The speed of a branching system of (L,1) random walks in random environment

Meijuan ZHANG

Central University of Finance and Economics

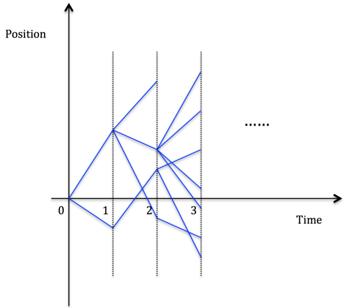
11 May 2017

Outline

- Model and background
- Main results
- A sketch of the proof
- References

Branching Random Walk

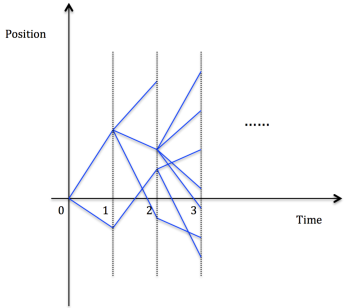
- An initial particle is at the origin of $\mathbb{R}$, which forms the 0th generation.
- It gives birth to offspring particles that form the first generation. Their displacement from their parent are described by a point process Θ.



- Résvése(1994), Shi Zhan(2016).

Branching Random Walk

- An initial particle is at the origin of $\mathbb{R}$, which forms the 0th generation.
- It gives birth to offspring particles that form the first generation. Their displacement from their parent are described by a point process Θ.



- R$\acute{e}$sv$\acute{e}$se(1994), Shi Zhan(2016).

Branching Random Walk

- Classical branching random walk: the number of any particle's children has a fixed distribution and the children's displacement also has a fixed distribution.
- Asmussen,S., Kaplan,N.(1976), Biggins,J.D.(1990).

Branching random walk in random environment

Branching random walk with random environment in time

- the distributions of offspring and the distribution of the displacement of the children may vary from generation to generation according to a random environment
- Biggins and Kyprianou (2004);
- Gao,Z., Liu,Q., Wang,H.(2012): Central limit theorems for a counting measure;
- Liu,Q.(2007): Limit properties for the rightmost particle.

Branching Random Walk

- Classical branching random walk: the number of any particle's children has a fixed distribution and the children's displacement also has a fixed distribution.
- Asmussen,S., Kaplan,N.(1976), Biggins,J.D.(1990).

Branching random walk in random environment

Branching random walk with random environment in time

- the distributions of offspring and the distribution of the displacement of the children may vary from generation to generation according to a random environment
- Biggins and Kyprianou (2004);
- Gao,Z., Liu,Q., Wang,H.(2012): Central limit theorems for a counting measure;
- Liu,Q.(2007): Limit properties for the rightmost particle.

Branching Random Walk

- Classical branching random walk: the number of any particle's children has a fixed distribution and the children's displacement also has a fixed distribution.
- Asmussen,S., Kaplan,N.(1976), Biggins,J.D.(1990).

Branching random walk in random environment

Branching random walk with random environment in time

- the distributions of offspring and the distribution of the displacement of the children may vary from generation to generation according to a random environment
- Biggins and Kyprianou (2004);
- Gao,Z., Liu,Q., Wang,H.(2012): Central limit theorems for a counting measure;
- Liu,Q.(2007): Limit properties for the rightmost particle.

Branching Random Walk

- Classical branching random walk: the number of any particle's children has a fixed distribution and the children's displacement also has a fixed distribution.
- Asmussen,S., Kaplan,N.(1976), Biggins,J.D.(1990).

Branching random walk in random environment

Branching random walk with random environment in time

- the distributions of offspring and the distribution of the displacement of the children may vary from generation to generation according to a random environment
- Biggins and Kyprianou (2004);
- Gao,Z., Liu,Q., Wang,H.(2012): Central limit theorems for a counting measure;
- Liu,Q.(2007): Limit properties for the rightmost particle.

Branching Random Walk

- Classical branching random walk: the number of any particle's children has a fixed distribution and the children's displacement also has a fixed distribution.
- Asmussen,S., Kaplan,N.(1976), Biggins,J.D.(1990).

Branching random walk in random environment

Branching random walk with random environment in time

- the distributions of offspring and the distribution of the displacement of the children may vary from generation to generation according to a random environment
- Biggins and Kyprianou (2004);
- Gao,Z., Liu,Q., Wang,H.(2012): Central limit theorems for a counting measure;
- Liu,Q.(2007): Limit properties for the rightmost particle.

Branching random walk with random environment in location

◇ the offspring distribution of a particle situated at z depends on a random environment indexed by its situation z, while the moving mechanism is controlled by a fixed deterministic law, e.g. Greven,A., den Hollander, F.(1992);

◇ the reproduction law depends on the location, and each particle has almost surely at least one offspring.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(1998):

  If $m > m_c$, there is infinitely often at least one particle with positive location;
  If $m \leq m_c$, there is no particle in $\mathbb{N}$ at time n for n large enough.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(2007)—multidimensional random walks in random environment.

Branching Random Walk in random environment

Branching random walk with random environment in location

◇ the offspring distribution of a particle situated at z depends on a random environment indexed by its situation z, while the moving mechanism is controlled by a fixed deterministic law, e.g. Greven,A., den Hollander, F.(1992);

◇ the reproduction law depends on the location, and each particle has almost surely at least one offspring.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(1998):

  If $m > m_c$, there is infinitely often at least one particle with positive location;

  If $m \leq m_c$, there is no particle in $\mathbb{N}$ at time n for n large enough.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(2007)—multidimensional random walks in random environment.

Branching Random Walk in random environment

Branching random walk with random environment in location

◇ the offspring distribution of a particle situated at z depends on a random environment indexed by its situation z, while the moving mechanism is controlled by a fixed deterministic law, e.g. Greven,A., den Hollander, F.(1992);

◇ the reproduction law depends on the location, and each particle has almost surely at least one offspring.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(1998):

  If $m > m_c$, there is infinitely often at least one particle with positive location;
  If $m \leq m_c$, there is no particle in $\mathbb{N}$ at time n for n large enough.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(2007)—multidimensional random walks in random environment.

Branching random walk with random environment in location

◇ the offspring distribution of a particle situated at z depends on a random environment indexed by its situation z, while the moving mechanism is controlled by a fixed deterministic law, e.g. Greven,A., den Hollander, F.(1992);

◇ the reproduction law depends on the location, and each particle has almost surely at least one offspring.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(1998):

  If $m > m_c$, there is infinitely often at least one particle with positive location;
  If $m \leq m_c$, there is no particle in $\mathbb{N}$ at time n for n large enough.

- Comets,F., Menshikov,M.V.,Popov, S.Y.(2007)—multidimensional random walks in random environment.

◇ the step transition probability at the situation z depends on a random environment indexed by the situation z, while the offspring distribution is controlled by a fixed reproduction law.

- Devulder,A.(2007): the speed of the rightmost particle for a branching system of (1,1) random walks in random environment.

  If $m < m_c$, the rightmost particle goes to $-\infty$ with a negative speed;

  If $m > m_c$, the rightmost particle goes to $+\infty$ with a positive speed.

- Zhang(2013): Central limit theorem for the counting measure $Z_n(\cdot)$ in the annealed case.

$\diamond$ the step transition probability at the situation z depends on a random environment indexed by the situation z, while the offspring distribution is controlled by a fixed reproduction law.

- Devulder,A.(2007): the speed of the rightmost particle for a branching system of (1,1) random walks in random environment.

  If $m < m_c$, the rightmost particle goes to $-\infty$ with a negative speed;

  If $m > m_c$, the rightmost particle goes to $+\infty$ with a positive speed.

- Zhang(2013): Central limit theorem for the counting measure $Z_n(\cdot)$ in the annealed case.

Branching random walk with random environment in time and in location

- Li,Y., Li,X., Liu,Q.(2007)—the particles reproduce with a random environment in time and move with another random environment in location: Limit properties of the rightmost particle;
- Hu,Y., Yoshidab,N.(2009)—branching random walk with time-space i.i.d. offspring distributions: Limit theorems for localization;
- Yoshidab,N.(2008): Central limit theorem.

Branching random walk with random environment in time and in location

- Li,Y., Li,X., Liu,Q.(2007)—the particles reproduce with a random environment in time and move with another random environment in location: Limit properties of the rightmost particle;
- Hu,Y., Yoshidab,N.(2009)—branching random walk with time-space i.i.d. offspring distributions: Limit theorems for localization;
- Yoshidab,N.(2008): Central limit theorem.

Branching Random Walk in random environment

Branching random walk with random environment in time and in location

- Li,Y., Li,X., Liu,Q.(2007)—the particles reproduce with a random environment in time and move with another random environment in location: Limit properties of the rightmost particle;
- Hu,Y., Yoshidab,N.(2009)—branching random walk with time-space i.i.d. offspring distributions: Limit theorems for localization;
- Yoshidab,N.(2008): Central limit theorem.

Branching Random Walk in random environment

Branching random walk with random environment in time and in location

- Li,Y., Li,X., Liu,Q.(2007)—the particles reproduce with a random environment in time and move with another random environment in location: Limit properties of the rightmost particle;
- Hu,Y., Yoshidab,N.(2009)—branching random walk with time-space i.i.d. offspring distributions: Limit theorems for localization;
- Yoshidab,N.(2008): Central limit theorem.

Branching Random Walk in random environment

⋆ Consider a branching system of (L,1) random walks in random environment(BSRWRE)

- branching random walk with random environment in location
- particles reproduce with a fixed reproduction law;
- move as (L,1) random walk in random environment X_n .

⋆ If $X_n \to -\infty$, there is for our model a competition between the environment and the branching process.

