Some convergence results related to a stable branching random walk

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Contents

This talk is based on two joint works. One is with H. He and J. Liu, the other is with J. Liu.

Branching random walk

- It starts with an initial particle located at the origin.
- At time 1, the particle dies, producing some new particles, positioned according to Θ.
- At time 2, these particles die, each giving birth to new particles positioned (with respect to the birth place) according to Θ.
- The process goes on with the same mechanism. We assume each particle produces new particles independently.

For each vertex x on the branching tree, we denote the position by $V(x)$. The family of random variables $(V(x))$ is referred as a branching random walk (Biggins ('10)).

We assume

$$\mathbf{E}(\sum_{|x|=1} 1) > 1, \quad \text{(supercritical)}$$

$$\mathbf{E}(\sum_{|x|=1} e^{-V(x)}) = 1, \ \mathbf{E}(\sum_{|x|=1} V(x)e^{-V(x)}) = 0, \quad \text{(boundary case)} \tag{1}$$

where $|x|$ denotes the generation of x. $(V(x))$ can be reduced to this case by some renormalization, if Θ is not bounded from below (see Jaffuel ('12)).

Additive martingale

$$W_n := \sum_{|x|=n} e^{-V(x)}, \quad \text{(additive martingale)}$$

- $V(x) \equiv 0$: (W_n) degenerates to a supercritical GW process.
- $V(x) \neq 0$: (W_n) converges almost surely to 0 under (1) (supercritical + boundary case, Biggins ('77), Lyon ('97)).

It is natural to ask

At which rate W_n goes to 0?

Related work:

- Galton-Watson processes: Seneta ('68), Heyde ('70).
- Branching random walk:
  - General case: Biggins and Kyprianou ('96, '97).
  - Boundary case: Hu and Shi ('09), Aidekon and Shi ('14).

Derivative martingale

$$D_n := \sum_{|x|=n} V(x)e^{-V(x)}, \quad \text{(derivative martingale)}$$

Related work:

- Non-boundary case: Barral ('00), Biggins ('91,'92);
- Boundary case: Biggins and Kyprianou ('04), Chen ('15), etc.

Related results on derivative martingale

Suppose that

$$\mathbf{E}(\sum_{|x|=1} V^2(x)e^{-V(x)}) < \infty. \tag{2}$$

Theorem A (Biggins and Kyprianou ('04))

Assume (1) and (2) hold. Then there exists a nonnegative random variable D_∞ such that

$$D_n \to D_\infty, \ \ \mathbf{P}-a.s.$$

Theorem B (Chen ('15))

Assume (1) and (2) hold. Then $\mathbf{P}(D_\infty > 0) > 0$ if and only if the following condition holds:

$$\mathbf{E}[X(\log_+ X)^2 + \widetilde{X} \log_+ \widetilde{X}] < \infty, \tag{3}$$

where $\log_+ y := 0 \vee \log y$ and

$$X := \sum_{|x|=1} e^{-V(x)}, \quad \widetilde{X} := \sum_{|x|=1} V(x)_+ e^{-V(x)}, \tag{4}$$

with $V(x)_+ := V(x) \vee 0$.

When D_∞ is nontrivial, $\mathbf{P}(D_\infty > 0)$ equals to the non-extinction probability of the branching tree. Define

$$\mathbf{P}^*(\cdot) := \mathbf{P}(\cdot\,|\text{non extinction})$$

Obviously $W_n \to 0,\ \mathbf{P}^*\text{–}a.s.$

Theorem C (Aidekon and Shi (2014))

