Inhomogeneous random graphs and their scaling limits

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A model of inhomogeneous random graphs

- For $n \geq 1$, let $\mathbf{w} = (w_1, w_2, \cdots, w_n) \in (0, \infty)^n$: w_i = **weight** of vertex i.

  For $r \geq 0$, denote $\sigma_r(\mathbf{w}) = \sum_{i=1}^n w_i^r$.

- Let $\mathcal{G}_{\mathbf{w}}$ be the graph with n vertices (labelled $1, 2, \cdots, n$) and where

  $(\mathbf{1}_{\text{edge } \{i,j\} \in \mathcal{G}_{\mathbf{w}}})_{1 \leq i < j \leq n}$ are **independent** and have probabilities $f\left(\frac{w_i w_j}{\sigma_1(\mathbf{w})}\right)$,

  where $f : \mathbb{R}_+ \to [0, 1]$ and $f(x) = x + \mathcal{O}(x^2)$, $x \to 0$.

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e.g. $f(x) = 1 - e^{-x}$: Poisson random graph Aldous '97, Norros & Reittu '05
$f(x) = x \wedge 1$: Chung & Lu '02
$f(x) = x/(1 + x)$: Britton & Deijfen & Martin-Löf '06

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  Special case (homogenous case): (w_i) all **equal**, then $\mathcal{G}_{\mathbf{w}} = G(n, p)$.
  In general, $\deg(i) \approx \text{Poisson}(w_i)$.

Scaling limits of $\mathcal{G}_{\mathbf{w}}$

- $\mathcal{G}_{\mathbf{w}}^k \equiv k$-th largest connected components of $\mathcal{G}_{\mathbf{w}}$.
- Each $\mathcal{G}_{\mathbf{w}}^k$ is a metric space equipped with $d_{\mathrm{gr}} =$ graph distance of $\mathcal{G}_{\mathbf{w}}$.

Aim: Under suitable conditions on $(\mathbf{w}_n)$, find $\epsilon_n \to 0$ such that

$$\left(\mathcal{G}_{\mathbf{w}}^k,\ \epsilon_n \cdot d_{\mathrm{gr}}\right)_{k\geq 1} \xrightarrow[n\to\infty]{(d)} (\mathbf{G}_k)_{k\geq 1}$$

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[Janson '09] Asymptotic equivalence of random graphs: The choice of f is irrelevant.

From now on, take $f(x) = 1 - e^{-x}$.

Outline

$\rightsquigarrow$ Encoding $\mathcal{G}_{\mathbf{w}}$ with stochastic processes

$\rightsquigarrow$ Convergence of the coding processes

$\rightsquigarrow$ Identifying the limit graphs.

Outline

- A LIFO queue representation of $\mathcal{G}_{\mathbf{w}}$
  $\rightsquigarrow$ Encoding $\mathcal{G}_{\mathbf{w}}$ with stochastic processes
- Embed $\mathcal{G}_{\mathbf{w}}$ into Galton–Watson trees
  $\rightsquigarrow$ Convergence of the coding processes
- Construction of the graphs from the limit coding processes
  $\rightsquigarrow$ Identifying the limit graphs.

A LIFO queue representation of $\mathcal{G}_{\mathbf{w}}$

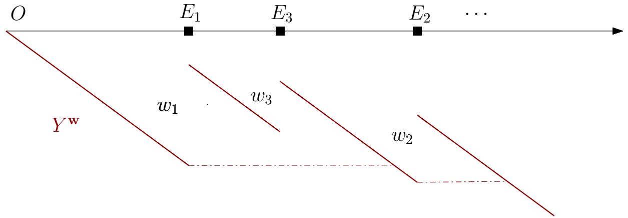
A Last-In-First-Out queueing system:

- a single server
- n clients (labelled $1, \ldots, n$): Client i arrives at time E_i and requests service time w_i.
- clients in the queue are served in Last-In-First-Out order:
  whenever a new customer arrives, the server interrupts the current service (if any) and serves the new-comer. When the latter quits the queue, the server then resumes the previous service.

