Survival-extinction behaviors for nonlinear continuous state branching processes

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Outline of the Talk

1. Introduction
   - Galton-Watson branching process
   - Continuous state branching process
   - Continuous state branching process with nonlinear branching
   - Survive or die out?
2. The extinction and explosion behaviors of nonlinear branching process
   - The extinction
   - Possible explosion
   - About the proofs

Galton-Watson branching process

- Let $(\xi_{n,i})$ be i.i.d. random variables representing the number of children of the i-th individual in the $(n-1)$-th generation.
- Let

$$X_n := \sum_{i=1}^{X_{n-1}} \xi_{n,i}$$

be the population of the n-th generation. (X_n) a Galton-Watson branching process.

- Let ϕ be a nonnegative integer-valued function on $\mathbb{N}$.

$$X_n := \sum_{i=1}^{\phi(X_{n-1})} \xi_{n,i}.$$

is a ϕ-controlled Galton-Watson branching process.

From Galton-Watson process to Feller diffusion

- Write $\mu = \mathbb{E}\xi_{1,1} < \infty$ and $b = 1 - \mu$. Then

$$X_k - X_{k-1} = (\mu - 1)X_{k-1} + \sum_{i=1}^{X_{k-1}} (\xi_{k,i} - \mu)$$

and

$$X_n - X_0 = -\sum_{k=1}^{n} bX_{k-1} + \sum_{k=1}^{n} \sum_{i=1}^{X_{k-1}} (\xi_{k,i} - \mu).$$

- If $b = 0$, the branching is critical.

- Taking a time-space scaling limit, under certain conditions we have the Feller diffusion process

$$X_t = X_0 - \int_0^t bX_s \mathrm{d}s + \int_0^t \int_0^{\gamma X_s} W(\mathrm{d}s, \mathrm{d}u), \qquad (1)$$

where $W(\mathrm{d}s, \mathrm{d}u)$ is a time-space white noise.

- Solution of (1) has the same distribution as the solution to

$$X_t = X_0 - \int_0^t bX_s \mathrm{d}s + \int_0^t \sqrt{\gamma X_s} \mathrm{d}B_s,$$

where B is a Brownian motion. Here γ is the branching rate.

Continuous state branching process (CB processes)

- Feller diffusion is an example of continuous state branching processes.
- A continuous state branching process arises as a scaling limit of Galton-Watson processes.
- It is a nonnegative Markov process X (with possible positive jumps) satisfying the branching property, i.e. for any $\lambda, x, y > 0$
$$\mathbb{E}_{x+y} e^{-\theta X_t} = \mathbb{E}_x e^{-\theta X_t} \mathbb{E}_y e^{-\theta X_t}.$$
- The branching property plays a key role in analyzing CB processes.

- Its Laplace transform is determined by

$$\mathbb{E}_x e^{-\theta X_t} = e^{-x u_t(\theta)}$$

where function $u_t(\theta)$ satisfies the differential equation

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0$$

with $u_0(\theta) = \theta$ and

$$\psi(\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\pi(\mathrm{d}x)$$

for $q, \sigma \geq 0, a \in \mathbb{R}$ and for σ-finite measure π on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge z^2)\pi(\mathrm{d}z) < \infty$.
X is critical if $a = 0$.

- CB process is associated with a spectrally positive Lévy process via the Lamperti time change.

The extinction of a continuous state branching process X

- Let

  $$\tau_0^- := \inf\{t > 0 : X_t = 0\}.$$

  be the extinction time of X and

  $$p(x) := \mathbb{P}_x\{\tau_0^- < \infty\}$$

  be the extinction probability.
- (Grey condition) If $\psi(\infty) = \infty$, then $p(x) > 0$ for some (and then for all) $x > 0$ if and only if $\int^\infty 1/\psi(\xi)\mathrm{d}\xi < \infty$.

CB process as solution of SDE

- By Bertoin and Le Gall (2003, 2005, 2006), Dawson and Li (2006, 2012), a continuous state branching process solves the following SDE.

$$X_t = X_0 + \int_0^t X_s \mathrm{d}s + \sigma \int_0^t \sqrt{X_{s-}} \mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{X_{s-}} x \tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u),$$

where $\tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u)$ is a compensated Poisson random measure on $[0, \infty) \times (0, \infty) \times [0, \infty)$ with compensator $\mathrm{d}s\pi(\mathrm{d}x)\mathrm{d}u$, where the jump measure π satisfies $\int_0^\infty x \wedge x^2 \pi(\mathrm{d}x) < \infty$.

