quad \forall i.$$

By setting $H_t = e^{tQ}$ and $d''_{\text{TV}}(t) := \max_y \|H_t(y,\cdot) - \pi'\|_{\text{TV}}$, one has

$$\frac{1}{2} e^{-\lambda_1 t} \leq d''_{\text{TV}}(t) \leq \frac{1}{2} \left(\sum_{i=1}^{n} e^{-2\lambda_i t} \right)^{1/2}.$$

where $\lambda_i = 1 - \cos(\pi i/n)$. As a result of $\theta^2/5 \leq 1 - \cos\theta \leq \theta^2/2$ with $|\theta| \leq \pi$, there are $C(\epsilon) > 1$ for $\epsilon \in (0,1)$ such that

$$C(\epsilon)^{-1} n^2 \leq T''_{\text{TV}}(\epsilon) \leq C(\epsilon) n^2 \quad \text{for } n \text{ large enough.}$$

Some remarks on classical techniques

Remarks.

- A similar upper bound can be obtained by the tail probability of a random time, say the coupling time or the strong stationary time.
- A similar lower bound can be derived by the Kolmogorov inequality.
- For multi-dimensional walks, let $d \in \mathbb{N}$, $\mathcal{S} = (\mathbb{Z}_{2n})^d$, $\pi(x) = (2n)^{-d}$ and

$$Q(x,y) = \frac{1}{2d} \quad \forall |x-y| = \sum_{i=1}^{d} |x_i - y_i| = 1, \quad Q(x,x) = -1 \quad \forall x.$$

For $\epsilon \in (0,1)$, there are $C(\epsilon) > 1$ such that

$$C(\epsilon)^{-1}(d \log d)n^2 \le T_{\text{TV}}(\epsilon) \le C(\epsilon)(d \log d)n^2 \quad \text{for } n \text{ large enough.}$$

Cutoffs for Markov chains

A family of continuous time Markov chains, $\mathcal{F} = (\mathcal{S}_n, Q_n, \pi_n)_{n=1}^{\infty}$, with mixing times, $(T_{n,\mathrm{TV}})_{n=1}^{\infty}$, presents a cutoff in the total variation if

$$\lim_{n\to\infty} \frac{T_{n,\mathrm{TV}}(\epsilon)}{T_{n,\mathrm{TV}}(1-\epsilon)} = 1 \quad \forall \epsilon \in (0,1),$$

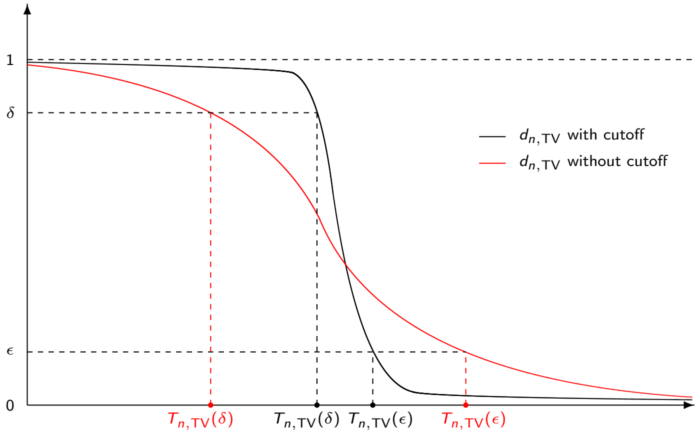
or, equivalently, there is a sequence of positive reals $(t_n)_{n=1}^{\infty}$ s.t.

$$\lim_{n\to\infty} d_{n,\mathrm{TV}}(at_n) = \begin{cases} 0 & \text{for } a > 1, \\ 1 & \text{for } 0 < a < 1. \end{cases}$$

The sequence $(t_n)_{n=1}^{\infty}$ is called a total variation cutoff time for $\mathcal{F}$.

Remark. For the d-dimensional constant speed random walks, the family indexed by the sizes of state spaces has no cutoff in the total variation.