LIFO queue & coding functions

Define $Y_t^{\mathbf{w}} \equiv -t + \sum_{1 \le i \le n} w_i \mathbf{1}_{[0,t]}(E_i)$. Then

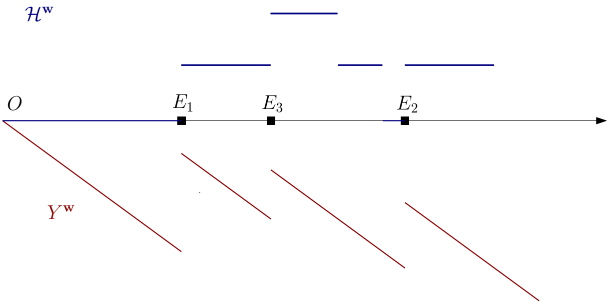
$$Y_t^{\mathbf{w}} - \inf_{s \le t} Y_s^{\mathbf{w}} = \text{ load of the server at } t.$$



LIFO queue & coding functions

$\mathcal{H}_t^{\mathbf{w}} \equiv$ length of the queue at time t. By the LIFO rule, we have

$$\forall\, t \geq 0: \quad \mathcal{H}_t^{\mathbf{w}} = \#\Big\{ s \leq t : \inf_{u \in [s,t]} Y_u^{\mathbf{w}} > Y_{s-}^{\mathbf{w}} \Big\}.$$



LIFO queue & trees

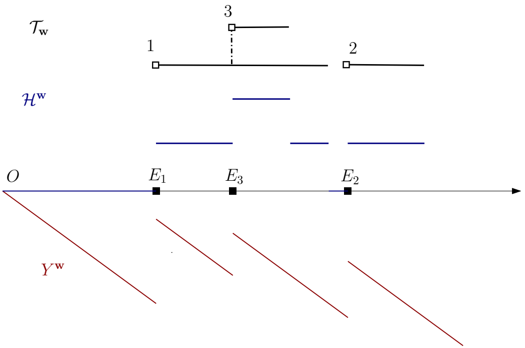
LIFO queue $\rightsquigarrow$ a sequence of family trees $\mathcal{T}_{\mathbf{w}}$:

- Vertices of $\mathcal{T}_{\mathbf{w}} = \{$ labels of the n clients $\}$.
- Children of i = labels of the clients interrupting the service of Client i.

We have

$$\mathcal{H}^{\mathbf{w}}_t = \text{height of } V_t, \quad \text{where} \quad V_t \equiv \text{label of the client served at time } t.$$

In particular, each excursion of $\mathcal{H}^{\mathbf{w}}$ above 0 encodes a tree component in $\mathcal{T}_{\mathbf{w}}$.



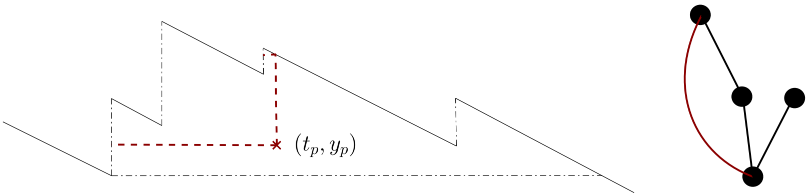
LIFO queue & graph $\mathcal{G}_{\mathbf{w}}$

To sample some additional edges, let

$$\mathcal{P}^{\mathbf{w}} = \sum_{p\geq 1} \delta_{(t_p, y_p)}$$

be a Poisson point measure on $\mathbb{R}_+^2$ of intensity $\frac{1}{\sigma_1(\mathbf{w})}\mathbf{1}_{\{0\leq x\leq Y_t^{\mathbf{w}} - \inf_{[0,t]} Y^{\mathbf{w}}\}} dtdx$. Set

$$\mathcal{A}^{\mathbf{w}} = \big\{\{V_{s_p}, V_{t_p}\} : 1 \leq p \leq |\mathcal{P}^{\mathbf{w}}|\big\} \quad \text{where} \quad s_p \equiv \inf\{s : \inf_{s\leq u\leq t_p} Y_u^{\mathbf{w}} - \inf_{s\leq t} Y_s^{\mathbf{w}} > y_p\}$$



LIFO queue & graph $\mathcal{G}_{\mathbf{w}}$

Denote by $\tilde{\mathcal{G}} = \mathcal{T}_{\mathbf{w}} \cup \mathcal{A}^{\mathbf{w}}$.

