Branching Brownian Motion with Catalytic Branching at the Origin

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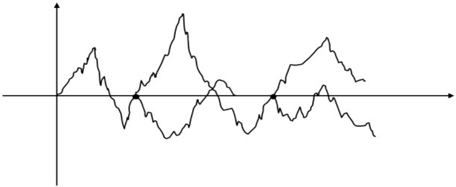
Outline

- Introduction of the model
- Main tool: spine decomposition
- Main results
- Sketch of the proof

Introduction: BBM



We begin with one Brownian particle ϕ starting from the origin. This particle has a lifetime σ_ϕ exponentially distributed as

$$\mathbf{P}(\sigma_\phi > t \mid X_s^\phi, s \le t) = e^{-\beta L_t^\phi}. \tag{1}$$

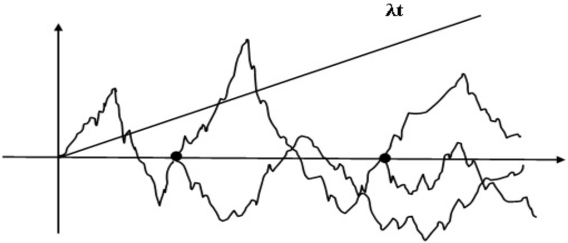
where $L_t^\phi = \int_0^t \delta_0(X_s^\phi)ds$, the local time at the origin of the initial particle.

When the particle dies, it is replaced in its position (the origin) by a random number A of offsprings,

$$p(A = k) = p_k, \; k = 0, 1, 2, \cdots. \tag{2}$$

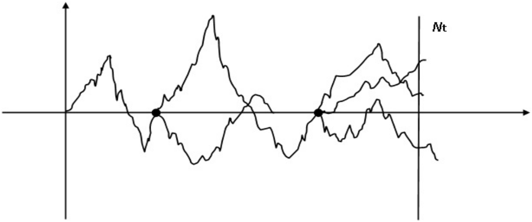
Background

Bocharov and Harris (2014): consider the case of binary branching, i.e., $P_2 = 1$. They obtain results regarding the asymptotic behaviour of the number of particles above λt at time t, for $\lambda > 0$. They also prove a SLLN for this catalytic BBM.



Question: Inspired by their paper, we consider the general case $p_0 > 0$ and assume $m = \sum_{k=0}^{\infty} kp_k > 1$.

Notations

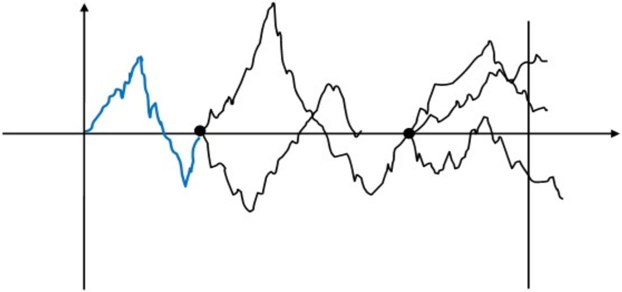


- N_t: the set of particles alive at time t, labelling particles according to the usual Ulam-Harris convention.
- X_t^u: the spacial position at time t.
- A_u: the number of the children of particle u, $A_u \overset{d}{=} A$.
- σ_u: lifetime of particle u, $\sigma_u \overset{d}{=} \sigma_\phi$. Then fission time $S_u = \sum_{\nu \le u} \sigma_\nu$.

Then the BBM can be represented as $X_t := \{X_t^u : u \in N_t\}$. Let $\mathbf{P}$ be the distribution of $X = (X_t : t \geq 0)$ with a single initial particle at origin.

Spine setup

Since $p_0 > 0$, the spine we will choose is a little different from the history literature, see Harris [2009].



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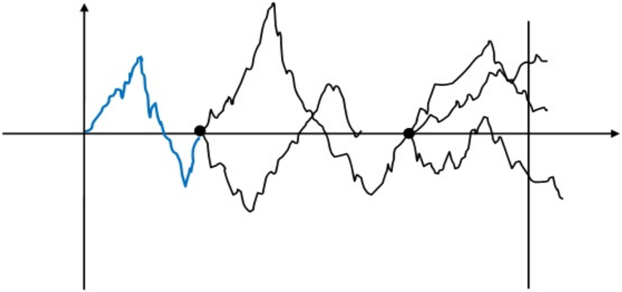