- random walk in random environment: pushing the particle to $-\infty$;
- branching process: creates new particle and then increases the possibility that some particles go very far on the right.

Objective

Study the asymptotic behavior of the rightmost particle, conditionally on the survival of the branching process.

Branching Random Walk in random environment

⋆ Consider a branching system of (L,1) random walks in random environment(BSRWRE)

- branching random walk with random environment in location
- particles reproduce with a fixed reproduction law;
- move as (L,1) random walk in random environment X_n .

⋆ If $X_n \to -\infty$, there is for our model a competition between the environment and the branching process.

- random walk in random environment: pushing the particle to $-\infty$;
- branching process: creates new particle and then increases the possibility that some particles go very far on the right.

Objective

Study the asymptotic behavior of the rightmost particle, conditionally on the survival of the branching process.

⋆ Consider a branching system of (L,1) random walks in random environment(BSRWRE)

- branching random walk with random environment in location
- particles reproduce with a fixed reproduction law;
- move as (L,1) random walk in random environment X_n .

⋆ If $X_n \to -\infty$, there is for our model a competition between the environment and the branching process.

- random walk in random environment: pushing the particle to $-\infty$;
- branching process: creates new particle and then increases the possibility that some particles go very far on the right.

Objective

Study the asymptotic behavior of the rightmost particle, conditionally on the survival of the branching process.

Branching Random Walk in random environment

⋆ Consider a branching system of (L,1) random walks in random environment(BSRWRE)

- branching random walk with random environment in location
- particles reproduce with a fixed reproduction law;
- move as (L,1) random walk in random environment X_n .

⋆ If $X_n \to -\infty$, there is for our model a competition between the environment and the branching process.

- random walk in random environment: pushing the particle to $-\infty$;
- branching process: creates new particle and then increases the possibility that some particles go very far on the right.

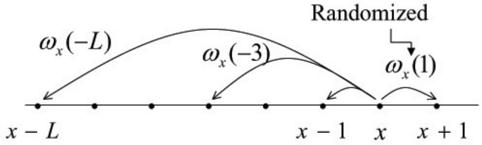
Objective

Study the asymptotic behavior of the rightmost particle, conditionally on the survival of the branching process.

⋆ (L,1) random walks in random environment

- Random environment: $(\Omega, \mathscr{F}, P, \theta)$, where $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$ i.i.d.
- Random walk: $\{X_n,\ n \geq 0\}$ is a time homogeneous Markov chain with transition probability

$$P_\omega(X_{n+1} = x + l | X_n = x) = \omega_x(l), \quad x \in \mathbb{Z}, l \in \{-L, \dots, -1, 1\}.$$

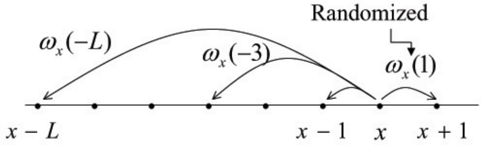


$$\{\omega_x\}_{x\in\mathbb{Z}} = \{\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1)\} \sim P$$

⋆ (L,1) random walks in random environment

- Random environment: $(\Omega, \mathscr{F}, P, \theta)$, where $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$ i.i.d.
- Random walk: $\{X_n,\ n \geq 0\}$ is a time homogeneous Markov chain with transition probability

$$P_\omega(X_{n+1} = x + l | X_n = x) = \omega_x(l), \quad x \in \mathbb{Z}, l \in \{-L, \dots, -1, 1\}.$$

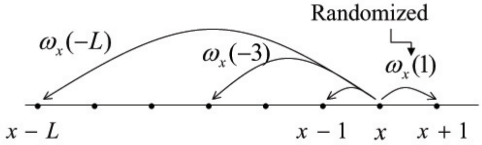


$$\{\omega_x\}_{x\in\mathbb{Z}} = \{\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1)\} \sim P$$

⋆ (L,1) random walks in random environment

- Random environment: $(\Omega, \mathscr{F}, P, \theta)$, where $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$ i.i.d.
- Random walk: $\{X_n,\ n \geq 0\}$ is a time homogeneous Markov chain with transition probability

$$P_\omega(X_{n+1} = x + l | X_n = x) = \omega_x(l), \quad x \in \mathbb{Z}, l \in \{-L, \dots, -1, 1\}.$$



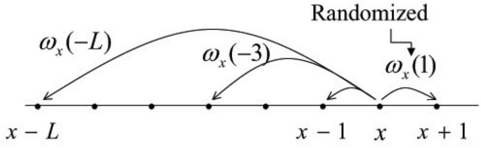
$$\{\omega_x\}_{x\in\mathbb{Z}} = \{\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1)\} \sim P$$

Model: BSRWRE

⋆ (L,1) random walks in random environment

- Random environment: $(\Omega, \mathscr{F}, P, \theta)$, where $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$ i.i.d.
- Random walk: $\{X_n,\ n \geq 0\}$ is a time homogeneous Markov chain with transition probability

$$P_\omega(X_{n+1} = x + l | X_n = x) = \omega_x(l), \quad x \in \mathbb{Z}, l \in \{-L, \ldots, -1, 1\}.$$

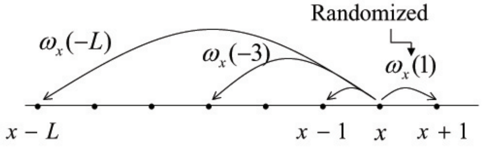


$$\{\omega_x\}_{x\in\mathbb{Z}} = \{\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1)\} \sim P$$

⋆ (L,1) random walks in random environment

- Random environment: $(\Omega, \mathscr{F}, P, \theta)$, where $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$ i.i.d.
- Random walk: $\{X_n,\ n \geq 0\}$ is a time homogeneous Markov chain with transition probability

$$P_\omega(X_{n+1} = x + l | X_n = x) = \omega_x(l), \quad x \in \mathbb{Z}, l \in \{-L, \ldots, -1, 1\}.$$

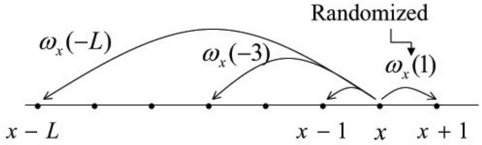


$$\{\omega_x\}_{x\in\mathbb{Z}} = \{\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1)\} \sim P$$

⋆ (L,1) random walks in random environment

- Random environment: $(\Omega, \mathscr{F}, P, \theta)$, where $(\omega_i)_{i\in\mathbb{Z}} \in \Omega$ i.i.d.
- Random walk: $\{X_n,\ n \geq 0\}$ is a time homogeneous Markov chain with transition probability

$$P_\omega(X_{n+1} = x + l | X_n = x) = \omega_x(l), \quad x \in \mathbb{Z}, l \in \{-L, \dots, -1, 1\}.$$



$$\{\omega_x\}_{x\in\mathbb{Z}} = \{\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1)\} \sim P$$

Model: BSRWRE

- quenched law: P_ω, which is defined on the path space $(\mathbb{Z}^{\mathbb{N}}, \mathscr{G})$.
- annealed law: $\mathbb{P} = P \otimes P_\omega$, which is defined on the space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathscr{F} \times \mathscr{G})$

$$\mathbb{P}(F \times G) = \int_F P_\omega(G) P(d\omega) \quad F \in \mathscr{F},\ G \in \mathscr{G}.$$

⋆ branching system: Galton-Watson process Γ.

- $p_k \geq 0$ and $\sum_{k \in \mathbb{N}} p_k = 1$;
- supercritical, $m > 1$, $0 < q \leq 1$.

Model: BSRWRE

- quenched law: P_ω, which is defined on the path space $(\mathbb{Z}^{\mathbb{N}}, \mathscr{G})$.
- annealed law: $\mathbb{P} = P \otimes P_\omega$, which is defined on the space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathscr{F} \times \mathscr{G})$

$$\mathbb{P}(F \times G) = \int_F P_\omega(G) P(d\omega) \quad F \in \mathscr{F},\ G \in \mathscr{G}.$$

$\star$ branching system: Galton-Watson process Γ.

- $p_k \geq 0$ and $\sum_{k \in \mathbb{N}} p_k = 1$;
- supercritical, $m > 1$, $0 < q \leq 1$.

Model: BSRWRE

- quenched law: P_ω, which is defined on the path space $(\mathbb{Z}^{\mathbb{N}}, \mathscr{G})$.
- annealed law: $\mathbb{P} = P \otimes P_\omega$, which is defined on the space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathscr{F} \times \mathscr{G})$

$$\mathbb{P}(F \times G) = \int_F P_\omega(G) P(d\omega) \quad F \in \mathscr{F},\ G \in \mathscr{G}.$$

⋆ branching system: Galton-Watson process Γ.

- $p_k \geq 0$ and $\sum_{k \in \mathbb{N}} p_k = 1$;
- supercritical, $m > 1$, $0 < q \leq 1$.

Model: BSRWRE

For each given environment, the particle system behaves like this

- At time $n = 0$, there is only one particle, located at 0.
- At time $n = 1$, the particle moves to 1 with probability $\omega_0(1)$, or to $-l$ with probability $\omega_0(-l)$, where $-l \in \{-L, \cdots, 1\}$. Arriving at the new location, it gives birth to k offspring with probability p_k, and dies.
- At time $n = 2$, each particle moves independently, according to the probabilities for random walk in random environment. Then it produces new offspring independently, with the same reproduction law as before, and dies.
- Iterating this procedure, we obtain a branching system of random walks in random environment.