Mixing times and cutoffs



Outline

Randomized products

For $n \geq 1$, let $\mathcal{X}_n = \{0, 1, ..., n\}$ and R_n be a Q-matrix on $\mathcal{X}_n$ given by

$$\begin{cases} R_n(i, i+1) = R_n(i, i-1) = 1/2, \ \forall 0 < i < n, \\ R_n(0, 1) = R_n(n, n-1) = 1, \ R_n(i, i) = -1. \end{cases}$$

Consider the family $(\mathcal{S}_n, Q_n, \pi_n)_{n=1}^{\infty}$, where $\mathcal{S}_n = \mathcal{X}_n^n$ and

$$Q_n(x, y) = \sum_{i=1}^{n} \xi_i \delta_{\check{x}_i}(\check{y}_i) R_n(x_i, y_i) \quad \forall x, y \in \mathcal{S}_n,$$

where $\check{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and $(\xi_n)_{n=1}^{\infty}$ is a sequence of positive random variables.

Convergence of randomized products

If ν_n is the stationary distribution of R_n, then

$$\pi_n(x) = \nu_n(x_1) \times \cdots \times \nu_n(x_n).$$

Let $P_{n,t} = e^{tR_n}$ and $H_{n,t} = e^{tQ_n}$. Then, one has

$$H_{n,t}(x,y) = P_{n,t\xi_1}(x_1,y_1) \times \cdots \times P_{n,t\xi_n}(x_n,y_n).$$

How to analyze the total variation of $(\mathcal{S}_n, Q_n, \pi_n)$?

- Coupling for multi-dimensional random walks.
- Spectral analysis with $(n+1)^n$ eigenvalues and eigenvectors.
- The total variation of $(\mathcal{X}_n, R_n, \nu_n)$.

The Hellinger distance

For any irreducible Markov chain $(\mathcal{S}, K, \pi)$, the Hellinger distance is defined by

$$d_H(t) = \sup_{x\in\mathcal{S}} \left(\frac{1}{2} \sum_{y\in\mathcal{S}} \left(\sqrt{K^t(x,y)} - \sqrt{\pi(y)} \right)^2 \right)^{1/2} \in [0,1].$$

(C. and Kumagai 2016) If d_{TV} is the total variation of $(\mathcal{S}, K, \pi)$, then

$$1 - \sqrt{1 - d_{\text{TV}}^2(t)} \le d_H^2(t) \le d_{\text{TV}}(t).$$

This implies

Cutoff in the total variation $\Leftrightarrow$ Cutoff in the Hellinger distance.

Hellinger distances of product chains

Let $d_{n,H}, D_{n,H}$ be Hellinger distances of $(\mathcal{X}_n, R_n, \nu_n), (\mathcal{S}_n, Q_n, \pi_n)$. Then,

$$1 - D_{n,H}^2(t) = \prod_{i=1}^{n}(1 - d_{n,H}^2(\xi_i t)) \le 1 - d_{n,H}^2(\zeta_n t),$$

where $\zeta_n = \min\{\xi_1, ..., \xi_n\}$, and there is a universal constant $C > 1$ s.t.

$$C^{-1}e^{-Ct/n^2} \le d_{n,H}(t) \le \frac{Ce^{-C^{-1}t/n^2}}{1 - e^{-C^{-1}t/n^2}} \quad \forall t > 0,\ n \ge 1.$$

This implies that, for $a > 0$,

$$-\log(1 - D_{n,H}^2(t)) \asymp \sum_{i=1}^{n} d_{n,H}^2(\xi_i t) \quad \forall t \ge a\zeta_n^{-1} n^2.$$

Cutoffs for randomized products

(C. and Kumagai 2017) The family $(\mathcal{S}_n, Q_n, \pi_n)_{n=1}^\infty$ has a cutoff if and only if there is a sequence of positive reals $(t_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} f_n(at_n) = \begin{cases} \infty & \forall a \in (0,1), \\ 0 & \forall a \in (1,\infty), \end{cases}, \quad f_n(t) = \sum_{i=1}^n e^{-\xi_i t/n^2}. \tag{1}$$

(C. and Saloff-Coste 2010) Let $(\xi_i^{(n)})_{i=1}^n$ be an order statistics of $(\xi_i)_{i=1}^n$. Then, (1) holds if and only if

$$\lim_{n\to\infty} \xi_1^{(n)} \max_{1\le i\le n} \frac{\log(i+1)}{\xi_i^{(n)}} = \infty.$$

Some examples

1. For some $c > 0$, $\mathbb{P}(\xi_n = c, \forall n) = 1$. In this case, there is a total variation cutoff for $(\mathcal{S}_n, Q_n, \pi_n)_{n=1}^{\infty}$ and $T_{n,\mathrm{TV}}(\epsilon) \asymp n^2 \log n$.