CB process with population dependent branching rate

- An continuous state branching process with population dependent branching rate can be identified as the solution (up to the first time of reaching 0) to the following SDE.

$$
\begin{aligned}
X_t =& X_0 + \int_0^t \gamma_0(X_s)\mathrm{d}s + \sigma \int_0^t \sqrt{\gamma_1(X_{s-})}\mathrm{d}B_s \\
&+ \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} x\tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u),
\end{aligned} \tag{2}
$$

  where $\gamma_0, \gamma_1, \gamma_2$ are nonnegative functions.
- Such a process is also called a nonlinear branching process.
- Informally it can be treated as a scaling limit of controlled Galton-Watson branching process.
- It does not have the branching property.

Previous work on continuous state nonlinear branching process

- For nonlinear CSBP with $\gamma_0(\cdot) = \gamma_1(\cdot) = \gamma_2(\cdot)$, using Lamperti transform Li (2016) obtained necessary and sufficient conditions of extinction and explosion for the nonlinear continuous state branching process.
- The approach of Li (2016) can not be applied to handle the case with non-identical rate functions $\gamma_i(\cdot)$, $i = 1, 2, 3$.

A critical CB process that dies out

- To see how nonlinear branching rate affects the extinction behavior, we first consider the critical Feller branching process determined by SDE $\mathrm{d}X_t = \sqrt{\gamma X_t}\mathrm{d}B_t$.
- It is well known that Feller branching process dies out within a finite time; i.e. with probability one, $X_t = 0$ for t large enough.
- The random fluctuation due to branching causes the population to die out.
- The Feller branching process has a constant branching rate. For a nonlinear branching process whose branching rate goes to 0 as the population size goes to 0, a natural question is whether the process still dies out within a finite time.

A nonlinear CB process that survives forever

- We consider a nonlinear Feller branching process that solves SDE $\quad \mathrm{d}X_t = \sqrt{X_t^r}\mathrm{d}B_t \quad$ for $r > 0$.
- It is not hard to believe that X_t converge as $t \to \infty$.
- For $r = 1$, X is the Feller branching process and $X_t = 0$ for t large enough.
- On the other hand, this SDE is solvable if $r = 2$.

$$X_t = X_0 e^{B_t - \frac{1}{2}t}.$$

Clearly, $X_t \to 0$ as $t \to \infty$, but $X_t > 0$ for all t.

- What happens for $1 < r < 2$?

Recall that the continuous state nonlinear branching process X solves the SDE

$$\begin{aligned} X_t =& X_0 + \int_0^t \gamma_0(X_s)\mathrm{d}s + \sigma \int_0^t \sqrt{\gamma_1(X_{s-})}\mathrm{d}B_s \\ &+ \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} x\tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u), \end{aligned} \tag{3}$$

where $\gamma_1(x)/x$ and $\gamma_2(x)/x$ can be treated as the branching rates. We assume that the above SDE allows a unique weak solution up to the first time of hitting 0 or explosion.

Discussions

- We want to know whether the nonlinear CB process dies out with a positive probability.
- Since the process has no negative jumps, the extinction behavior depends on the values of $\gamma_i(x)$ for x close to 0. $\gamma_0(\cdot)$ tells us the direction and size of the drift, and $\gamma_1(\cdot), \gamma_2(\cdot)$ tell us the magnitude of fluctuation of the process near 0.
- Suppose that for $i = 0, 1, 2, \quad \gamma_i(x) \to 0$ as $x \to 0+$.
- Extinction can be caused either by a relatively large negative drift related to γ_0 or by relatively large fluctuations related to γ_1 and γ_2. Otherwise, the process survives.

Extinction behaviors

$$G_a(u) = \frac{a-1}{u}\gamma_0(u) - \frac{a(a-1)}{2u^2}\sigma^2\gamma_1(u) - \gamma_2(u)u^{a-1}\int_0^\infty [(u+z)^{1-a} - u^{1-a} - (1-a)zu^{-a}]\pi(\mathrm{d}z).$$

Theorem

(Survive with probability one) If there exist $a > 1$ and $r < 1$ such that $G_a(u) \geq -(\ln u^{-1})^r$ for all small $u > 0$, then $\mathbb{P}_x\{\tau_0^- = \infty\} = 1$ for all small $x > 0$.

Theorem

(Extinction with a positive probability) If there exist $0 < a < 1$ and $r > 1$ such that $G_a(u) \geq (\ln u^{-1})^r$ for all small $u > 0$, then $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ for all small $x > 0$.