Preliminary

Let T_n be the hitting time of (L,1) random walk in random environment, and

$$\lambda_{crit} = \sup\{\lambda : \quad \lim_{n\to\infty} \frac{1}{n} \log E_\omega(e^{\lambda T_n},\ T_n < +\infty) < +\infty,$$
$$\lim_{n\to\infty} \frac{1}{n} \log E_\omega(e^{\lambda T_{-n}},\ T_{-n} < +\infty) < +\infty\}.$$

Proposition 1

λ_{crit} is deterministic.

The idea of the proof. By the decomposition for stopping time T_1, we have

$$\begin{aligned} E_{\theta\omega}(e^{\lambda T_1}) &= \omega_1(1)e^{\lambda} + \omega_1(-1)e^{\lambda} E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-1})}) E_{\theta\omega}(e^{\lambda T_1}) \\ &+ \omega_1(-2)e^{\lambda} E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-2})}) E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-1})}) E_{\theta\omega}(e^{\lambda T_1}). \end{aligned}$$

Preliminary

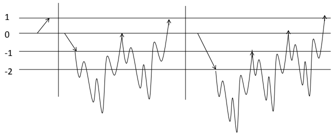
Let T_n be the hitting time of (L,1) random walk in random environment, and

$$\lambda_{crit} = \sup\{\lambda : \quad \lim_{n\to\infty} \frac{1}{n}\log E_\omega(e^{\lambda T_n},\ T_n < +\infty) < +\infty,$$
$$\lim_{n\to\infty} \frac{1}{n}\log E_\omega(e^{\lambda T_{-n}},\ T_{-n} < +\infty) < +\infty\}.$$

Proposition 1

λ_{crit} is deterministic.

The idea of the proof. By the decomposition for stopping time T_1, we have



$$\begin{aligned} E_{\theta\omega}(e^{\lambda T_1}) &= \omega_1(1)e^{\lambda} + \omega_1(-1)e^{\lambda}E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-1})})E_{\theta\omega}(e^{\lambda T_1}) \\ &+ \omega_1(-2)e^{\lambda}E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-2})})E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-1})})E_{\theta\omega}(e^{\lambda T_1}). \end{aligned}$$

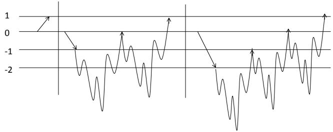
Let T_n be the hitting time of (L,1) random walk in random environment, and

$$\lambda_{crit} = \sup\{\lambda : \quad \lim_{n\to\infty} \frac{1}{n} \log E_\omega(e^{\lambda T_n},\ T_n < +\infty) < +\infty, \\ \lim_{n\to\infty} \frac{1}{n} \log E_\omega(e^{\lambda T_{-n}},\ T_{-n} < +\infty) < +\infty\}.$$

Proposition 1

λ_{crit} is deterministic.

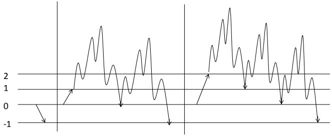
The idea of the proof. By the decomposition for stopping time T_1, we have



$$\begin{aligned} E_{\theta\omega}(e^{\lambda T_1}) &= \omega_1(1)e^{\lambda} + \omega_1(-1)e^{\lambda} E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-1})}) E_{\theta\omega}(e^{\lambda T_1}) \\ &+ \omega_1(-2)e^{\lambda} E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-2})}) E_{\theta\omega}(e^{\lambda(T_1\circ\theta^{-1})}) E_{\theta\omega}(e^{\lambda T_1}). \end{aligned}$$

Let $\lambda_c(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_1}) < +\infty\}$. Then $\lambda_c(\theta\omega) = \lambda_c(\omega)$, for η-a.s. ω. We can obtain that λ_c is deterministic.

Similarly, by decomposing the stopping time T_1' for (1,R) random walk in random environment,

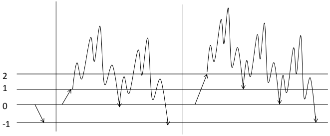


and the intrinsic branching structure for (1,R) random walk in random environment, we can also obtain that $\lambda_c'(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_{-1}}) < +\infty\}$ is deterministic.

Thus λ_{crit} is deterministic.

Let $\lambda_c(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_1}) < +\infty\}$. Then $\lambda_c(\theta\omega) = \lambda_c(\omega)$, for η-a.s. ω. We can obtain that λ_c is deterministic.

Similarly, by decomposing the stopping time T_1' for (1,R) random walk in random environment,

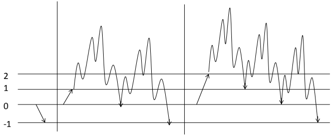


and the intrinsic branching structure for (1,R) random walk in random environment, we can also obtain that $\lambda_c'(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_{-1}}) < +\infty\}$ is deterministic.

Thus λ_{crit} is deterministic.

Let $\lambda_c(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_1}) < +\infty\}$. Then $\lambda_c(\theta\omega) = \lambda_c(\omega)$, for η-a.s. ω. We can obtain that λ_c is deterministic.

Similarly, by decomposing the stopping time T_1' for (1,R) random walk in random environment,



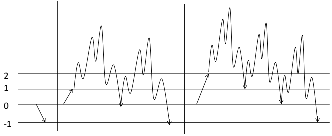
and the intrinsic branching structure for (1,R) random walk in random environment, we can also obtain that $\lambda_c'(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_{-1}}) < +\infty\}$ is deterministic.

Thus λ_{crit} is deterministic.

Preliminary

Let $\lambda_c(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_1}) < +\infty\}$. Then $\lambda_c(\theta\omega) = \lambda_c(\omega)$, for η-a.s. ω. We can obtain that λ_c is deterministic.

Similarly, by decomposing the stopping time T_1' for (1,R) random walk in random environment,



and the intrinsic branching structure for (1,R) random walk in random environment, we can also obtain that $\lambda_c'(\omega) = \sup\{\lambda : E_\omega(e^{\lambda T_{-1}}) < +\infty\}$ is deterministic.

Thus λ_{crit} is deterministic.

Preliminary

Define

$$\overline{M}_i = \begin{pmatrix} a_i(1) & \cdots & a_i(L-1) & a_i(L) \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},$$

where $a_i(l) = \frac{\omega_i(-l)+\cdots+\omega_i(-L)}{\omega_i(1)}, \ i \in \mathbb{Z}$.

Let $\gamma_L = \lim_{n\to\infty} \frac{1}{n} \log \|M(n-1,0)\|, \ P-\text{a}.s.$ be the top Lyapunov exponent of random matrix M.

Lemma (Brémont(2002))

For (L,1) random walk in random environment,

(i) If $\gamma_L > 0$, $X_n \to -\infty, \ \mathbb{P}$-a.s.

(ii) Under the condition of (IM),

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}(\pi(\omega))}, \quad \mathbb{P}-a.s.$$

Main result

Let m_n^* denote the location of the rightmost particle at time n, and

$$m_c = \exp(\lambda_{crit}).$$

Theorem

Suppose $\gamma_L > 0$, Γ be the Galton-Watson process governing the branching system.

(i) If $1 < m < m_c$, then

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} < 0 | \ \Gamma \mathit{survives}\right) = 1;$$

(ii) If $m > m_c$, then

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{m_n^*}{n} > 0 | \ \Gamma \mathit{survives}\right) = 1;$$

(iii) If $m = m_c$, then

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} \leq 0 | \ \Gamma \mathit{survives}\right) = 1.$$

Precise formulation of the model

- $\lambda(x,n)$ denotes the number of particles located at x at time n;
- $m_n^* = \max\{x \in \mathbb{Z},\ \lambda(x,n) > 0\}$;
- $\{Z(x,n,\mu)\}$ i.i.d., independent of ω, s.t. $\mathbb{P}(Z(x,n,\mu) = p_k)$;
- $\{X(x,n,\mu)\} \sim U[0,1]$ i.i.d.;
- $\mathscr{F}_\omega(n) = \sigma(\lambda(x,k),\ 0 \le k \le n,\ x \in \mathbb{Z})$.

Then

$$\lambda(x,n+1) = \sum_{\mu=1}^{\lambda(x-1,n)} \mathbf{1}_{X(x-1,n,\mu)\le\omega_{x-1}} Z(x-1,n,\mu) + \sum_{\mu=1}^{\lambda(x+1,n)} \mathbf{1}_{X(x+1,n,\mu)\le\omega_{x+1}} Z(x+1,n,\mu)$$

Lemma (Révész(1994), Devulder,A.(2007))

For $N \in \mathbb{N}$, $\forall 0 \le n \le N$, $\forall x \in \mathbb{Z}$,

$$\begin{aligned} E_\omega(\lambda(x,N)|\mathscr{F}_\omega(n)) &= m^{N-n} \sum_{y\in\mathbb{Z}} \lambda(y,n) P_\omega(X_{p+N-n} = x | X_p = y); \\ E_\omega(\lambda(x,N)) &= m^N P_\omega(X_N = x). \end{aligned}$$

Precise formulation of the model

- $\lambda(x,n)$ denotes the number of particles located at x at time n;
- $m_n^* = \max\{x \in \mathbb{Z},\ \lambda(x,n) > 0\}$;
- $\{Z(x,n,\mu)\}$ i.i.d., independent of ω, s.t. $\mathbb{P}(Z(x,n,\mu) = p_k)$;
- $\{X(x,n,\mu)\} \sim U[0,1]$ i.i.d.;
- $\mathscr{F}_\omega(n) = \sigma(\lambda(x,k),\ 0 \le k \le n,\ x \in \mathbb{Z})$.