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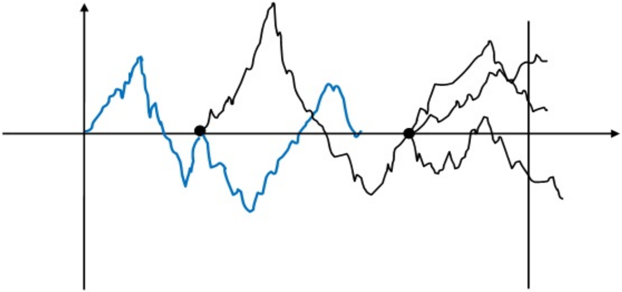

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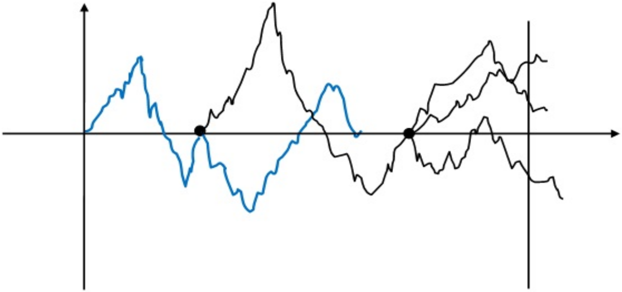

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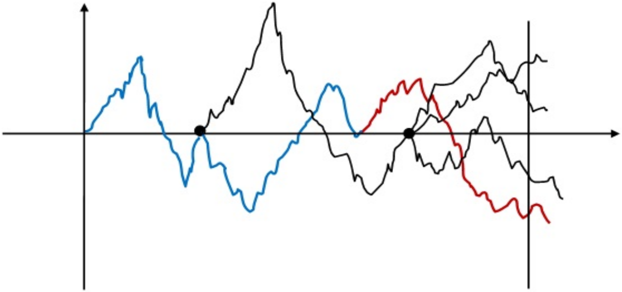

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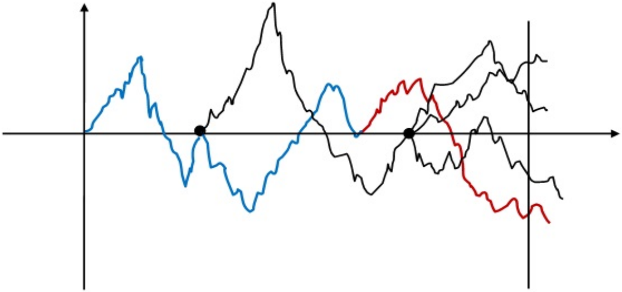
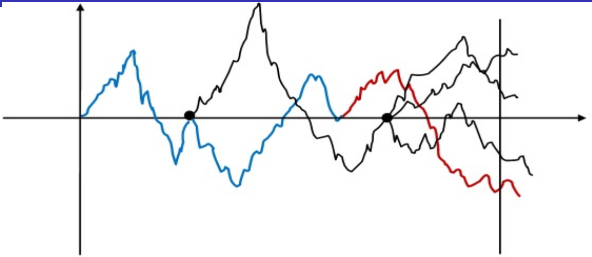


Spine setup

1. The spine starts with the initial particle ϕ.
2. The spine undergoes fission into particles with rate $\beta\delta_0(\cdot)dt$;
3. A spine particle u, once death, gives birth to A_u particles.
4. If $A_u > 0$, one of the A_u children is chosen uniformly at the fission time of u to continue the spine.
   If $A_u = 0$, the particle u will be sent to a "revive state" and continue the path as if it were still in the system, so that the spine can continue.
5. Each of the remaining $A_u - 1(A_u > 0)$ particles gives rise to an independent copy of a **P**-branching Brownian motion started at its space-time point of creation.

Technically, All the particles, die with no offspring, will be sent to the "revive state".

Spine



Let $N_t(R)$ be the set of particles in the "revive state" and let $\bar{N}_t = N_t \cup N_t(R)$.

For each $\nu \in \bar{N}_t$, define

$$\bar{A}_\nu := \begin{cases} A_\nu, & \text{if } \nu \text{ is alive,} \\ 1, & \text{if } \nu \text{ is in the revive state.} \end{cases}$$

Thus for $u \in \bar{N}_t$,

$$\mathrm{Prob}(u \in \xi) = \prod_{\nu < u} \frac{1}{\bar{A}_\nu} = \prod_{\nu < u} \frac{1}{A_\nu} 1_{\{u \in N_t\}} + \prod_{\nu < u} \frac{1}{\bar{A}_\nu} 1_{\{u \in N_t(R)\}}.$$

Then it is easy to show that $\sum_{u \in \bar{N}_t} \mathrm{Prob}(u \in \xi) = 1$.