For simplicity we only present the results for a CB process with rate functions of the forms

$$\gamma_i(x) = b_i x^{r_i}, i = 0, 1, 2, \ \pi(\mathrm{d}x) = x^{-1-\alpha}\mathrm{d}x \ \ \text{for} \ \ x \approx 0$$

and $1 < \alpha < 2$, $b_0 \in \mathbb{R}, b_1, b_2, r_0, r_1, r_2 \geq 0$.
If r_i is bigger, $\gamma_i(x)$ is smaller.
Let

$$\tau_0^- = \inf\{t : X_t = 0\}$$

be the extinction time.

Corollary

$\mathbb{P}_x\{\tau_0^- = \infty\} = 1$ for all $x > 0$ if one of the following two sets of conditions holds.

- *(i) $b_0 < 0$, $r_0 \geq 1$ and*
  - *(ia) if $\sigma > 0$, then $r_1 \geq 2$;*
  - *(ib) if $\pi \neq 0$, then $r_2 \geq \alpha$;*
- *(ii) $b_0 > 0$ and*
  - *(iia) if $\sigma > 0$, then either $r_1 \geq 2$ or $r_1 > r_0 + 1$ or $r_1 = r_0 + 1$ and b_0 is much bigger than b_1;*
  - *(iib) if $\pi \neq 0$, then either $r_2 \geq \alpha$ or $r_2 > r_0 - 1 + \alpha$ or $r_2 = r_0 + \alpha - 1$ and b_0 is much bigger than b_2.*

Corollary

$\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ for all $x > 0$ if one of the following two sets of conditions holds.

- *(i) $b_0 \le 0$ and one of the following holds.*
  - *(ia) $0 < r_0 < 1$ if $b_0 < 0$.*
  - *(ib) $0 < r_1 < 2$ if $\sigma \neq 0$.*
  - *(ic) $0 < r_2 < \alpha$ if $\pi \neq 0$.*
- *(ii) $b_0 > 0$ and one of the following holds.*
  - *(iia) If $\sigma > 0$, then either $r_1 < (r_0 + 1) \wedge 2$ or $r_1 = r_0 + 1 < 2$ and b_1 is big enough.*
  - *(iib) If $\pi \neq 0$, then either $r_2 < (r_0 + \alpha - 1) \wedge \alpha$ or $r_2 = r_0 + \alpha - 1 < \alpha$ and b_2 is big enough.*

Remark

For $b_0 = 0$, $\pi = 0$ *and* $\gamma_1(x) = x^{r_1}$, *then combining the above Theorems, for any* $x > 0$ *we have* $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ *for* $0 \le r_1 < 2$ *and* $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ *but* $X_t \to 0$ $\mathbb{P}_x$*-a.s. for* $r_1 \ge 2$.
This answers the question early in the talk.

Remark

If $b_0 = 0$, $\sigma = 0$ *and* $\gamma_2(x) = x^{r_2}, \pi(\mathrm{d}x) = x^{1+\alpha}\mathrm{d}x$ *with* $1 < \alpha < 2$ *in SDE (3), combining the Theorems we have for any* $x > 0$, $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ *for* $0 \le r_2 < \alpha$ *and* $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ *but* $X_t \to 0$ $\mathbb{P}_x$*-a.s. for* $r_2 \ge \alpha$.

The explosion of nonlinear CB process

Let $\tau_\infty^+ = \lim_{n\to\infty} \tau_n^+$ be the explosion time for process X.

Theorem

(No explosion) If there exist $0 < a < 1$ and $r < 1$ such that $G_a(u) \geq -(\ln u^{-1})^r$ for all large u, then $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ for all large x.

Theorem

(Explosion with positive probability) If there exist $a > 1$ and $r > 1$ such that $G_a(u) \geq (\ln u^{-1})^r$ for all large u, then $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ for all large x.

Suppose that $\gamma_i(x) = b_i x^{r_i}$, $i = 0, 1, 2$ for $b_1, b_2 > 0$ and for large x.

Corollary

(No explosion)

$$\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$$

for x large enough if $b_0 \leq 0$ or one of the followings is true.

(i) $b_0 > 0$ *and* $r_0 \leq 1$.

(ii) $b_0 > 0$, $r_0 > 1$ *and one of the followings holds.*

- (iia) *If* $\sigma > 0$, *then either* $r_1 > r_0 + 1$ *or* $r_1 = r_0 + 1$ *and* b_1 *is much bigger than* b_0.
- (iib) *If* $\pi \neq 0$, *then either* $r_2 > r_0 + \alpha - 1$ *or* $r_2 = r_0 + \alpha - 1$ *and* b_2 *is much bigger than* b_0.