Then

$$\lambda(x,n+1) = \sum_{\mu=1}^{\lambda(x-1,n)} \mathbf{1}_{X(x-1,n,\mu) \le \omega_{x-1}} Z(x-1,n,\mu) + \sum_{\mu=1}^{\lambda(x+1,n)} \mathbf{1}_{X(x+1,n,\mu) \le \omega_{x+1}} Z(x+1,n,\mu)$$

Lemma (Révész(1994), Devulder,A.(2007))

For $N \in \mathbb{N}$, $\forall 0 \le n \le N$, $\forall x \in \mathbb{Z}$,

$$\begin{aligned} E_\omega(\lambda(x,N) | \mathscr{F}_\omega(n)) &= m^{N-n} \sum_{y \in \mathbb{Z}} \lambda(y,n) P_\omega(X_{p+N-n} = x | X_p = y); \\ E_\omega(\lambda(x,N)) &= m^N P_\omega(X_N = x). \end{aligned}$$

Precise formulation of the model

- $\lambda(x,n)$ denotes the number of particles located at x at time n;
- $m_n^* = \max\{x \in \mathbb{Z},\ \lambda(x,n) > 0\}$;
- $\{Z(x,n,\mu)\}$ i.i.d., independent of ω, s.t. $\mathbb{P}(Z(x,n,\mu) = p_k)$;
- $\{X(x,n,\mu)\} \sim U[0,1]$ i.i.d.;
- $\mathscr{F}_\omega(n) = \sigma(\lambda(x,k),\ 0 \le k \le n,\ x \in \mathbb{Z})$.

Then

$$\lambda(x,n+1) = \sum_{\mu=1}^{\lambda(x-1,n)} \mathbf{1}_{X(x-1,n,\mu)\le\omega_{x-1}} Z(x-1,n,\mu) + \sum_{\mu=1}^{\lambda(x+1,n)} \mathbf{1}_{X(x+1,n,\mu)\le\omega_{x+1}} Z(x+1,n,\mu)$$

Lemma (Révész(1994), Devulder,A.(2007))

For $N \in \mathbb{N}$, $\forall 0 \le n \le N$, $\forall x \in \mathbb{Z}$,

$$E_\omega(\lambda(x,N)|\mathscr{F}_\omega(n)) = m^{N-n} \sum_{y\in\mathbb{Z}} \lambda(y,n) P_\omega(X_{p+N-n} = x | X_p = y);$$

$$E_\omega(\lambda(x,N)) = m^N P_\omega(X_N = x).$$

Precise formulation of the model

- $\lambda(x,n)$ denotes the number of particles located at x at time n;
- $m_n^* = \max\{x \in \mathbb{Z},\ \lambda(x,n) > 0\}$;
- $\{Z(x,n,\mu)\}$ i.i.d., independent of ω, s.t. $\mathbb{P}(Z(x,n,\mu) = p_k)$;
- $\{X(x,n,\mu)\} \sim U[0,1]$ i.i.d.;
- $\mathscr{F}_\omega(n) = \sigma(\lambda(x,k),\ 0 \le k \le n,\ x \in \mathbb{Z})$.

Then

$$\lambda(x,n+1) = \sum_{\mu=1}^{\lambda(x-1,n)} \mathbf{1}_{X(x-1,n,\mu)\le\omega_{x-1}} Z(x-1,n,\mu) + \sum_{\mu=1}^{\lambda(x+1,n)} \mathbf{1}_{X(x+1,n,\mu)\le\omega_{x+1}} Z(x+1,n,\mu)$$

Lemma (Révész(1994), Devulder,A.(2007))

For $N \in \mathbb{N},\ \forall 0 \le n \le N,\ \forall x \in \mathbb{Z},$

$$\begin{aligned} E_\omega(\lambda(x,N)|\mathscr{F}_\omega(n)) &= m^{N-n} \sum_{y\in\mathbb{Z}} \lambda(y,n) P_\omega(X_{p+N-n} = x | X_p = y); \\ E_\omega(\lambda(x,N)) &= m^N P_\omega(X_N = x). \end{aligned}$$

Lemma (Yilmaz 2009)

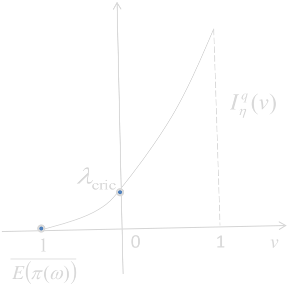
Suppose
(A1) there exists a $r > 0$*, such that* $\int |\log \omega_0(j)|^{1+r} d\mathbb{P} > \infty$ *for each* $j \in \{-L, \dots, -1, 1\}$*;*
(A2) there exists a $\delta > 0$*, such that* $\mathbb{P}(\omega_0(\pm 1) \geq \delta) = 1$*.*
Then (L,1) random walk in random environment X_n *satisfies a quenched large deviation principle with deterministic, convex and continuous rate function* I_η^q*,*

$$I_\eta^q(v) = \begin{cases} \sup_{\lambda \in \mathbb{R}} \{\lambda - v \lim_{n\to\infty} \frac{1}{n} \log E_\omega(e^{\lambda T_n},\ T_n < +\infty) < +\infty\} & v > 0, \\ \lambda_{crit} & v = 0, \\ \sup_{\lambda \in \mathbb{R}} \{\lambda - |v| \lim_{n\to\infty} \frac{1}{n} \log E_\omega(e^{\lambda \bar{T}_{-n}},\ \bar{T}_{-n} < +\infty) < +\infty\} & v < 0 \end{cases}$$

Properties of the rate function

We need the properties of the rate function $I_\eta^q(v)$ on $(u,1)$, where $u<0$.

- $I_\eta^q(v)$ is convex and continuous at 0.
- Note that
$$\frac{X_n}{n} \to \frac{1}{\mathbb{E}(\pi(\omega))}, \quad \mathbb{P}-\text{a.s.}$$
Then $I_\eta^q(\frac{1}{\mathbb{E}(\pi(\omega))})=0$, where $\frac{1}{\mathbb{E}(\pi(\omega))}<0$.
- The shape of the I_η^q



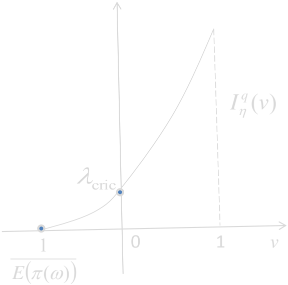
Properties of the rate function

We need the properties of the rate function $I_\eta^q(v)$ on $(u, 1)$, where $u < 0$.

- $I_\eta^q(v)$ is convex and continuous at 0.
- Note that

$$\frac{X_n}{n} \to \frac{1}{\mathbb{E}(\pi(\omega))}, \quad \mathbb{P} - \text{a.s.}$$

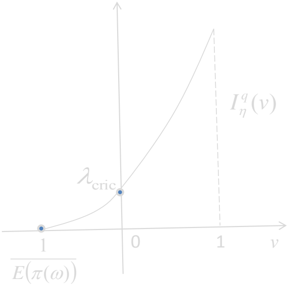
  Then $I_\eta^q(\frac{1}{\mathbb{E}(\pi(\omega))}) = 0$, where $\frac{1}{\mathbb{E}(\pi(\omega))} < 0$.
- The shape of the I_η^q



Properties of the rate function

We need the properties of the rate function $I_\eta^q(v)$ on $(u, 1)$, where $u < 0$.

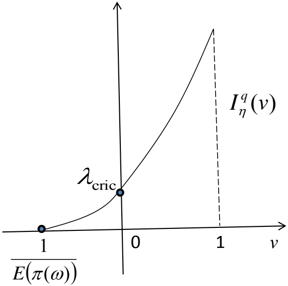
- $I_\eta^q(v)$ is convex and continuous at 0.
- Note that
$$\frac{X_n}{n} \to \frac{1}{\mathbb{E}(\pi(\omega))}, \quad \mathbb{P} - \text{a.s.}$$
Then $I_\eta^q(\frac{1}{\mathbb{E}(\pi(\omega))}) = 0$, where $\frac{1}{\mathbb{E}(\pi(\omega))} < 0$.
- The shape of the I_η^q



Properties of the rate function

We need the properties of the rate function $I^q_\eta(v)$ on $(u, 1)$, where $u < 0$.

- $I^q_\eta(v)$ is convex and continuous at 0.
- Note that
$$\frac{X_n}{n} \to \frac{1}{\mathbb{E}(\pi(\omega))}, \quad \mathbb{P} - \text{a.s.}$$
Then $I^q_\eta(\frac{1}{\mathbb{E}(\pi(\omega))}) = 0$, where $\frac{1}{\mathbb{E}(\pi(\omega))} < 0$.
- The shape of the I^q_η



In the case of $1 < m < m_c = \exp(I_\eta^q(0))$,

- Note that I_η^q is increase on $[-\alpha, 1]$. Then for n large enough,

$$P_\omega(X_n \geq -n\alpha) \leq \exp\{-n[I_\eta^q(-\alpha) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- $\lambda_{crit} \neq 0$ since $m_c > 1$. Then $I_\eta^q(v)$ is strictly increase on $(\frac{1}{\mathbb{E}(\pi(\omega))}, 1]$ and is continuous at 0. Hence $\exists \alpha > 0$, $\varepsilon > 0$, s.t. $I_\eta^q(0) < I_\eta^q(-\alpha) - \varepsilon$. Therefore

$$\log m < I_\eta^q(-\alpha) - \varepsilon$$

- Thus

$$\begin{aligned} P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1\} &\leq E_\omega[\sum_{x=-\alpha n}^{+\infty} \lambda(x, n)] \\ &= m^n P_\omega(X_n \geq -n\alpha) \leq \exp\{n[\log m - I_\eta^q(-\alpha) + \varepsilon]\}, \ \mathbb{P}\text{-a.s.} \end{aligned}$$