2. Suppose that there is an increasing sequence $(a_n)_{n=1}^{\infty}$ such that ξ_n/a_n converges a.s. to a positive random variable. Then, $(\mathcal{S}_n, Q_n, \pi_n)_{n=1}^{\infty}$ has a cutoff with probability one if and only if $\sup_n \log(n+1)/a_n = \infty$.

3. If $(\theta_n)_{n=1}^{\infty}$ is a i.i.d. sequence of positive random variables with finite mean and $\xi_n = \theta_1 + \cdots + \theta_n$, then, in the setting of **2.**, one has $a_n = n$ and thus $(\mathcal{S}_n, Q_n, \pi_n)_{n=1}^{\infty}$ has no cutoff with probability one.

Outline

Random walks with random hopping times

For $n \in \mathbb{N}$, let $\mathcal{S}_n = \{0, 1, ..., n\}$ and K_n be given by

$$K_n(i, i \pm 1) = 1/2 \quad \forall 0 < i < n, \quad K_n(0, 1) = K_n(n, n-1) = 1.$$

Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of positive random variables and set

$$Q_n(i, j) = \xi_i K_n(i, j) \quad \forall i \neq j, \quad Q_n(i, i) = -\xi_i.$$

The continuous time Markov chain with Q-matrix Q_n has stationary distribution

$$\pi_n(i) = \pi_n(0) \frac{2\xi_0}{\xi_i} \quad \forall 0 < i < n, \quad \pi_n(n) = \pi_n(0) \frac{\xi_0}{\xi_n}.$$

Review of cutoff criterion

(C. and Saloff-Coste, 2015) Consider the family $\mathcal{F} = (\mathcal{S}_n, Q_n)_{n=1}^{\infty}$, where $\mathcal{S}_n = \{0, ..., n\}$ and Q_n is a Q-matrix satisfying $Q_n(i, i \pm k) = 0$ for $k > 1$.

Notations:

1. Let $\tau_i^{(n)}$ be the first hitting time to i of the chain $(\mathcal{S}_n, Q_n)$.

2. Let M_n be a state such that $\pi_n([0, M_n]) \asymp \pi_n([M_n, n])$.

3. Set $t_n = \mathbb{E}_0 \tau_{M_n}^{(n)} \vee \mathbb{E}_n \tau_{M_n}^{(n)}$ and $b_n^2 = \mathrm{Var}_0 \tau_{M_n}^{(n)} \vee \mathrm{Var}_n \tau_{M_n}^{(n)}$.

Then, one has

$$\mathcal{F} \text{ has a cutoff in the total variation} \quad \Leftrightarrow \quad b_n = o(t_n).$$

Further, t_n is the cutoff time.

Computations of means and variances of hitting times

By writing $Q_n = (q_{i,j}^{(n)})$, one has

$$\mathbb{E}_i\tau_{i+1}^{(n)} = \frac{\pi_n([0,i])}{\pi_n(i)q_{i,i+1}^{(n)}}, \quad \mathrm{Var}_i\tau_{i+1}^{(n)} = \frac{1}{\pi_n(i)q_{i,i+1}^{(n)}}\sum_{j=0}^{i}\pi_n(j)(\mathbb{E}_j\tau_i^{(n)} + \mathbb{E}_j\tau_{i+1}^{(n)}).$$

In the case of VSRW, we have

$$\mathbb{E}_0\tau_i^{(n)} = i\xi_0^{-1} + 2\sum_{k=1}^{i-1}(\xi_1^{-1} + \cdots + \xi_k^{-1}) \asymp \sum_{k=0}^{i-1}(\xi_0^{-1} + \cdots + \xi_k^{-1})$$

and

$$\mathrm{Var}_0\tau_i^{(n)} \asymp \sum_{j=0}^{i-1}\sum_{k=0}^{j}(\xi_0^{-1} + \cdots + \xi_k^{-1})^2.$$

Cutoffs for variable speed birth and death chains

(C. and Kumagai, 2017, ongoing)

Assumption: $(\xi_n^{-1})_{n=0}^{\infty}$ is an i.i.d. sequence with finite mean, say μ.