Corollary

(Explosion within finite time)

$$\mathbb{P}_x\{\tau_\infty^+ < \infty\} > 0$$

for x large enough if $r_0 > 1$ and both of the followings hold.

(i) *If $\sigma > 0$, then either $r_1 < r_0 + 1$ or $r_1 = r_0 + 1$ and b_0 is much bigger than b_1.*

(ii) *If $\pi \neq 0$, then either $r_2 < r_0 + \alpha - 1$ or $r_2 = r_0 + \alpha - 1$ and b_0 is much bigger than b_2.*

Discussions

From the previous results on explosion we see that

- explosion is caused by large enough positive drift;
- fluctuations can not cause explosion, but large enough fluctuations can prevent explosion.

A short proof for a special case

Proposition

Suppose there are constants $C, \delta > 0$ *and* $\alpha \in (1, 2)$ *so that* $\pi(\mathrm{d}z)$ *is absolutely continuous with respect to Lebesgue measure when restricted to interval* $(0, \delta)$ *and*

$$\pi(\mathrm{d}z) \le Cz^{-1-\alpha}\mathrm{d}z, \quad z \in (0, \delta).$$

Further, $\sup_{x\in(0,\delta)}(-\gamma_0(x))x^{-1} < \infty$,

$$\sup_{x\in(0,\delta)} \gamma_1(x)x^{-2} < \infty, \quad \sup_{x\in(0,\delta)} \gamma_2(x)x^{-\alpha} < \infty.$$

Then $\mathbb{P}\{\tau_0^- = \infty\} = 1$.

A short proof on a special case of non-extinction

For $0 < x < \delta$, let $\tau_\delta^+ := \inf\{t : X_t > \delta\}$. Applying Ito's formula,

$$
\begin{aligned}
&e^{-\lambda X_{t\wedge\tau_\delta^+\wedge\tau_0^-}} \\
&= e^{-\lambda x} - \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} \lambda\gamma_0(X_{s-})e^{-\lambda X_{s-}}\mathrm{d}s \\
&\quad + \frac{\sigma^2}{2}\int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} \lambda^2 e^{-\lambda X_s}\gamma_1(X_s)\mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} e^{-\lambda X_{s-}}\gamma_2(X_{s-})\mathrm{d}s\int_0^\infty \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z) \\
&\quad + \text{martingale}.
\end{aligned}
$$

Since

$$\int_0^\infty \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z) \le C\int_0^\infty \left[(\lambda z)\wedge(\lambda z)^2\right] z^{-1-\alpha}\mathrm{d}z$$

$$= C\lambda^\alpha \int_0^\infty \left[z \wedge z^2\right] z^{-1-\alpha}\mathrm{d}z = C(\alpha)\lambda^\alpha,$$

then

$$\sup_{\lambda\ge 1, s\in[0,\tau_u^+)} \Big[-\lambda\gamma_0(X_{s-})e^{-\lambda X_{s-}} + \frac{\sigma^2}{2}\lambda^2 e^{-\lambda X_{s-}}\gamma_1(X_{s-})$$

$$+ e^{-\lambda X_{s-}}\gamma_2(X_{s-})\mathrm{d}s \int_0^\infty \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z)\Big]$$

$$\le C \sup_{\lambda\ge 1, s\in[0,\tau_u^+)} \left(\lambda X_{s-}e^{-\lambda X_{s-}} + (\lambda X_{s-})^2 e^{-\lambda X_{s-}} + (\lambda X_{s-})^\alpha e^{-\lambda X_{s-}}\right)$$

$$\le C \sup_{\lambda\ge 0} \left(\lambda e^{-\lambda} + \lambda^2 e^{-\lambda} + \lambda^\alpha e^{-\lambda}\right)$$

$$< C.$$

$$
\begin{aligned}
\mathbb{E}_x e^{-\lambda X_{t\wedge\tau_\delta^+\wedge\tau_0^-}} &= e^{-\lambda x} - \mathbb{E}_x \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} \lambda\gamma_0(X_{s-})e^{-\lambda X_{s-}}\mathrm{d}s \\
&+ \mathbb{E}_x \frac{\sigma^2}{2}\int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} \lambda^2 e^{-\lambda X_s}\gamma_1(X_s)\mathrm{d}s \\
&+ \mathbb{E}_x \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} e^{-\lambda X_{s-}}\gamma_2(X_{s-})\mathrm{d}s \int_0^\infty \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z) \\
&\le e^{-\lambda x} + C\mathbb{E}_x \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} [\lambda X_{s-} + (\lambda X_{s-})^2 + (\lambda X_{s-})^\alpha]e^{-\lambda X_s}\mathrm{d}s. \\
&\to 0 \quad \text{as } \lambda\to\infty.
\end{aligned}
$$