In the case of $1 < m < m_c = \exp(I_\eta^q(0))$,

- Note that I_η^q is increase on $[-\alpha, 1]$. Then for n large enough,

$$P_\omega(X_n \geq -n\alpha) \leq \exp\{-n[I_\eta^q(-\alpha) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- $\lambda_{crit} \neq 0$ since $m_c > 1$. Then $I_\eta^q(v)$ is strictly increase on $(\frac{1}{\mathbb{E}(\pi(\omega))}, 1]$ and is continuous at 0. Hence $\exists \alpha > 0$, $\varepsilon > 0$, s.t. $I_\eta^q(0) < I_\eta^q(-\alpha) - \varepsilon$. Therefore

$$\log m < I_\eta^q(-\alpha) - \varepsilon$$

- Thus

$$\begin{aligned} P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1\} &\leq E_\omega[\sum_{x=-\alpha n}^{+\infty} \lambda(x, n)] \\ &= m^n P_\omega(X_n \geq -n\alpha) \leq \exp\{n[\log m - I_\eta^q(-\alpha) + \varepsilon]\}, \ \mathbb{P}\text{-a.s.} \end{aligned}$$

In the case of $1 < m < m_c = \exp(I_\eta^q(0))$,

- Note that I_η^q is increase on $[-\alpha, 1]$. Then for n large enough,

$$P_\omega(X_n \geq -n\alpha) \leq \exp\{-n[I_\eta^q(-\alpha) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- $\lambda_{crit} \neq 0$ since $m_c > 1$. Then $I_\eta^q(v)$ is strictly increase on $(\frac{1}{\mathbb{E}(\pi(\omega))}, 1]$ and is continuous at 0. Hence $\exists \alpha > 0$, $\varepsilon > 0$, s.t. $I_\eta^q(0) < I_\eta^q(-\alpha) - \varepsilon$. Therefore

$$\log m < I_\eta^q(-\alpha) - \varepsilon$$

- Thus

$$\begin{aligned} & P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1\} \leq E_\omega[\sum_{x=-\alpha n}^{+\infty} \lambda(x, n)] \\ = \ & m^n P_\omega(X_n \geq -n\alpha) \leq \exp\{n[\log m - I_\eta^q(-\alpha) + \varepsilon]\}, \ \mathbb{P}\text{-a.s.} \end{aligned}$$

In the case of $1 < m < m_c$

Then

$$\sum_{n=1}^{+\infty} P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1\} < +\infty, \ \mathbb{P}\text{-a.s.}$$

By Borel-Cantelli Lemma, we have

$$P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1, \ i.o.\} = 0, \ \mathbb{P}\text{-a.s.}$$

That is,

- P_ω-a.s. for n large enough, there is no particle in $(-\alpha n, +\infty)$.
- $\mathbb{P}$-a.s. for n large enough, $m_n^* < -\alpha n$.

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} < 0 | \ \Gamma \text{survives}\right) = 1;$$

Then

$$\sum_{n=1}^{+\infty} P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1\} < +\infty, \ \mathbb{P}\text{-a.s.}$$

By Borel-Cantelli Lemma, we have

$$P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1, \ i.o.\} = 0, \ \mathbb{P}\text{-a.s.}$$

That is,

- P_ω-a.s. for n large enough, there is no particle in $(-\alpha n, +\infty)$.
- $\mathbb{P}$-a.s. for n large enough, $m_n^* < -\alpha n$.

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} < 0 | \ \Gamma\text{survives}\right) = 1;$$

Then

$$\sum_{n=1}^{+\infty} P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1\} < +\infty, \ \mathbb{P}\text{-a.s.}$$

By Borel-Cantelli Lemma, we have

$$P_\omega\{\lambda[(-\alpha n, +\infty), n] \geq 1, \ i.o.\} = 0, \ \mathbb{P}\text{-a.s.}$$

That is,

- P_ω-a.s. for n large enough, there is no particle in $(-\alpha n, +\infty)$.
- $\mathbb{P}$-a.s. for n large enough, $m_n^* < -\alpha n$.

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} < 0 | \ \Gamma \text{survives}\right) = 1;$$

In the case of $m > m_c = \exp(I_\eta^q(0))$,

Step 1. Aim: Construct a supercritical Galton-Watson tree T, whose vertices of the nth generation are particles which are at a positive location at time nk_ω.

Basic idea: Hammersley(1974), Biggins(1977), Devulder(2007).

We first fix constants k_ω and Λ_ω:

- Fix $\varepsilon > 0$, s.t. $\log m > I_\eta^q(0) + \varepsilon$;
- For $k \geq n_\omega$,

$$E_\omega[\sum_{x \in \mathbb{N}} \lambda(x, k)] \geq \exp\{k[\log m - I_\eta^q(0) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- We can fix a integer $k_\omega > 0$ s.t.

$$E_\omega[\sum_{x \in \mathbb{N}} \lambda(x, k_\omega)] := \Lambda_\omega > 2.$$

In the case of $m > m_c = \exp(I_\eta^q(0))$,

Step 1. Aim: Construct a supercritical Galton-Watson tree T, whose vertices of the nth generation are particles which are at a positive location at time nk_ω.

Basic idea: Hammersley(1974), Biggins(1977), Devulder(2007).

We first fix constants k_ω and Λ_ω:

- Fix $\varepsilon > 0$, s.t. $\log m > I_\eta^q(0) + \varepsilon$;
- For $k \geq n_\omega$,

$$E_\omega[\sum_{x\in\mathbb{N}} \lambda(x,k)] \geq \exp\{k[\log m - I_\eta^q(0) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- We can fix a integer $k_\omega > 0$ s.t.

$$E_\omega[\sum_{x\in\mathbb{N}} \lambda(x,k_\omega)] := \Lambda_\omega > 2.$$

In the case of $m > m_c = \exp(I_\eta^q(0))$,

Step 1. Aim: Construct a supercritical Galton-Watson tree T, whose vertices of the nth generation are particles which are at a positive location at time nk_ω.

Basic idea: Hammersley(1974), Biggins(1977), Devulder(2007).

We first fix constants k_ω and Λ_ω:

- Fix $\varepsilon > 0$, s.t. $\log m > I_\eta^q(0) + \varepsilon$;
- For $k \geq n_\omega$,

$$E_\omega[\sum_{x\in\mathbb{N}} \lambda(x,k)] \geq \exp\{k[\log m - I_\eta^q(0) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- We can fix a integer $k_\omega > 0$ s.t.

$$E_\omega[\sum_{x\in\mathbb{N}} \lambda(x,k_\omega)] := \Lambda_\omega > 2.$$

In the case of $m > m_c$

In the case of $m > m_c = \exp(I_\eta^q(0))$,

Step 1. Aim: Construct a supercritical Galton-Watson tree T, whose vertices of the nth generation are particles which are at a positive location at time nk_ω.

Basic idea: Hammersley(1974), Biggins(1977), Devulder(2007).

We first fix constants k_ω and Λ_ω:

- Fix $\varepsilon > 0$, s.t. $\log m > I_\eta^q(0) + \varepsilon$;
- For $k \geq n_\omega$,

$$E_\omega[\sum_{x \in \mathbb{N}} \lambda(x,k)] \geq \exp\{k[\log m - I_\eta^q(0) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- We can fix a integer $k_\omega > 0$ s.t.

$$E_\omega[\sum_{x \in \mathbb{N}} \lambda(x,k_\omega)] := \Lambda_\omega > 2.$$

In the case of $m > m_c = \exp(I_\eta^q(0))$,

Step 1. Aim: Construct a supercritical Galton-Watson tree T, whose vertices of the nth generation are particles which are at a positive location at time nk_ω.

Basic idea: Hammersley(1974), Biggins(1977), Devulder(2007).

We first fix constants k_ω and Λ_ω:

- Fix $\varepsilon > 0$, s.t. $\log m > I_\eta^q(0) + \varepsilon$;
- For $k \geq n_\omega$,

$$E_\omega[\sum_{x\in\mathbb{N}} \lambda(x,k)] \geq \exp\{k[\log m - I_\eta^q(0) - \varepsilon]\}, \ \mathbb{P}\text{-a.s.}$$

- We can fix a integer $k_\omega > 0$ s.t.

$$E_\omega[\sum_{x\in\mathbb{N}} \lambda(x,k_\omega)] := \Lambda_\omega > 2.$$

Construct recursively a supercritical tree $\{Y_n\}$, s.t. at each time nk_ω, there are at least Y_n particles located in $\mathbb{N}$.

- $Y_0 = 1, \quad Y_1 = \lambda(\mathbb{N}, k_\omega)$.
- Suppose that at time nk_ω there are at least Y_n particles in $\mathbb{N}$.
  (We only consider there Y_n particles and ignore all the other particles which are possibly surviving at time nk_ω.)
- The number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$ and generated by these Y_n particles is greater than or equal to the number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$, and is greater than or equal to the number of particles generated by Y_n particles all of which are located at 0 at time nk_ω.
- Thus, at time $(n+1)k_\omega$, there are at least Y_{n+1} particles in $\mathbb{N}$,

$$Y_{n+1} := \sum_{i=1}^{Y_n} X_{n,i}, \quad \text{where } X_{n,i} =^d Y_1$$

  and are independent (given ω).

Construct recursively a supercritical tree $\{Y_n\}$, s.t. at each time nk_ω, there are at least Y_n particles located in $\mathbb{N}$.