Theorem. Suppose $\frac{w_i}{\sigma_1(\mathbf{w})} E_i, 1 \leq i \leq n$, are i.i.d. $\mathbf{Exp}(1)$. Then $\tilde{\mathcal{G}} \stackrel{(\mathrm{d})}{=} \mathcal{G}_{\mathbf{w}}$.

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In consequence, $\mathcal{G}_{\mathbf{w}}$ is encoded by $Y^{\mathbf{w}}$, $\mathcal{H}^{\mathbf{w}}$ and $\mathcal{P}^{\mathbf{w}}$. Therefore, to find the scaling limit of $\mathcal{G}_{\mathbf{w}}$, we look at

- the convergence of $Y^{\mathbf{w}}$: ✓
- the convergence of $\mathcal{P}^{\mathbf{w}}$: ✓
- the convergence of $\mathcal{H}^{\mathbf{w}}$: not a continuous funct. of $Y^{\mathbf{w}}$...

Preview of the embedding

To embed $\mathcal{G}_{\mathbf{w}}$ into Galton–Watson trees,

- introduce a Markovian LIFO queue $\rightsquigarrow$ Galton–Watson trees
- embed the previous n-client queue into the Markovian one.

Markovian LIFO queue & GW trees

From now on, **suppose** $\sigma_2(\mathbf{w}) \le \sigma_1(\mathbf{w})$.

A Markovian LIFO queueing system:

- A single server treats the clients in Last-In-First-Out order
- An ∞-sequence of clients arrive at rate 1:
  - the k-th client arrives at time τ_k,
  - chooses his type $J_k \in \{1, \dots, n\}$ with probabilities $\mathbf{P}(J_k = i) = \frac{w_i}{\sigma_1(\mathbf{w})}$
  - requests service time w_i if $J_k = i$.

- Load of the server: $X_t^{\mathbf{w}} \equiv -t + \sum_{k \ge 1} w_{J_k} \mathbf{1}_{[0,t]}(\tau_k)$.

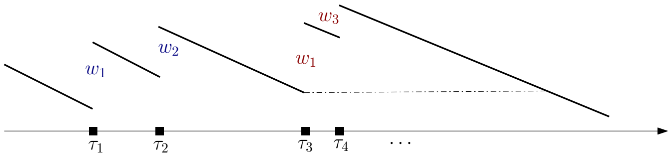
  $X^{\mathbf{w}}$ is a Lévy process of Lévy measure $\nu_{\mathbf{w}} \equiv \frac{1}{\sigma_1(\mathbf{w})} \sum_i w_i \delta_{w_i}$ and ≤ 0 drift.
- As before, we can associate with the queue a sequence of family trees $\mathbf{T_w}$.

  It turns out $\mathbf{T_w}$ = i.i.d. Galton–Watson trees with offspring distribution **Poisson**(W_n), where $W_n \sim \nu_{\mathbf{w}}$.
- $\mathbf{H^w} \equiv$ length of the queue = height process of $\mathbf{T_w}$.

Embedding of the n-client queue

Color the clients in blue or red according to the following rule:

- If the type of the k-th client $\in$ {types of previous blue clients}, the client is in red;
- otherwise, the client inherits his color from his parent, with the convention that if there is no parent, the client is in blue.



Embedding of the n-client queue

$\mathcal{B} \equiv \{t \geq 0$: either a blue client is served at t or the server is idle at $t\}$
Let

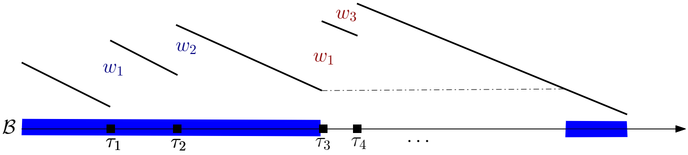
$$X^{\mathbf{w}} \circ \theta^{\mathbf{w}} = \text{ process obtained by } \textbf{restricting } X^{\mathbf{w}} \text{ to } \mathcal{B},$$

namely, $\Lambda_t^{\mathbf{w}} \equiv \int_0^t \mathbf{1}_{\mathcal{B}}(s)ds$ and then $\theta_t^{\mathbf{w}} \equiv \inf\{s : \Lambda_s^{\mathbf{w}} > t\}$.