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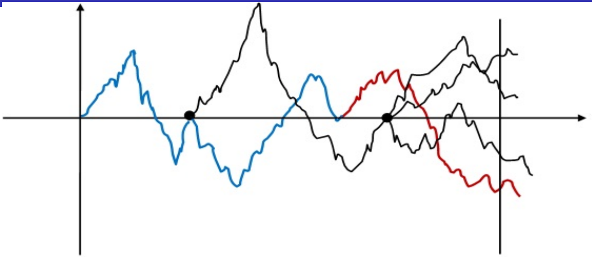

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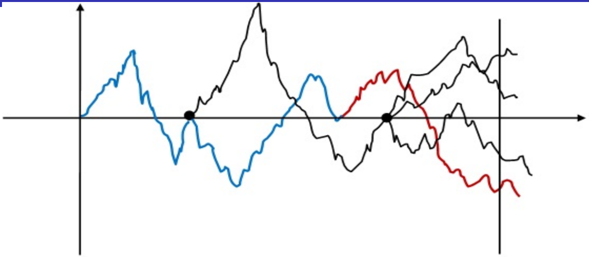

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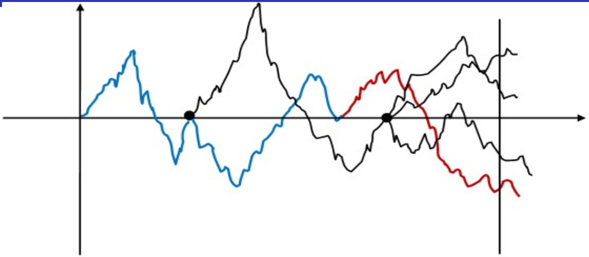

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Filtration

- $\xi = \{\phi, \xi_1, \xi_2, \ldots\}$: the selected line of decent in the spine
- $node_t(\xi)$: the node in the spine that is alive at time t
- $n = (n_t : t \geq 0)$: the number of fission times along the spine
- $\mathcal{F}_t$: the natural filtration of this branching process.
- $\tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t, (node_s(\xi), s < t))$.
- $\mathcal{G}_t = \sigma(\xi_s; s \leq t)$.
- $\tilde{\mathcal{G}}_t := \sigma(\mathcal{G}_t, (node_s(\xi), s < t))$.
- $\mathcal{F}_\infty := \bigcup_{t\geq 0} \mathcal{F}_t$, $\tilde{\mathcal{F}}_\infty := \bigcup_{t\geq 0} \tilde{\mathcal{F}}_t$, $\tilde{\mathcal{G}}_\infty := \bigcup_{t\geq 0} \tilde{\mathcal{G}}_t$.

We extend $\mathbf{P}$ to $\tilde{\mathbf{P}}$ so that the spine is a single genealogical line chosen from the underlying process.

In detail, the measure $\widetilde{\mathbf{P}}$ on $\tilde{\mathcal{F}}_t$ is defined by

$$d\widetilde{\mathbf{P}}(\tau, M, \xi)|_{\tilde{\mathcal{F}}_t} = dP(\xi)dL^{\beta(\xi)}(\mathbf{n}) \prod_{\nu<\xi_{n_t}} p_{A_\nu} \prod_{\nu<\xi_{n_t}} \frac{1}{\bar{A}_\nu} \prod_{j:\nu j\in O_\nu} d\mathbf{P}^{\xi_{S_\nu}}_{t-S_\nu}((\tau, M)^\nu_j)$$

where $L^{\beta(\xi)}(\mathbf{n})$ is the law of the Poisson random measure $\mathbf{n} = \{\{\sigma_i : i = 1, 2, \cdots, n_t\} : t \geq 0\}$ with intensity βdL_t along the path of ξ, $P(\xi)$ is the law of ξ starting from the origin, and $p_{A_\nu} = \sum_{k\geq 0} p_k I_{(A_\nu=k)}$ is the probability that individual ν has A_ν offsprings.

