Spine decomposition and $L \log L$ criterion for superprocesses with non-local branching mechanisms

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2017-05-10, Beijing Normal University.

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- L: number of children given by one particle.
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- Q: When does, for a supercritical G-W process, the mean $EZ_n = m^n$ give the right growth rate?
- It is easy to see that $\{\frac{Z_n}{m^n} : n \geq 1\}$ is a non-negative martingale. Let W_∞ be the martingale limit.

Theorem (Kesten-Stigum 1966)

Suppose $m > 1$. Then W_∞ is nondegenerate if and only if

$$E\left(L \log^+ L\right) < +\infty. \qquad (1)$$

Moreover, under condition (1),

$$\frac{Z_n}{m^n} \to W_\infty \quad \text{a.s. and in } L^1.$$

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- For **branching Markov processes**: Hardy-Harris(09), Engländer-Harris-Kyprianou(10), Liu-Ren-Song(09,11).

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- Q: What about a more general non-local branching mechanism?

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ϕ^L is called the **local branching mechanism** and takes the form

$$\phi^L(x,\lambda) = a(x)\lambda + b(x)\lambda^2 + \int_{(0,+\infty)} \left(e^{-\lambda\theta} - 1 + \lambda\theta\right) \Pi^L(x, d\theta),$$

where $a(x) \in \mathcal{B}_b(E)$, $b(x) \in \mathcal{B}_b^+(E)$ and $(\theta \wedge \theta^2)\Pi^L(x, d\theta)$ is a bounded kernel from E to $(0, +\infty)$.

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ϕ^{NL} is called the **non-local branching mechanism** and takes the form

$$\phi^{NL}(x,f) = -c(x)\pi(x,f) - \int_{(0,+\infty)} \left(1 - e^{-\theta\pi(x,f)}\right) \Pi^{NL}(x, d\theta),$$

where $c(x) \in \mathcal{B}_b^+(E)$, $\pi(x, dy)$ is a probability kernel on E with $\pi(x, \{x\}) \not\equiv 1$ and $\theta\Pi^{NL}(x, d\theta)$ is a bounded kernel from E to $(0, +\infty)$.

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- X is an $\mathcal{M}(E)$-valued Markov process such that for every $f \in \mathcal{B}_b^+(E)$ and every $\mu \in \mathcal{M}(E)$,

$$\mathrm{P}_\mu\left(e^{-\langle f,X_t\rangle}\right) = e^{-\langle u_f(\cdot,t),\mu\rangle} \quad \text{for } t \geq 0,$$

where $u_f(x,t) := -\log \mathrm{P}_{\delta_x}\left(e^{-\langle f,X_t\rangle}\right)$ is the unique non-negative locally bounded solution to the integral equation

$$\begin{aligned} u_f(x,t) &= P_t f(x) - \Pi_x\left[\int_0^t \phi^L(\xi_s, u_f(t-s,\xi_s))ds\right] \\ &\quad -\Pi_x\left[\int_0^t \phi^{NL}(\xi_s, u_f^{t-s})ds\right]. \end{aligned}$$

Example: multitype CSBP

Suppose $E = \{1, 2, \cdots, K\}$ $(K \geq 2)$, m is the counting measure on E and $P_t f(i) = f(i)$ for all $i \in E$, $t \geq 0$ and $f \in \mathcal{B}^+(E)$. Suppose

$$\phi^L(i,\lambda) := a(i)\lambda + b(i)\lambda^2 + \int_{(0,+\infty)} \left(e^{-\lambda r} - 1 + \lambda r\right) \Pi^L(i,dr),$$

$$\phi^{NL}(i,f) := -c(i)\pi(i,f) - \int_{(0,+\infty)} \left(1 - e^{-r\pi(i,f)}\right) \Pi^{NL}(i,dr),$$

There exists a Markov process $\{(X_t^{(1)}, \cdots, X_t^{(K)})^{\mathrm{T}} : t \geq 0\}$ in $[0,+\infty)^K$, which is called a K-type CB process such that for $f \in \mathcal{B}_b^+(E)$

$$\mathrm{P}_\mu \left[\exp\left(-\sum_{i\in E} f(i) X_t^{(i)}\right)\right] = \exp\left(-\sum_{i\in E} \mu_i V_t f(i)\right),$$

where $V_t f(i)$ is the unique non-negative locally bounded solution to the following integral equation:

$$V_t f(i) = f(i) - \int_0^t \left(\phi^L(i, V_s f(i)) + \phi^{NL}(i, V_s f)\right) ds \quad \text{for } t \geq 0,\ i \in E.$$