By the SLLN, we may select $M_n = [n/2]$ almost surely and obtain

$$\mathbb{E}_0 \tau_{M_n}^{(n)} \asymp \sum_{k=0}^{M_n-1} (\xi_0^{-1} + \cdots + \xi_k^{-1}) \sim \frac{\mu n^2}{4} \quad \text{almost surely,}$$

and

$$\mathrm{Var}_0 \tau_{M_n}^{(n)} \asymp \sum_{j=0}^{M_n-1} \sum_{k=0}^{j} (\xi_0^{-1} + \cdots + \xi_k^{-1})^2 \sim \frac{\mu^2 n^4}{192} \quad \text{alomost surely.}$$

Cutoffs for variable speed birth and death chains

By writing

$$\xi_{n-k}^{-1}+\cdots+\xi_n^{-1}=(\xi_0^{-1}+\cdots+\xi_n^{-1})-(\xi_0^{-1}+\cdots+\xi_{n-k-1}^{-1}),$$

one may apply the same computation as before to achieve

$$\mathbb{E}_n\tau_{M_n-1}^{(n)}\asymp\sum_{k=0}^{n-M_n-1}(\xi_{n-k}^{-1}+\cdots+\xi_n^{-1})\sim\frac{\mu n^2}{4}\quad\text{almost surely,}$$

and

$$\mathrm{Var}_n\tau_{M_n}^{(n)}\asymp\sum_{j=0}^{n-M_n-1}\sum_{k=0}^{j}(\xi_{n-k}^{-1}+\cdots+\xi_n^{-1})^2\sim\frac{\mu^2 n^4}{192}\quad\text{alomost surely.}$$

This implies that $\mathcal{F}$ has no cutoff with probability one.

Some remarks

- (C. and Saloff-Coste 2013) Let λ_n be the spectral gap of Q_n, i.e. the smallest non-zero eigenvalue of $-Q_n$. Then,

$$\mathcal{F} \text{ has a cutoff} \quad \Leftrightarrow \quad t_n\lambda_n \to \infty, \quad t_n = \mathbb{E}_0\tau_{M_n}^{(n)} \vee \mathbb{E}_n\tau_{M_n}^{(n)}.$$

  As no subfamily of $(\mathcal{S}_n, Q_n)_{n=1}^{\infty}$ presents a cutoff, $\lambda_n \asymp t_n^{-1} \asymp n^{-2}$ almost surely.

- (M.-F. Chen) Let $C = \pi_n([0, M_n]) \wedge \pi_n([M_n, n])$ and

$$\ell_n = \max\left\{\max_{j<M_n}\sum_{k=j}^{M_n-1}\frac{\pi_n([0,j])}{\pi_n(k)q_{k,k+1}^{(n)}}, \max_{j>M_n}\sum_{k=M_n+1}^{j}\frac{\pi_n([j,n])}{\pi_n(k)q_{k,k-1}^{(n)}}\right\}.$$

  Then,

$$\frac{1}{4\ell_n} \le \lambda_n \le \frac{2}{C\ell_n}.$$

Outline

Perspectives of future work

What could we say about:

- the cutoff for the randomized product of random walks with i.i.d. $(\xi_n)_{n=1}^\infty$;
- the cutoff for the one-dimensional VSRW with i.i.d. $(\xi_n^{-1})_{n=1}^\infty$ and $\mathbb{E}\xi_1^{-1} = \infty$;
- the cutoff for the randomized product of random walks with random hopping times;
- the scaling limit process and its mixing time.

Reference

1. Chen, G.-Y. and Saloff-Coste, L. *The L^2-cutoff for reversible Markov processes*, 2010.
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3. Chen, G.-Y. and Kumagai, T. *Cutoffs for product chains*, 2017.
4. Chen, G.-Y. and Kumagai, T. *Products of random walks on finite groups with moderate growth*, 2017.

Thank you for your attention!