- $Y_0 = 1, \quad Y_1 = \lambda(\mathbb{N}, k_\omega)$.
- Suppose that at time nk_ω there are at least Y_n particles in $\mathbb{N}$. (We only consider there Y_n particles and ignore all the other particles which are possibly surviving at time nk_ω.)
- The number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$ and generated by these Y_n particles is greater than or equal to the number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$, and is greater than or equal to the number of particles generated by Y_n particles all of which are located at 0 at time nk_ω.
- Thus, at time $(n+1)k_\omega$, there are at least Y_{n+1} particles in $\mathbb{N}$,

$$Y_{n+1} := \sum_{i=1}^{Y_n} X_{n,i}, \quad \text{where } X_{n,i} =^d Y_1$$

and are independent (given ω).

In the case of $m > m_c$

Construct recursively a supercritical tree $\{Y_n\}$, s.t. at each time nk_ω, there are at least Y_n particles located in $\mathbb{N}$.

- $Y_0 = 1, \quad Y_1 = \lambda(\mathbb{N}, k_\omega)$.
- Suppose that at time nk_ω there are at least Y_n particles in $\mathbb{N}$.
  (We only consider there Y_n particles and ignore all the other particles which are possibly surviving at time nk_ω.)
- The number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$ and generated by these Y_n particles is greater than or equal to the number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$, and is greater than or equal to the number of particles generated by Y_n particles all of which are located at 0 at time nk_ω.
- Thus, at time $(n+1)k_\omega$, there are at least Y_{n+1} particles in $\mathbb{N}$,

$$Y_{n+1} := \sum_{i=1}^{Y_n} X_{n,i}, \quad \text{where } X_{n,i} =^d Y_1$$

and are independent (given ω).

Construct recursively a supercritical tree $\{Y_n\}$, s.t. at each time nk_ω, there are at least Y_n particles located in $\mathbb{N}$.

- $Y_0 = 1, \quad Y_1 = \lambda(\mathbb{N}, k_\omega)$.
- Suppose that at time nk_ω there are at least Y_n particles in $\mathbb{N}$. (We only consider there Y_n particles and ignore all the other particles which are possibly surviving at time nk_ω.)
- The number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$ and generated by these Y_n particles is greater than or equal to the number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$, and is greater than or equal to the number of particles generated by Y_n particles all of which are located at 0 at time nk_ω.
- Thus, at time $(n+1)k_\omega$, there are at least Y_{n+1} particles in $\mathbb{N}$,

$$Y_{n+1} := \sum_{i=1}^{Y_n} X_{n,i}, \quad \text{where } X_{n,i} =^d Y_1$$

and are independent (given ω).

In the case of $m > m_c$

Note that

$$E_\omega(Y_1) = E_\omega(\lambda(\mathbb{N}, k_\omega)) = \Lambda_\omega > 2.$$

The constructed G-W tree is supercritical, and

Lemma

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) > 0, \ \mathbb{P}\text{-}a.s.$$

Remark:

- When T survives, $Y_n \geq 2^n$ as n large enough.
- With a positive probability, there is an exponential number of particles in $\mathbb{N}$ at time nk_ω, $n \in \mathbb{N}$..

Note that

$$E_\omega(Y_1) = E_\omega(\lambda(\mathbb{N}, k_\omega)) = \Lambda_\omega > 2.$$

The constructed G-W tree is supercritical, and

Lemma

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) > 0, \ \mathbb{P}\text{-}a.s.$$

Remark:

- When T survives, $Y_n \geq 2^n$ as n large enough.
- With a positive probability, there is an exponential number of particles in $\mathbb{N}$ at time nk_ω, $n \in \mathbb{N}$..

Note that

$$E_\omega(Y_1) = E_\omega(\lambda(\mathbb{N}, k_\omega)) = \Lambda_\omega > 2.$$

The constructed G-W tree is supercritical, and

Lemma

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) > 0, \ \mathbb{P}\text{-}a.s.$$

Remark:

- When T survives, $Y_n \geq 2^n$ as n large enough.
- With a positive probability, there is an exponential number of particles in $\mathbb{N}$ at time $nk_\omega, \ n \in \mathbb{N}$..

Proof. Let $B(k_\omega)$ denote the total number of particles at time k_ω, satisfies $E_\omega(B(k_\omega)^2) < \infty$. As a consequence, we have $E_\omega((Y_1)^2) < \infty$.

Since $E_\omega(Y_1) = \Lambda_\omega > 2$, which is greater than 1, there exists a random variable W_ω, satisfying $P_\omega(W_\omega > 0) > 0$, $\mathbb{P}$-a.s., and

$$\lim_{n\to\infty} \frac{Y_n}{(\Lambda_\omega)^n} = W_\omega, \ \mathbb{P}\text{-a.s.}$$

Note that $\Lambda_\omega > 2$. We have $Y_n \sim (\Lambda_\omega)^n W_\omega \geq 2^n$ for n large enough if $W_\omega > 0$.

Accordingly,

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) = P_\omega(W_\omega > 0) > 0, \ \mathbb{P}\text{-a.s.}$$

Proof. Let $B(k_\omega)$ denote the total number of particles at time k_ω, satisfies $E_\omega(B(k_\omega)^2) < \infty$. As a consequence, we have $E_\omega((Y_1)^2) < \infty$.

Since $E_\omega(Y_1) = \Lambda_\omega > 2$, which is greater than 1, there exists a random variable W_ω, satisfying $P_\omega(W_\omega > 0) > 0$, $\mathbb{P}$-a.s., and

$$\lim_{n\to\infty} \frac{Y_n}{(\Lambda_\omega)^n} = W_\omega, \ \mathbb{P}\text{-a.s.}$$

Note that $\Lambda_\omega > 2$. We have $Y_n \sim (\Lambda_\omega)^n W_\omega \geq 2^n$ for n large enough if $W_\omega > 0$. Accordingly,

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) = P_\omega(W_\omega > 0) > 0, \ \mathbb{P}\text{-a.s.}$$

Proof. Let $B(k_\omega)$ denote the total number of particles at time k_ω, satisfies $E_\omega(B(k_\omega)^2) < \infty$. As a consequence, we have $E_\omega((Y_1)^2) < \infty$.

Since $E_\omega(Y_1) = \Lambda_\omega > 2$, which is greater than 1, there exists a random variable W_ω, satisfying $P_\omega(W_\omega > 0) > 0$, $\mathbb{P}$-a.s., and

$$\lim_{n\to\infty} \frac{Y_n}{(\Lambda_\omega)^n} = W_\omega, \ \mathbb{P}\text{-a.s.}$$

Note that $\Lambda_\omega > 2$. We have $Y_n \sim (\Lambda_\omega)^n W_\omega \geq 2^n$ for n large enough if $W_\omega > 0$.

Accordingly,

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) = P_\omega(W_\omega > 0) > 0, \ \mathbb{P}\text{-a.s.}$$

In the case of $m > m_c$

Proof. Let $B(k_\omega)$ denote the total number of particles at time k_ω, satisfies $E_\omega(B(k_\omega)^2) < \infty$. As a consequence, we have $E_\omega((Y_1)^2) < \infty$.

Since $E_\omega(Y_1) = \Lambda_\omega > 2$, which is greater than 1, there exists a random variable W_ω, satisfying $P_\omega(W_\omega > 0) > 0$, $\mathbb{P}$-a.s., and

$$\lim_{n\to\infty} \frac{Y_n}{(\Lambda_\omega)^n} = W_\omega, \ \mathbb{P}\text{-a.s.}$$

Note that $\Lambda_\omega > 2$. We have $Y_n \sim (\Lambda_\omega)^n W_\omega \geq 2^n$ for n large enough if $W_\omega > 0$.

Accordingly,

$$\lim_{n\to\infty} P_\omega(\cap_{l\geq n}\{Y_l \geq 2^l\}) = P_\omega(W_\omega > 0) > 0, \ \mathbb{P}\text{-a.s.}$$

Step 2. Aim: Some of the particles originated from T will go far enough.

For $n \in \mathbb{N}$, $A \in \mathbb{N}$, and any integer N,

$$P_\omega\{\lambda([A,+\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\}$$

$$= \prod_{x\in\mathbb{Z}} \prod_{l=1}^{\lambda(x,nk_\omega)} P_\omega\{\lambda([A,+\infty), N) = 0\} = \prod_{x\in\mathbb{Z}} [P_\omega^x(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)}$$

By coupling, we have for $x \geq 0$,

$$P_\omega^x(\lambda([A,+\infty), N) = 0) \leq P_\omega^0(\lambda([A,+\infty), N) = 0).$$

Thus

$$\prod_{x\in\mathbb{Z}} [P_\omega^x(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)}$$

$$\leq \prod_{x\in\mathbb{N}} [P_\omega^0(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)} \leq [P_\omega^0(\lambda([A,+\infty), N) = 0)]^{Y_n}.$$

Step 2. Aim: Some of the particles originated from T will go far enough.