Mean semigroup

- Let

$$\gamma(x) := c(x) + \int_{(0,+\infty)} \theta \Pi^{NL}(x, d\theta) \quad \text{and} \quad \gamma(x, dy) := \gamma(x)\pi(x, dy).$$

For every $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_b(E)$,

$$\mathrm{P}_\mu\left(\langle f, X_t\rangle\right) = \langle \mathfrak{P}_t f, \mu\rangle,$$

where $\mathfrak{P}_t f(x)$ is the unique locally bounded solution to the following integral equation:

$$\mathfrak{P}_t f(x) = P_t f(x) - \Pi_x\left[\int_0^t a(\xi_s)\mathfrak{P}_{t-s} f(\xi_s) ds\right] + \Pi_x\left[\int_0^t \gamma(\xi_s, \mathfrak{P}_{t-s} f) ds\right].$$

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- The corresponding **bilinear form** $(\mathcal{Q}, \mathcal{F})$ of $\mathfrak{P}_t$ is given by
$$\mathcal{Q}(u, v) := \mathcal{E}(u, v) + \int_E a(x)u(x)v(x)m(dx) - \int_E \int_E u(y)v(x)\gamma(x, dy)m(dx),$$
where $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form on $L^2(E, m)$ with respect to P_t.

Basic assumptions

- **Assumption 1.** For every x such that $b(x) > 0$, there exists $\mathbb{N}_x$ such that

$$\mathrm{P}_{\delta_x}\left(e^{-\langle f, X_t\rangle}\right) = e^{-\mathbb{N}_x\left(1-e^{-\langle f, X_t\rangle}\right)}.$$

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- Assumption 1 is true for superdiffusions with a local branching mechanism but **NOT** for all superprocesses. (sufficient conditions can be found in Li (2011).)

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- Kunita(1969) ensures that under Assumption 2, the bilinear form $(\mathcal{Q}, \mathcal{F})$ is closed on $L^2(E, m)$ and corresponds to a pair of strongly continuous dual semigroups $\{T_t : t \geq 0\}$ and $\{\widehat{T}_t : t \geq 0\}$ on $L^2(E, m)$.

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- Further we can show that for all $f \in \mathcal{B}_b(E) \cap L^2(E,m)$,

$$\mathfrak{P}_t f = T_t f \ m\text{-a.e.} \quad \text{for all } t > 0.$$

T_t can be regarded as the unique bounded linear operator on $L^2(E,m)$ extended by $\mathfrak{P}_t$.

- **Assumption 3.** There exist a constant $\lambda_1 \in (-\infty, +\infty)$ and positive functions $h, \widehat{h} \in \mathcal{F}$ with h bounded continuous, $\|h\|_{L^2(E,m)} = 1$ and $(h, \widehat{h}) = 1$ such that

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- We can define a new probability measure Q_μ by the following formula:

$$d\mathrm{Q}_\mu|_{\mathcal{F}_t} := \frac{1}{\langle h, \mu \rangle} W_t^h(X) d\mathrm{P}_\mu \Big|_{\mathcal{F}_t} \quad \text{for all } t \geq 0.$$

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- Q: How to construct X_t under Q_μ?

For every $\mu \in \mathcal{M}(E)$, $g \in \mathcal{B}_b^+(E)$,

$$\mathrm{Q}_\mu \left(e^{-\langle g, X_t \rangle} \right) = \frac{e^{\lambda_1 t}}{\langle h, \mu \rangle} e^{-\langle V_t g, \mu \rangle} \langle V_t^h g, \mu \rangle,$$

where $V_t g(x) := u_g(x,t)$ is the Log-Laplace exponent of original superprocess, and $V_t^h g(x)$ is the unique locally bounded solution to the following integral equation

$$V_t^h g(x) = P_t h(x) - \Pi_x \left[\int_0^t \Psi(\xi_s, V_{t-s} g, V_{t-s}^h g) ds \right],$$

where

$$\begin{aligned} \Psi(x, f, g) \quad := \quad & g(x) \left(a(x) + 2b(x) f(x) + \int_{(0,+\infty)} \theta \left(1 - e^{-f(x)\theta} \right) \Pi^L(x, d\theta) \right) \\ & -\pi(x, g) \left(c(x) + \int_{(0,+\infty)} \theta e^{-\theta \pi(x,f)} \Pi^{NL}(x, d\theta) \right). \end{aligned}$$