Then $\mathbb{P}_x\{X_{t\wedge\tau_\delta^+\wedge\tau_0^-} \neq 0\} = 1$ for all $t > 0$. Letting $t\to\infty$, we have $\mathbb{P}_x\{\tau_\delta^+ < \tau_0^-$ or $\tau_\delta^+ = \tau_0^- = \infty\} = 1$. Then by Markov property and lack of negative jumps, $\mathbb{P}_x\{\tau_0^- = \infty\} = 1$.

The proofs for the general cases are more involved and we can work with the following martingale.
Recall that

$$G_a(u) = \frac{a-1}{u}\gamma_0(u) - \frac{a(a-1)}{2u^2}\sigma^2\gamma_1(u) - \gamma_2(u)u^{a-1}\int_0^\infty [(u+z)^{1-a} - u^{1-a} - (1-a)zu^{-a}]\pi(\mathrm{d}z).$$

Then

$$X_t^{1-a}\exp\left\{\int_0^t G_a(X_s)\mathrm{d}s\right\}$$

is a martingale.
We choose $a > 1$ to prove the non-extinction result and explosion result, and $a < 1$ to show the extinction result and the non-explosion result.

Proof of non-extinction

let $T_n := \tau^-(\epsilon^n) \wedge \tau_b^+$ for small enough $0 < \epsilon < b$. For $a > 1$ consider martingale $X_{t\wedge T_n}^{1-a} \exp\left\{ \int_0^{t\wedge T_n} G_a(X_s)\mathrm{d}s \right\}$. By optional stopping and letting $t\to\infty$, we have

$$
\begin{aligned}
\epsilon^{1-a} &= \mathbb{E}_\epsilon\left[X_{\tau^-(\epsilon^n)\wedge\tau^+(b)}^{1-a} \exp\left\{ \int_0^{\tau^-(\epsilon^n)\wedge\tau_b^+} G_a(X_s)\mathrm{d}s \right\}\right. \\
&\geq \mathbb{E}_\epsilon\left[X_{\tau^-(\epsilon^n)\wedge\tau^+(b)}^{1-a} \exp\left\{ -(\ln\epsilon^{-n})^r(\tau^-(\epsilon^n)\wedge\tau^+(b)\right\}\right] \\
&\geq \mathbb{E}_\epsilon\left[X_{\tau^-(\epsilon^n)}^{1-a} \exp\left\{ -(\ln\epsilon^{-n})^r d_n\right\} 1_{\{\tau_{\epsilon^n}^- < \tau_b^+\wedge d_n\}}\right] \\
&= \epsilon^{(1-a)n} \exp\{\ln\epsilon^{n(a-1)/2}\} \mathbb{P}_\epsilon\left\{\tau_{\epsilon^n}^- < \tau_b^+ \wedge d_n\right\} \\
&= \epsilon^{(1-a)n/2} \mathbb{P}_\epsilon\left\{\tau_{\epsilon^n}^- < \tau_b^+ \wedge d_n\right\},
\end{aligned}
$$

where $d_n := \frac{\ln\epsilon^{n(a-1)/2}}{-(\ln\epsilon^{-n})^r} \to\infty$ as $n\to\infty$.

Then

$$\mathbb{P}_\epsilon \left\{ \tau^-_{\epsilon^n} < \tau^+_b \wedge d_n \right\} \leq \epsilon^{(a-1)(n-2)/2}.$$

Since $\sum_{i=1}^{\infty} \mathbb{P}_\epsilon \left\{ \tau^-_{\epsilon^n} < \tau^+_b \wedge d_n \right\} < \infty$, by Borel-Cantelli Lemma we have

$$\mathbb{P}_\epsilon \Big\{ \tau^-(\epsilon^n) < \tau^+_b \wedge d_n \ \text{ infinitely often } \Big\} = 0.$$

Then $\tau^-(\epsilon^n) \geq \tau^+_b \wedge d_n$ for all n large enough, and we can show that $\mathbb{P}_\epsilon\{\tau^-_0 = \infty\} = 1$.

Thank you for your attention!