Assume (1), (2) and (3) hold. Under $\mathbf{P}^*$, we have

$$n^{1/2} W_n \xrightarrow{\mathsf{P}^*} \Big(\frac{2}{\pi\sigma^2}\Big)^{1/2} D_\infty, \tag{5}$$

where $D_\infty > 0$ is the random variable in Theorem A, and

$$\sigma^2 := \mathbf{E}\Big(\sum_{|x|=1} V(x)^2 e^{-V(x)}\Big) < \infty.$$

Stable branching random walk

In this paper, instead of (2) and (3), we shall study $(V(x))$ under (1) and ($\alpha \in (1,2)$):

$$(i)\ \ \mathbf{E}\Big(\sum_{|x|=1} I_{\{V(x)\le -y\}} e^{-V(x)}\Big) = o(y^{-\alpha}), \quad y \to +\infty, \tag{6}$$

$$(ii)\ \ \mathbf{E}\Big(\sum_{|x|=1} I_{\{V(x)\ge y\}} e^{-V(x)}\Big) \sim C y^{-\alpha}, \quad y \to +\infty, \tag{7}$$

$$(iii)\ \ \mathbf{E}\big(X(\log_+ X)^{\alpha} + \widetilde{X}(\log_+ \widetilde{X})^{\alpha-1}\big) < \infty. \tag{8}$$

Under (6) and (7), the step of the one-dimensional random walk (S_n) associated with $(V(x))$ belongs to the domain of attraction of a stable law.

(The many-to-one formula)

$$\mathbf{E}\Big[\sum_{|x|=n} g(V(x_1),\ldots,V(x_n))\Big] = \mathbf{E}\big[e^{S_n} g(S_1,\ldots,S_n)\big],$$

where (S_n) is a random walk, $E(S_1)=0$ (by (1)) and S_1 belongs to the domain of attraction of a stable law with characteristic function (spectrally positive). see Biggins and Kyprianou (1997), Lyons (1997), Lyons et al (1995).

$$G(t) := \exp\Big\{-c|t|^{\alpha}\Big(1 - i\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\Big)\Big\},\ c>0.$$

$$(2) \Leftrightarrow E(S_1^2) < \infty$$

$$(6)+(7) \Leftrightarrow \begin{cases} P(S_1 > y) \sim Cy^{-\alpha}, \\ P(S_1 < -y) = o(y^{-\alpha}), \end{cases} \quad E(S_1^2) = \infty$$

Main results – derivative martingale

Theorem 1 (HLZ ('17)) Assume (1), (6), (7). Then there exists a nonnegative random variable D_∞ such that

$$D_n \to D_\infty, \quad \mathbf{P}-a.s.$$

Moreover, if condition (8) holds, then $\mathbf{P}^*(D_\infty > 0) = 1$.

Main results – additive martingale

Theorem 2 (HLZ ('17)) Assume (1), (6)–(8). We have

$$n^{\frac{1}{\alpha}} W_n \xrightarrow{\mathbf{P}^*} \frac{\theta}{\Gamma(1-1/\alpha)} D_\infty.$$

where $D_\infty > 0$ is given in Theorem 1, and θ is a positive constant related to the renewal function of (S_n).

Theorem 3 (LZ ('17)) Assume (1), (6)–(8). For any function $f \uparrow \infty$, we have $\mathbf{P}^*$–a.s.

$$\limsup_{n\to\infty} \frac{n^{\frac{1}{\alpha}} W_n}{f(n)} = \begin{cases} 0 \\ \infty \end{cases} \Leftrightarrow \int_0^\infty \frac{1}{t\, f(t)} dt \begin{cases} < \infty \\ = \infty. \end{cases}$$

$$\overline{\lim}_{n\to\infty}\, n^{\frac{1}{\alpha}} W_n = \infty \quad \mathbf{P}^* - a.s. \tag{9}$$

Minimal position

Denote $M_n = \min_{|x|=n} V(x)$.

The asymptotic behaviors of M_n: Biggins ('77), Lyons ('97), Addario-Berry and Reed ('09), Aidékon ('13), Bramson and Zeitouni ('06), Hu and Shi ('09), etc.

$$\liminf_{n\to\infty} \left(M_n - \frac{1}{2}\log n\right) = -\infty, \quad a.s. \quad \text{(Aidékon and Shi ('14))}$$

$$\liminf_{n\to\infty} \frac{M_n - \frac{1}{2}\log n}{\log\log n} = -1, \quad a.s. \quad \text{(Hu ('15))}$$