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- Transfer kernel $\kappa(\omega, dy) := \pi^h(\widehat{\xi}_{\widehat{\zeta}(\omega)-}(\omega), dy)$ where

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- The spine $\widetilde{\xi}$ evolves as the process ξ^h until the killing time $\widehat{\zeta}$, it is then revived by means of the kernel $\kappa(\omega, dy)$ and evolves again as ξ^h and so on.

- For every $f(s,x,y)\in\mathcal{B}([0,+\infty)\times E\times E),\ t>0$ and $x\in E$, we have

$$\mathrm{E}_x\left[\sum_{\tau_i\le t} f(\tau_i,\widetilde{\xi}_{\tau_i-},\widetilde{\xi}_{\tau_i})\right]=\mathrm{E}_x\left[\int_0^t q(\widetilde{\xi}_s)ds\int_E f(s,\widetilde{\xi}_s,y)\pi^h(\widetilde{\xi}_s,dy)\right].$$

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- **"Many-to-One formula":** for $f\in\mathcal{B}_b^+(E)$ and $t\ge 0$,

$$\frac{\mathrm{P}_{\delta_x}\left[\langle fh,X_t\rangle\right]}{\mathrm{P}_{\delta_x}\left[\langle h,X_t\rangle\right]}=\mathrm{E}_{\delta_x}\left[f(\widetilde{\xi}_t)\right].$$

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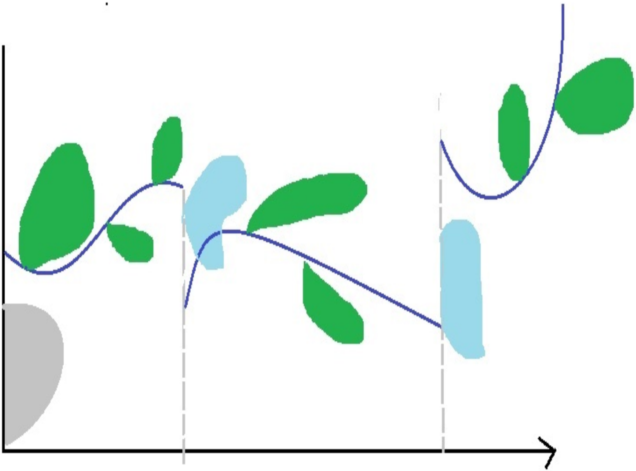
$$\theta\Pi^L(\widetilde{\xi}_t, d\theta) \times d\mathrm{P}_{\theta\delta_{\widetilde{\xi}_t}} \times dt.$$
- **(revival-caused immigration)** At each revival time τ_i of $\widetilde{\xi}$, "immigrate" an independent copy of (X, P_{π_i}) with initial distribution $\pi_i(\cdot) := \Theta_i\pi(\widetilde{\xi}_{\tau_i-}, \cdot)$ and Θ_i is a $[0, +\infty)$-valued random variable with distribution $\eta(\widetilde{\xi}_{\tau_i-}, d\theta)$ given by
$$\eta(x, d\theta) := \left(\frac{c(x)}{\gamma(x)}1_A(x) + 1_{E\setminus A}(x)\right)\delta_0(d\theta) + \frac{1}{\gamma(x)}1_A(x)1_{(0,+\infty)}(\theta)\theta\Pi^{NL}(x, d\theta).$$



- Remark: If branching mechanism is **local**, the **revival-caused immigration** does **NOT** occur. The piecing-out procedure and the revival-cased immigration are consequences of non-local branching.

- Remark: If branching mechanism is **local**, the **revival-caused immigration** does **NOT** occur. The piecing-out procedure and the revival-cased immigration are consequences of non-local branching.
- Similar phenomenon has also been observed in Kyprianou-Paulau(16+) for super-Markov chain and in Chen-Ren-Song (16+) for multitype superdiffusions.