Lemma. We have

$$X^{\mathbf{w}} \circ \theta^{\mathbf{w}} \overset{(\mathrm{d})}{=} Y^{\mathbf{w}}, \quad \mathbf{H}^{\mathbf{w}} \circ \theta^{\mathbf{w}} \overset{(\mathrm{d})}{=} \mathcal{H}^{\mathbf{w}}.$$

Problem: difficult to prove directly the convergence of $\theta^{\mathbf{w}}$.



Another description of $\theta^{\mathbf{w}}$

Recall the kth client arrives at τ_k and is of type J_k. Let

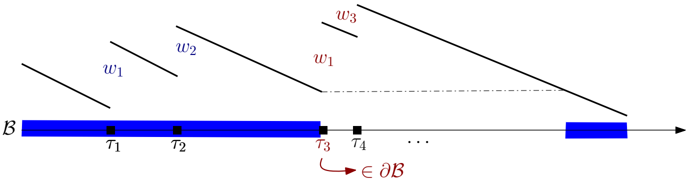
$$X_t^{b,\mathbf{w}} \equiv X^{\mathbf{w}}_{\theta_t^{\mathbf{w}}} + \overbrace{\sum_{k\geq 1} w_{J_k} \mathbf{1}_{\{\tau_k \in \partial\mathcal{B}\}} \mathbf{1}_{\{\Lambda^{b,\mathbf{w}}_{\tau_k} \leq t\}}}^{A_t^{\mathbf{w}}} \quad \text{and} \quad X^{r,\mathbf{w}} \equiv X^{\mathbf{w}} - X^{b,\mathbf{w}}.$$

Lemma. $X^{b,w}, X^{r,\mathbf{w}}$ are two independent copies of $X^{\mathbf{w}}$. Moreover,

$$\theta_t^{\mathbf{w}} = t + \gamma^{\mathbf{w}}_{A_t^{\mathbf{w}}},$$

where $\gamma_x^{\mathbf{w}} = \inf\{t : X_s^{r,\mathbf{w}} < -x\}$ and

$$A_t^{\mathbf{w}} = \sum_i w_i (N_i(t) - 1)_+, \quad \text{where} \quad N_i(t) = \#\{\text{jumps of } X^{b,\mathbf{w}} \text{ of type } i \text{ before } t\}$$



Construction of $\mathcal{G}_{\mathbf{w}}$ from Lévy processes

- Sample $X^{b,\mathbf{w}}, X^{r,\mathbf{w}}$. Let $X^{b,\mathbf{w}}, X^{r,\mathbf{w}}$ be two independent copies of a (sub-)critical spectrally positive Lévy process such that

$$\psi_n(\lambda) \equiv \log \mathbb{E}[e^{-\lambda X_1^b}] = \Big(1 - \tfrac{\sigma_2(\mathbf{w})}{\sigma_1(\mathbf{w})}\Big)\lambda + \int (e^{-\lambda u} - 1 + \lambda u)\nu_{\mathbf{w}}(du),$$

  where $\nu_{\mathbf{w}} = \frac{1}{\sigma_1(\mathbf{w})}\sum_{i\geq 1} w_i \delta_{w_i}$. Set $X^{\mathbf{w}} \equiv X^{b,\mathbf{w}} + X^{r,\mathbf{w}}$.

- Define $Y^{\mathbf{w}}$. Let

$$Y^{\mathbf{w}} \equiv X^{b,\mathbf{w}} - A^{\mathbf{w}}, \quad \text{where} \quad A_t^{\mathbf{w}} \equiv \Sigma_i w_i (N_i(t) - 1)_+.$$

- Define $\theta^{\mathbf{w}}$. Set $\theta_t^{\mathbf{w}} = t + \gamma^{\mathbf{w}}_{A_t^{\mathbf{w}}}$, where $\gamma_x^{\mathbf{w}} = \inf\{t : X_t^{r,\mathbf{w}} < -x\}$.