When $\mathbf{E}(\sum_{|x|=1}(V(x)_+)^3 e^{-V(x)}) < \infty$:

$$\limsup_{n\to\infty} \frac{M_n - \frac{3}{2}\log n}{\log\log\log n} = 1. \quad a.s. \quad \text{(Hu ('13))}$$

The second order limits of M_n

Theorem 4 (LZ ('17)) Assume (1), (6)–(8). For any function f satisfying $f \uparrow \infty$, we have

$$\mathbf{P}^*(M_n - \frac{1}{\alpha}\log n < -f(n), \text{i.o.}) = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int_0^\infty \frac{dt}{te^{f(t)}} \begin{cases} < \infty \\ = \infty. \end{cases}$$

Choose $f_1 = \log\log n$ and $f_2 = (1+\varepsilon)\log\log n$. Then

$$\liminf_{n\to\infty} \frac{M_n - \frac{1}{\alpha}\log n}{\log\log n} = -1, \ \mathbf{P}^*\text{–a.s.}$$

Theorem 5 (LZ ('17)) Assume (1), (6)–(8).

$$\limsup_{n\to\infty} \frac{M_n - (1+\frac{1}{\alpha})\log n}{\log\log\log n} \geq 1, \ \mathbf{P}^*\text{–a.s.}$$

Sketch of proofs – estimates for (S_n)

In the proofs, we depend heavily on the probability estimations for (S_n), and some properties for (S_n) conditioned to stay in $[-x, \infty)$. We refer to Vatutin and Wachtel ('09) and prove more inequalities. For example,

$$\sum_{l \geq 0} \mathbf{P}_z(S_l \leq x,\, \underline{S}_l \geq 0) \leq c\,(1+x)^{\alpha-1}(1+\min(x,z));$$

$$E\Big(f\big(\frac{S_n + x}{n^{1/\alpha}}\big)\mathbf{1}_{\{\underline{S}_n \geq -x\}}\Big) = \frac{R(x)}{\Gamma(1-\frac{1}{\alpha})n^{1/\alpha}}\Big(\int_0^\infty f(t)p_\alpha(t)dt + o_n(1)\Big).$$

uniformly in $x \in [0, d_n]$ with $d_n = o(n^{1/\alpha})$. And $\mathbf{E}\,(M_\alpha) = \frac{\Gamma(1-\frac{1}{\alpha})}{\theta}$ $(M_\alpha \leftrightarrow p_\alpha)$.

Sketch of proofs – truncating argument

It originated from Harris ('99), was formalized for BBM by Kyprianou ('04), and later be used for BRW by Biggins and Kyprianou ('04) and then Aidekon and Shi ('14).

Define $\underline{V}(x) := \min_{y \in \langle \varnothing, x \rangle} V(y)$. We use the renewal function $R(u)$ of (S_n) to introduce the truncated processes $(\beta > 0)$:

$$W_n^\beta := \sum_{|x|=n} e^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\beta\}}, \qquad (\sim W_n)$$

$$D_n^\beta := \sum_{|x|=n} R(V(x)+\beta) e^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\beta\}}. \qquad (\sim \theta D_n)$$

Note that $\lim_{u \to \infty} \frac{R(u)}{u} = \theta \in (0, \infty)$ for $S_1 \in D(\alpha, -1)$.

Sketch of proofs – Spine decomposition (1)

$$\left.\frac{d\,\mathbf{P}^{\beta}}{d\,\mathbf{P}}\right|_{\mathcal{F}_n} := \frac{D_n^{\beta}}{R(\beta)}, \quad \text{(change of probabilities)}$$

$\hat{\Theta}$ is the point process $(V(x), |x| = 1)$ under $\mathbf{P}^{\beta}$. The branching random walk $(V(x))$ under $\mathbf{P}^{\beta}$ can be described as follows:

At time 0, $V(\omega_0^{\beta}) = 0$. At each step n, all particles produce according to Θ, except ω_n^{β} according to $\hat{\Theta}$. ω_{n+1}^{β} is chosen to be y among the children of ω_n^{β} proportional to $R_{\beta}(V(y))e^{-V(y)}\mathbf{1}_{\{\underline{V}(y)\geq -\beta\}}$. Hence there is a "spine" in the branching tree.