For $n \in \mathbb{N}$, $A \in \mathbb{N}$, and any integer N,

$$P_\omega\{\lambda([A,+\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\}$$
$$= \prod_{x\in\mathbb{Z}} \prod_{l=1}^{\lambda(x,nk_\omega)} P_\omega\{\lambda([A,+\infty), N) = 0\} = \prod_{x\in\mathbb{Z}} [P_\omega^x(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)}$$

By coupling, we have for $x \geq 0$,

$$P_\omega^x(\lambda([A,+\infty), N) = 0) \leq P_\omega^0(\lambda([A,+\infty), N) = 0).$$

Thus

$$\prod_{x\in\mathbb{Z}} [P_\omega^x(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)}$$
$$\leq \prod_{x\in\mathbb{N}} [P_\omega^0(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)} \leq [P_\omega^0(\lambda([A,+\infty), N) = 0)]^{Y_n}.$$

Step 2. Aim: Some of the particles originated from T will go far enough.
For $n \in \mathbb{N}$, $A \in \mathbb{N}$, and any integer N,

$$P_\omega\{\lambda([A,+\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\}$$
$$= \prod_{x\in\mathbb{Z}} \prod_{l=1}^{\lambda(x,nk_\omega)} P_\omega\{\lambda([A,+\infty), N) = 0\} = \prod_{x\in\mathbb{Z}} [P_\omega^x(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)}$$

By coupling, we have for $x \geq 0$,

$$P_\omega^x(\lambda([A,+\infty), N) = 0) \leq P_\omega^0(\lambda([A,+\infty), N) = 0).$$

Thus

$$\prod_{x\in\mathbb{Z}} [P_\omega^x(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)}$$
$$\leq \prod_{x\in\mathbb{N}} [P_\omega^0(\lambda([A,+\infty), N) = 0)]^{\lambda(x,nk_\omega)} \leq [P_\omega^0(\lambda([A,+\infty), N) = 0)]^{Y_n}.$$

Let $a \in (0,1)$, $\varepsilon' > 0$. Then there exist $M_\omega \in \mathbb{N}$, s.t. $\forall\, N \geq M_\omega$,

$$P_\omega(X_N \geq Na) \geq \exp\{[-(I_\eta^q(a) + \varepsilon')N]\}, \ \mathbb{P}\text{-a.s.}$$

If q_N denotes the probability that the Galton-Watson tree Γ extinct before time N, we notice that for $N \geq M_\omega$,

$$\begin{aligned} P_\omega^0(\lambda([aN, +\infty), N) = 0) &\leq q_N + (1 - q_N) P_\omega(X_N \geq Na) \\ &\leq 1 - (1 - q_N) \exp[-(I_\eta^q(a) + \varepsilon')N]. \end{aligned}$$

Therefore $\forall N \geq M_\omega$,

$$\begin{aligned} & P_\omega\{\lambda([aN, +\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq \ & [P_\omega^0(\lambda([aN, +\infty), N) = 0)]^{Y_n} \leq [1 - (1 - q_N) \exp[-(I_\eta^q(a) + \varepsilon')N]]^{Y_n}. \end{aligned}$$

Let $a \in (0,1)$, $\varepsilon' > 0$. Then there exist $M_\omega \in \mathbb{N}$, s.t. $\forall\, N \geq M_\omega$,

$$P_\omega(X_N \geq Na) \geq \exp\{[-(I_\eta^q(a) + \varepsilon')N]\},\ \mathbb{P}\text{-a.s.}$$

If q_N denotes the probability that the Galton-Watson tree Γ extinct before time N, we notice that for $N \geq M_\omega$,

$$\begin{aligned} P_\omega^0(\lambda([aN, +\infty), N) = 0) &\leq q_N + (1 - q_N) P_\omega(X_N \geq Na) \\ &\leq 1 - (1 - q_N)\exp[-(I_\eta^q(a) + \varepsilon')N]. \end{aligned}$$

Therefore $\forall N \geq M_\omega$,

$$\begin{aligned} & P_\omega\{\lambda([aN, +\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq\ & [P_\omega^0(\lambda([aN, +\infty), N) = 0)]^{Y_n} \leq [1 - (1 - q_N)\exp[-(I_\eta^q(a) + \varepsilon')N]]^{Y_n}. \end{aligned}$$

Let $a \in (0,1)$, $\varepsilon' > 0$. Then there exist $M_\omega \in \mathbb{N}$, s.t. $\forall\, N \geq M_\omega$,

$$P_\omega(X_N \geq Na) \geq \exp\{[-(I_\eta^q(a) + \varepsilon')N]\},\ \mathbb{P}\text{-a.s.}$$

If q_N denotes the probability that the Galton-Watson tree Γ extinct before time N, we notice that for $N \geq M_\omega$,

$$\begin{aligned} P_\omega^0(\lambda([aN,+\infty),N)=0) &\leq q_N + (1-q_N)P_\omega(X_N \geq Na) \\ &\leq 1-(1-q_N)\exp[-(I_\eta^q(a)+\varepsilon')N]. \end{aligned}$$

Therefore $\forall N \geq M_\omega$,

$$\begin{aligned} & P_\omega\{\lambda([aN,+\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq\ & [P_\omega^0(\lambda([aN,+\infty),N)=0)]^{Y_n} \leq [1-(1-q_N)\exp[-(I_\eta^q(a)+\varepsilon')N]]^{Y_n}. \end{aligned}$$

In the case of $m > m_c$

Let $E_1(\omega, n) = \{Y_n \geq 2^n\}$, and notice that $q_N \leq q_\infty \in [0, 1)$.
As a consequence, on $E_1(\omega, n)$, we obtain for $N \geq M_\omega$,

$$\begin{aligned} & \log P_\omega\{\lambda([aN, +\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq \quad & 2^n \log[1 - (1 - q_N) \exp[-(I_\eta^q(a) + \varepsilon')N]] \\ \leq \quad & -(1 - q_\infty) \exp[n \log 2 - (I_\eta^q(a) + \varepsilon')N]]. \end{aligned}$$

Let $N_n = 2\lfloor \frac{n \log 2}{4(I_\eta^q(a) + \varepsilon')} \rfloor$. For all large n, we obtain on $E_1(\omega, n)$,

$$\begin{aligned} & P_\omega\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq \quad & \exp\{-(1 - q_\infty) C \exp[n(\log 2)/2]\}, \end{aligned}$$

where $C > 0$ is a constant. Hence

$$\sum_{n \in \mathbb{N}} P_\omega(\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0\} \bigcap E_1(\omega, n)) < +\infty.$$

In the case of $m > m_c$

Let $E_1(\omega, n) = \{Y_n \geq 2^n\}$, and notice that $q_N \leq q_\infty \in [0, 1)$.
As a consequence, on $E_1(\omega, n)$, we obtain for $N \geq M_\omega$,

$$\begin{aligned}
& \log P_\omega\{\lambda([aN, +\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\
\leq \quad & 2^n \log[1 - (1 - q_N) \exp[-(I_\eta^q(a) + \varepsilon')N]] \\
\leq \quad & -(1 - q_\infty) \exp[n \log 2 - (I_\eta^q(a) + \varepsilon')N]].
\end{aligned}$$

Let $N_n = 2\lfloor \frac{n \log 2}{4(I_\eta^q(a)+\varepsilon')} \rfloor$. For all large n, we obtain on $E_1(\omega, n)$,

$$\begin{aligned}
& P_\omega\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\
\leq \quad & \exp\{-(1 - q_\infty) C \exp[n(\log 2)/2]\},
\end{aligned}$$

where $C > 0$ is a constant. Hence

$$\sum_{n \in \mathbb{N}} P_\omega(\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0\} \bigcap E_1(\omega, n)) < +\infty.$$

In the case of $m > m_c$

Let $E_1(\omega, n) = \{Y_n \geq 2^n\}$, and notice that $q_N \leq q_\infty \in [0, 1)$.
As a consequence, on $E_1(\omega, n)$, we obtain for $N \geq M_\omega$,

$$\begin{aligned} & \log P_\omega\{\lambda([aN, +\infty), nk_\omega + N) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq & \; 2^n \log[1 - (1 - q_N) \exp[-(I_\eta^q(a) + \varepsilon')N]] \\ \leq & \; -(1 - q_\infty) \exp[n \log 2 - (I_\eta^q(a) + \varepsilon')N]]. \end{aligned}$$

Let $N_n = 2\lfloor \frac{n \log 2}{4(I_\eta^q(a)+\varepsilon')} \rfloor$. For all large n, we obtain on $E_1(\omega, n)$,

$$\begin{aligned} & P_\omega\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0 | \mathscr{F}_\omega(nk_\omega)\} \\ \leq & \; \exp\{-(1 - q_\infty) C \exp[n(\log 2)/2]\}, \end{aligned}$$

where $C > 0$ is a constant. Hence

$$\sum_{n \in \mathbb{N}} P_\omega(\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0\} \bigcap E_1(\omega, n)) < +\infty.$$

- Let $E_2(\omega, n) = \bigcap_{l \geq n} E_1(\omega, l)$. Then for almost all environment ω, there exists an integer n_ω, s.t. $P_\omega(E_2(\omega, n_\omega)) > 0$.
- By the Borel-Cantellic lemma, we obtain P_ω-a.s. on $E_2(\omega, n_\omega)$, for n large enough,
$$\lambda([aN_n, +\infty), nk_\omega + N_n) \geq 1.$$
Then P_ω-a.s. on $E_2(\omega, n)$, for all large n, there exists a particle p_n in $[aN_n, +\infty)$ at time $K_n = nk_\omega + N_n$.
- At any time $l \in (K_{n-1}, K_n] \bigcap \mathbb{Z}$, the ancestor of the particle p_n is located in $[aN_n - (K_n - K_{n-1}), +\infty)$, which is contained in $[S_\omega, +\infty)$ for some constant $S_\omega > 0$.