- Define $\mathcal{H}^{\mathbf{w}}$. Let

$$\mathcal{H}^{\mathbf{w}} = \mathbf{H}^{\mathbf{w}} \circ \theta^{\mathbf{w}} \quad \text{where} \quad \mathbf{H}_t^{\mathbf{w}} = \#\{s \leq t : X_{s-}^{\mathbf{w}} < \inf_{[s,t]} X^{\mathbf{w}}\}.$$

- Sample additional edges according to $\mathcal{P}^{\mathbf{w}}$, which is a Poisson point measure on $\mathbb{R}_+^2$ of intensity $\frac{1}{\sigma_1(\mathbf{w})}\mathbf{1}_{\{0\leq x \leq Y_t^{\mathbf{w}} - \inf_{[0,t]} Y^{\mathbf{w}}\}} dtdx$.

Construction of the limit graph: Part I

- Sample X^b, X^r. Let X^b, X^r be two independent copies of a (sub-)critical spectrally positive Lévy process such that
$$\psi(\lambda) \equiv \log \mathbb{E}[e^{-\lambda X_1^b}] = \alpha\lambda + \tfrac{1}{2}\beta\lambda^2 + \int \kappa(e^{-\lambda u} - 1 + \lambda u)\pi(du),$$
where $\alpha, \beta \geq 0$, $\kappa > 0$ and $\pi = \sum_{j\geq 1} c_i \delta_{c_i}$ with $c_1 \geq c_2 \geq \cdots \geq 0$ satisfying $\sum_i c_i^3 < \infty$. Namely,
$$X_t^b = -\alpha t + \sqrt{\beta} B_t + \Sigma_{i\geq 1} c_i (N_i(t) - c_i \kappa t),$$
where $B =$ standard Brownian Motion and $N_i =$ Poisson process of rate κc_i.
- Define Y. Let
$$Y \equiv X^b - A, \quad \text{where} \quad A_t \equiv \tfrac{1}{2}\beta\kappa t^2 + c_i(N_i(t) - 1)_+.$$
- Define θ. Set $\theta_t = t + \gamma_{A_t}$, where $\gamma_x = \inf\{t : X_t^r < -x\}$.

Construction of the limit graph: Part II

- Height process of a Lévy process. For the Lévy process X, we **can** define an analogue of the discrete height process $\mathbf{H}^{\mathbf{w}}$.

  Le Gall & Le Jan '98: Suppose $\int^{\infty} d\lambda/\psi(\lambda) < \infty$. Then there exists a continuous process $\mathbf{H}$ such that

  $$\mathbf{H}_t = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{\{X_s - \inf_{[s,t]} X < \epsilon\}} \quad \text{exists in probability}$$

  $\mathbf{H}$ is in fact the **height process** of the Lévy tree with branching mechanism ψ.

- Define $\mathcal{H}$. Let
  $$\mathcal{H} \equiv \mathbf{H} \circ \theta,$$

- Sample additional edges according to $\mathcal{P}$, which is a Poisson point measure on $\mathbb{R}_+^2$ of intensity $\mathbf{1}_{\{x \leq Y_t - \inf_{[0,t]} Y\}} dtdx$.

Convergence of the graphs $\mathcal{G}_{\mathbf{w}_n}$

Theorem. Let $\mathbf{w}_n = (w_{n1}, \ldots, w_{nn})$, $n \geq 1$. Denote $\psi_n(\lambda) = \log \mathbf{E}[e^{-\lambda X_1^{\mathbf{w}_n}}]$. Suppose

$$a_n \to \infty, \quad b_n/a_n \to \infty \quad \text{and} \quad b_n/a_n^2 \to \beta_0 \in [0, \beta] \quad \text{satisfying that}$$

$$(1) \qquad \forall\, \lambda \geq 0: \quad b_n \psi_n(\lambda/a_n) \to \psi(\lambda) \quad \text{and} \quad \lim_{y\to\infty} \limsup_{n\to\infty} \frac{a_n}{b_n} \int_{y/a_n}^{1} \frac{d\lambda}{\psi_n(\lambda)} = 0.$$