- Q: Will the superprocess extinct when $\lambda_1 \leq 0$?

$L \log L$ criterion for superprocesses

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- [Ren-Song-Y.(16+)] (Under some technical conditions) for every non-trivial $\mu \in \mathcal{M}(E)$, if $\lambda_1 \leq 0$, then $\mathrm{P}_\mu \left(W^h_\infty(X) = 0 \right) = 1$.

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- In this case, X exhibits weak local extinction, that is, for every nonempty relatively compact open subset B of E,

$$\mathrm{P}_\mu \left(\lim_{t \to +\infty} X_t(B) = 0 \right) = 1.$$

- Q: When does, for $\lambda_1 > 0$, $e^{\lambda_1 t}$ give the right growth rate of the superprocess?

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- [Ren-Song-Y.(16+)]If $\lambda_1 > 0$, then $W^h_\infty(X)$ is non-degenerate if and only if

$$\left(\int_{(0,+\infty)} rh(\cdot)\log^+(rh(\cdot))\Pi^L(\cdot,dr),\widehat{h}\right) < +\infty$$

and

$$\left(1_A(\cdot)\int_{(0,+\infty)} r\pi(\cdot,h)\log^+(r\pi(\cdot,h))\Pi^{NL}(\cdot,dr),\widehat{h}\right) < +\infty.$$

Application: multitype CSBP

- Suppose $(X_t^{(1)}, \cdots, X_t^{(K)})$ is a K-type CB process.
- Define the mean matrix $M(t) = (M(t)_{ij})_{ij}$ by $M(t)_{ij} := \mathrm{P}_{\delta_i}\left(X_t^{(j)}\right)$ for $i, j \in E$.
- Markov property implies that $M(t)$ has a formal matrix generator $A := (A_{ij})_{ij}$ given by

$$M(t) = e^{At}, \quad \text{and } A_{ij} = \gamma(i)p_{ij} - a(i)\delta_i(j) \text{ for } i, j \in E.$$

- Assume A is an irreducible matrix. Let $\Lambda := \sup_{\lambda \in \sigma(A)} \mathrm{Re}(\lambda)$ where $\sigma(A)$ denotes the set of eigenvalues of A.
- By **Perron-Frobenius theory**, for every $t > 0$, $e^{\Lambda t}$ is a simple eigenvalue of $M(t)$, and there exist a unique positive right eigenvector $\mathbf{u} = (u_1, \cdots, u_K)^{\mathrm{T}}$ and a unique positive left eigenvector $\mathbf{v} = (v_1, \cdots, v_K)^{\mathrm{T}}$ such that

$$\sum_{i=1}^{K} u_i = \sum_{i=1}^{K} u_i v_i = 1, \quad M(t)\mathbf{u} = e^{\Lambda t}\mathbf{u}, \quad \mathbf{v}^T M(t) = e^{\Lambda t}\mathbf{v}.$$

- $W_t(X) := e^{-\Lambda t} \sum_{i=1}^{K} u_i X_t^{(i)}$ is a non-negative martingale.
- Under the martingale change of measure the spine process $\widetilde{\xi}$ is a continuous-time Markov chain with Q-matrix $Q = (q_{ij})_{ij}$ given by

$$q_{ii} := -(\Lambda + a(i)), \quad q_{ij} := \frac{\gamma(i)\pi(i,j)u_j}{u_i} \quad \text{for } i \neq j.$$

- for every non-trivial $\mu \in \mathcal{M}(E)$, the martingale limit $W_\infty(X)$ is non-degenerate if and only if $\Lambda > 0$ and

$$\int_{(0,+\infty)} r \log^+ r \Pi^L(i, dr) + \int_{(0,+\infty)} r \log^+ r \Pi^{NL}(i, dr) < +\infty \quad \text{for every } i \in E.$$

- In particular, under the above condition, $\mathrm{P}_\mu \left(\lim_{t \to +\infty} X_t^{(i)} = 0 \right) = 1$ for every $i \in E$ and every non-trivial $\mu \in \mathcal{M}(E)$ if and only if $\Lambda \leq 0$.

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Thank you!