The spine process $(V(\omega_n^{\beta}), n \geq 0)$ under $\mathbf{P}^{\beta}$ is distributed as $(S_n)_{n\geq 0}$ conditioned to stay above $-\beta$ under $\mathbf{P}$ (Biggins and Kyprianou ('04)).

Sketch of proofs – Spine decomposition (2)

Denote the natural filtration of $(V(x))$ by $(\mathcal{F}_n)$. With W_n, define $\mathbf{Q}$ such that

$$\mathbf{Q}\big|_{\mathcal{F}_n} := W_n \cdot \mathbf{P}\big|_{\mathcal{F}_n}, \quad n \geq 1.$$

We give a description of $(V(x))$ under $\mathbf{Q}$.

$V(\omega_0) = 0$. each particle v in the nth generation dies and produces independently as $(V(x), |x| = 1)$ under $\mathbf{P}_{V(v)}$, except ω_n producing as $(V(x), |x| = 1)$ under $\mathbf{Q}_{V(\omega_n)}$. ω_{n+1} is chosen to be x among the children of ω_n, proportionally to $e^{-V(x)}$.

$(V(\omega_n))_{n\geq 0}$ under $\mathbf{Q}$ has the distribution of $(S_n)_{n\geq 0}$ under $\mathbf{P}$ (Lyons ('97)).

Proof of Theorem 1:

- prove $D_n^\beta \to D_\infty^\beta$ (truncated martingale convergence)
- prove $D_n \to D_\infty$.
- prove $\mathbf{P}(D_\infty^\beta > 0) > 0$. ($(D_n^\beta)$ is uniformly integrable)
- prove $\mathbf{P}^*(D_\infty > 0) = 1$. ($D_\infty^\beta \le cD_\infty$, a.s.)

Proof of Theorem 2: We first have

$$\mathbf{E}^{\beta}\left(\frac{W_n^{\beta}}{D_n^{\beta}}\right) \sim \frac{1}{\Gamma(1-1/\alpha)n^{\frac{1}{\alpha}}},$$

$$\mathbf{E}^{\beta}\left(\left(\frac{W_n^{\beta}}{D_n^{\beta}}\right)^2\right) \sim \frac{1}{\big(\Gamma(1-1/\alpha)\big)^2 n^{\frac{2}{\alpha}}}.$$

Therefore $\lim_{n\to\infty} n^{\frac{1}{\alpha}}\left(\frac{W_n^{\beta}}{D_n^{\beta}}\right) = \Gamma(1-\frac{1}{\alpha}), \quad in\ probability \quad (\mathbf{P}^{\beta})$

Finally, we manage to change the setting from $\mathbf{P}^{\beta}$ to $\mathbf{P}$.

Proof of Theorems 3 and 4:

We only need to prove the the convergence part in Theorem 4, i.e.,

$$\int_0^\infty \frac{dt}{tf(t)} < \infty \Rightarrow \limsup_{n\to\infty} \frac{n^{\frac{1}{\alpha}} W_n}{f(n)} = 0, \quad \mathbf{P}^*\text{–a.s.},$$

and the divergence part in Theorem 3, i.e.,

$$\int_0^\infty \frac{dt}{te^{f(t)}} = \infty \Rightarrow \mathbf{P}^*(M_n - \frac{1}{\alpha}\log n < -f(n), \quad \text{i.o.}) = 1.$$

We use estimations of (S_n), (S_n) conditioned to stay above $-x$, and spine decompositions.

Proof of Theorem 5: We mainly use the following estimation and Borel-Cantelli lemma.

For any $\lambda > 0$, there is $c_9 > 0$ such that for each $n \geq 1$,

$$\mathbf{P}\Big(M_n < (1+\frac{1}{\alpha})\log n - \lambda\Big) \leq c_9(1+\lambda)e^{-\lambda}.$$

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Thank you !