Then we have

$$(2) \qquad \Big(\big(\tfrac{1}{a_n} Y^{\mathbf{w}_n}_{b_n t}\big)_{t\geq 0}, \big(\tfrac{a_n}{b_n} \mathcal{H}^{\mathbf{w}_n}_{b_n t}\big)_{t\geq 0}, \tilde{\mathcal{P}}^{\mathbf{w}_n} \Big) \xrightarrow{d} (Y, \mathcal{H}, \mathcal{P}) \quad \text{in} \quad \mathbb{D} \times \mathbb{C} \times \mathcal{M}(\mathbb{R}_+^2),$$

where $\langle f, \tilde{\mathcal{P}}^{\mathbf{w}_n} \rangle = \sum_{p\geq 1} f(\frac{t_p}{b_n}, \frac{y_p}{a_n})$, $\forall\, f : \mathbb{R}_+^2 \to \mathbb{R}_+$ measurable.

Essential ingredient of the proof:

Duquesne & Le Gall '02 Under Condition (1), we have the convergence of the Poisson(W_n)-Galton–Watson trees to ψ-Lévy trees.

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We can deduce from (2) the convergence of the graph $\mathcal{G}_{\mathbf{w}_n}$:

$$(\mathcal{G}^k_{\mathbf{w}_n}, \tfrac{a_n}{b_n} d_{\mathrm{gr}})_{k\geq 1} \xrightarrow{d} (\mathbf{G}_k)_{k\geq 1} \quad \text{in Gromov–Hausdorff topology,}$$

where $\mathbf{G}_k$, $k \geq 1$, are the connected components of the limit graph constructed from $(Y, \mathcal{H}, \mathcal{P})$.

Convergence of the graphs $\mathcal{G}_{\mathbf{w}_n}$

From the previous theorem, we can recover

- Addario-Berry & Broutin & Goldschmidt '12 $G(n,p)$: (w_{ni}) all equal, then $\psi(\lambda) = \alpha\lambda + \lambda^2/2$.
- Bhamidi & van der Hofstad & Sen '17+ Power-law case: $w_{ni} \sim (n/i)^{\gamma}$, $\gamma \in (\frac{1}{3}, \frac{1}{2})$, then $\beta = 0$ and $c_i = i^{-\gamma}$, $i \geq 1$. Moreover, they show $\dim_h(\mathbf{G}_k) \leq \frac{1-\gamma}{1-2\gamma}$, a.s.

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[Duquesne & Le Gall '05] fractal properties of Lévy trees + our construction of $\mathbf{G}_k$

Proposition. Let

$$\overline{\gamma} := \inf\{s \geq 0 : \lim_{j\to\infty} j^s c_j = \infty\} \quad \text{and} \quad \underline{\gamma} \equiv \sup\{s \geq 0 : \lim_{j\to\infty} j^s c_j = 0\}$$

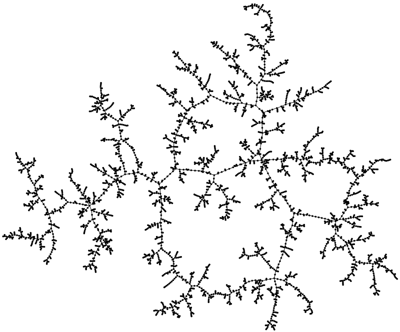
Suppose $\beta = 0$ and $\overline{\gamma} < 1/2$. Then $\mathbf{P}$-a.s. for all $k \geq 1$,

$$\dim_p(\mathbf{G}_k) = \frac{1-\underline{\gamma}}{1-2\underline{\gamma}} \quad \text{and} \quad \dim_h(\mathbf{G}_k) = \frac{1-\overline{\gamma}}{1-2\overline{\gamma}}.$$

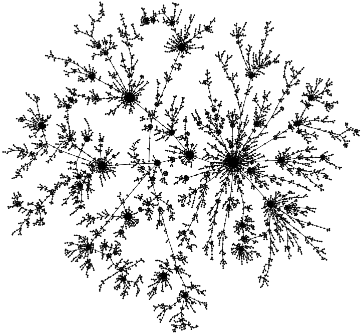
In particular, if (c_j) varies regularly with index γ, then $\dim_p(\mathbf{G}_k) = \dim_h(\mathbf{G}_k) = \frac{1-\gamma}{1-2\gamma}$.

Simulation of large $\mathcal{G}_{\mathbf{w}}^{k}$

Homogeneous case



Power law case



THANK YOU!