In the case of $m > m_c$

- Let $E_2(\omega, n) = \bigcap_{l \geq n} E_1(\omega, l)$. Then for almost all environment ω, there exists an integer n_ω, s.t. $P_\omega(E_2(\omega, n_\omega)) > 0$.
- By the Borel-Cantellic lemma, we obtain P_ω-a.s. on $E_2(\omega, n_\omega)$, for n large enough,

  $$\lambda([aN_n, +\infty), nk_\omega + N_n) \geq 1.$$

  Then P_ω-a.s. on $E_2(\omega, n)$, for all large n, there exists a particle p_n in $[aN_n, +\infty)$ at time $K_n = nk_\omega + N_n$.
- At any time $l \in (K_{n-1}, K_n] \bigcap \mathbb{Z}$, the ancestor of the particle p_n is located in $[aN_n - (K_n - K_{n-1}), +\infty)$, which is contained in $[S_\omega, +\infty)$ for some constant $S_\omega > 0$.

In the case of $m > m_c$

- Let $E_2(\omega, n) = \bigcap_{l \geq n} E_1(\omega, l)$. Then for almost all environment ω, there exists an integer n_ω, s.t. $P_\omega(E_2(\omega, n_\omega)) > 0$.
- By the Borel-Cantellic lemma, we obtain P_ω-a.s. on $E_2(\omega, n_\omega)$, for n large enough,

$$\lambda([aN_n, +\infty), nk_\omega + N_n) \geq 1.$$

  Then P_ω-a.s. on $E_2(\omega, n)$, for all large n, there exists a particle p_n in $[aN_n, +\infty)$ at time $K_n = nk_\omega + N_n$.
- At any time $l \in (K_{n-1}, K_n] \bigcap \mathbb{Z}$, the ancestor of the particle p_n is located in $[aN_n - (K_n - K_{n-1}), +\infty)$, which is contained in $[S_\omega, +\infty)$ for some constant $S_\omega > 0$.

- Thus, for all large l,

$$\lambda([S_\omega l, +\infty), l) \geq 1.$$

This means that P_ω-a.s. on $E_2(\omega, n)$, there are at any large time some particles with average speed greater than S_ω.

- Since $P_\omega(E_2(\omega, n_\omega)) > 0$, we can obtain

Lemma

Let m_n^* denote the location of the rightmost particle at time n. For almost all environment ω, there exists a real number $S_\omega > 0$, such that

$$P_\omega\left(\liminf_{n\to\infty} \frac{m_n^*}{n} \geq S_\omega\right) > 0.$$

- Thus, for all large l,

$$\lambda([S_\omega l, +\infty), l) \geq 1.$$

This means that P_ω-a.s. on $E_2(\omega, n)$, there are at any large time some particles with average speed greater than S_ω.

- Since $P_\omega(E_2(\omega, n_\omega)) > 0$, we can obtain

Lemma

Let m_n^* denote the location of the rightmost particle at time n. For almost all environment ω, there exists a real number $S_\omega > 0$, such that

$$P_\omega\left(\liminf_{n\to\infty} \frac{m_n^*}{n} \geq S_\omega\right) > 0.$$

Step 3. Aim: Although T only has a positive survival probability, there are always particles going very far, as long as the branching process Γ survives.

- Define the event

$$A(S) = \{\liminf_{n\to\infty} \frac{m_n^*}{n} \geq S\}.$$

- $\{P_{\theta^i\omega}(A(S))\}_{i\in\mathbb{Z}}$ is a stationary sequence, and nondecreasing. Thus $P_{\theta^i\omega}(A(S$ $P_\omega(A(S)),\ \mathbb{P}$-a.s., $\forall i \in \mathbb{Z}$.
- For a.a. ω, $\exists\ S_\omega > 0$, s.t.

$$P_{\theta^i\omega}(A(S_\omega)) = P_\omega(A(S_\omega)) > 0.$$

- Define $E_3(N) = \{B(N) \geq r^N\}$, where $B(N)$ denote the total number of particles at time N, $1 < r < m$. Then

$$\{\Gamma\text{survives}\} = \liminf_{N\to\infty} E_3(N),$$

Step 3. Aim: Although T only has a positive survival probability, there are always particles going very far, as long as the branching process Γ survives.

- Define the event
$$A(S) = \{\liminf_{n\to\infty} \frac{m_n^*}{n} \geq S\}.$$
- $\{P_{\theta^i\omega}(A(S))\}_{i\in\mathbb{Z}}$ is a stationary sequence, and nondecreasing. Thus $P_{\theta^i\omega}(A(S$ $P_\omega(A(S)),\ \mathbb{P}$-a.s., $\forall i \in \mathbb{Z}$.
- For a.a. ω, $\exists\ S_\omega > 0$, s.t.
$$P_{\theta^i\omega}(A(S_\omega)) = P_\omega(A(S_\omega)) > 0.$$
- Define $E_3(N) = \{B(N) \geq r^N\}$, where $B(N)$ denote the total number of particles at time N, $1 < r < m$. Then
$$\{\Gamma\text{survives}\} = \liminf_{N\to\infty} E_3(N),$$

Step 3. Aim: Although T only has a positive survival probability, there are always particles going very far, as long as the branching process Γ survives.

- Define the event

$$A(S) = \{\liminf_{n\to\infty} \frac{m_n^*}{n} \geq S\}.$$

- $\{P_{\theta^i\omega}(A(S))\}_{i\in\mathbb{Z}}$ is a stationary sequence, and nondecreasing. Thus $P_{\theta^i\omega}(A(S$ $P_\omega(A(S))$, $\mathbb{P}$-a.s., $\forall i \in \mathbb{Z}$.
- For a.a. ω, $\exists\ S_\omega > 0$, s.t.

$$P_{\theta^i\omega}(A(S_\omega)) = P_\omega(A(S_\omega)) > 0.$$

- Define $E_3(N) = \{B(N) \geq r^N\}$, where $B(N)$ denote the total number of particles at time N, $1 < r < m$. Then

$$\{\Gamma\text{survives}\} = \liminf_{N\to\infty} E_3(N),$$

Step 3. Aim: Although T only has a positive survival probability, there are always particles going very far, as long as the branching process Γ survives.

- Define the event
$$A(S) = \{\liminf_{n\to\infty} \frac{m_n^*}{n} \geq S\}.$$
- $\{P_{\theta^i\omega}(A(S))\}_{i\in\mathbb{Z}}$ is a stationary sequence, and nondecreasing. Thus $P_{\theta^i\omega}(A(S$ $P_\omega(A(S))$, $\mathbb{P}$-a.s., $\forall i \in \mathbb{Z}$.
- For a.a. ω, $\exists\ S_\omega > 0$, s.t.
$$P_{\theta^i\omega}(A(S_\omega)) = P_\omega(A(S_\omega)) > 0.$$
- Define $E_3(N) = \{B(N) \geq r^N\}$, where $B(N)$ denote the total number of particles at time N, $1 < r < m$. Then
$$\{\Gamma\text{survives}\} = \liminf_{N\to\infty} E_3(N),$$

- On $E_3(N)$,

$$P_\omega\left(\liminf_{n\to\infty}\frac{m_n^*}{n} < S|\mathscr{F}_\omega(N)\right)$$
$$\leq \prod_{x\in\mathbb{Z}}(1-P_{\theta^x\omega}[A(S)])^{\lambda(x,N)} \leq [1-P_\omega[A(S)]]^{r^N}.$$

- Since $P_\omega(A(S_\omega)) > 0$,

$$\lim_{N\to\infty}\left(P_\omega\{\liminf_{n\to\infty}\frac{m_n^*}{n} < S_\omega|\mathscr{F}_\omega(N)\}\cap E_3(N)\right)$$
$$\leq \lim_{N\to\infty}[1-P_\omega(A(S_\omega))]^{r^N} = 0.$$

- This yields

$$\mathbb{P}\left(\liminf_{n\to\infty}\frac{m_n^*}{n} > 0|\ \Gamma\text{survives}\right) = 1.$$

In the case of $m > m_c$

- On $E_3(N)$,

$$P_\omega\left(\liminf_{n\to\infty}\frac{m_n^*}{n} < S|\mathscr{F}_\omega(N)\right)$$

$$\leq \prod_{x\in\mathbb{Z}}(1 - P_{\theta^x\omega}[A(S)])^{\lambda(x,N)} \leq [1 - P_\omega[A(S)]]^{r^N}.$$

- Since $P_\omega(A(S_\omega)) > 0$,

$$\lim_{N\to\infty}\left(P_\omega\{\liminf_{n\to\infty}\frac{m_n^*}{n} < S_\omega|\mathscr{F}_\omega(N)\} \cap E_3(N)\right)$$

$$\leq \lim_{N\to\infty}[1 - P_\omega(A(S_\omega))]^{r^N} = 0.$$

- This yields

$$\mathbb{P}\left(\liminf_{n\to\infty}\frac{m_n^*}{n} > 0|\ \Gamma\text{survives}\right) = 1.$$

References

1. Athreya, K.B., Ney, P.E. (1972). *Branching Processes.* Springer-Verlag, Berlin.
2. Comets, F., Menshikov, M.V., Popov, S.Y. (1998). *One-dimensional branching random walk in a random environment: a classification.* Markov Process. Related Fields, Vol.4, 465-477.
3. Devulder, A. (2007). *The speed of a branching system of random walks in random environment.* Stat. Prob. Letters, Vol.77, 1712-1721.
4. Hong, W.M., Wang, H.M. (2013). *Branching structure for a random walk in random environment with bounded jumps and its application.* Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol.16(1), 1350006(14 pages).
5.Révész, P. (1994). *Random walks of infinitely many particles.* World Scientific, Singapore.
6. Yilmaz,A. (2009). Quenched large deviations for random walk in a random environment. Communications on Pure and Applied Mathematics, Vol.62(8): 1033 – 1075.
7. Zeitouni, O. (2004). Lectures notes on random walks in random environment. Lecture Notes in Math., Vol.1837, Springer, Berlin, 189-312.

Thanks for your attention