Girsanov change for spatial motion

Let

$$Z_t = e^{-\beta_2|\xi_t| + \beta_2 L_t - \frac{1}{2}\beta_2^2 t}.$$

Then $\{Z_t\}$ is a P-martingale so we can define a martingale change of measure by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{G}_t} = Z_t.$$

Then $\{\xi, Q\}$ is a Brownian Motion with drift β_2.

Change of measure for Possion process

Suppose that the Possion process $(n, L^{\beta(\xi)})$ where $n = \{\{\sigma_i : i = 1, 2, \ldots, n_t\} : t \geq 0\}$ has instantaneous rate $\beta\delta_0(\xi_t)$. Further, assume that n is adapted to $\{\mathcal{L}_t : t \geq 0\}$. Then under the change of measure

$$\left.\frac{L^{m\beta(\xi)}}{L^{\beta(\xi)}}\right|_{\mathcal{L}_t} = m^{n_t} \cdot e^{-\beta(m-1)L_t}$$

the process $(n, L^{m\beta(\xi)})$ is also a Possion process with instantaneous jump rate $m\beta\delta_0(\xi_t)$.

The spinal construction

Assume that $\sum_{k=1}^{\infty}(k\ln k)p_k < \infty$. Then it can be shown that

$$M_t = \sum_{u\in N_t} e^{-\beta_2|X_t^u|-\frac{1}{2}\beta_2^2 t}$$

is a $\mathbf{P}$-martingale and converges almost surely to an non-degenerate limit M_∞, i.e., $\mathbf{P}(M_\infty = 0) < 1$.

It can also be shown that

$$\widetilde{M}_t = e^{-\beta_2|\xi_t|-\frac{1}{2}\beta_2^2 t}\prod_{\nu<\xi_{n_t}} A_\nu 1_{\{\xi_{n_t}\in N_t\}}$$

defines a $\tilde{\mathcal{F}}_t$-measurable $\widetilde{\mathbf{P}}$-martingale.

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The spinal construction

Note that, Z, M and $\tilde{M}$ are all unit mean martingales and have the key relationships as follows:

$$M_t = \tilde{\mathbf{P}}(\tilde{M}_t \mid \mathcal{F}_t) \text{ and } Z_t = \tilde{\mathbf{P}}(\tilde{M}_t \mid \mathcal{G}_t). \tag{3}$$

Then we can define the Girsanov change of mesaures by

$$\left.\frac{d\tilde{\mathbf{Q}}}{d\tilde{\mathbf{P}}}\right|_{\tilde{\mathcal{F}}_t} = \tilde{M}_t \text{ and } \left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t} = M_t.$$

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Measure Q

$$
\begin{aligned}
d\widetilde{\mathbf{Q}} &= \tilde{M}_t d\widetilde{\mathbf{P}} \\
&= e^{-\beta_2|\xi_t|-\frac{1}{2}\beta_2^2 t} \prod_{\nu<\xi_{n_t}} A_\nu 1_{\{\xi_{n_t}\in N_t\}} dP dL^{\beta(\xi)}(\mathbf{n}) \prod_{\nu<\xi_{n_t}} p_{A_\nu} \prod_{\nu<\xi_{n_t}} \frac{1}{A_\nu} \prod_{j:\nu j\in O_\nu} dP^{\xi_{S_\nu}}_{t-S} \\
&= e^{-\beta_2|\xi_t|-\frac{1}{2}\beta_2^2 t+\beta_2 L_t} dP m^{n_t} e^{-\beta_2 L_t} dL^{\beta(\xi)}(\mathbf{n}) \prod_{\nu<\xi_{n_t}} 1_{\{\xi_{n_t}\in N_t\}} \frac{p_{A_\nu}}{m} \prod_{j:\nu j\in O_\nu} dP^{\xi_{S_\nu}}_{t-S_\nu}(\\
&= dQ dL^{m\beta(\xi)} \prod_{\nu<\xi_{n_t}} \hat{p}_{A_\nu} \prod_{\nu<\xi_{n_t}} \frac{1}{A_\nu} 1_{\{\xi_{n_t}\in N_t\}} \prod_{j:\nu j\in O_\nu} dP^{\xi_{S_\nu}}_{t-S_\nu}((\tau,M)^\nu_j)
\end{aligned}
$$

Then under $\widetilde{\mathbf{Q}}$, the branching process X can be constructed as follows:

1. The spine initially starting at the origin moves according to the measure Q;
2. The spine undergoes fission into particles at an accelerated intensity $\beta m\delta_0(\cdot)d$
3. A spine particle u, once death, gives birth to A_u particles with size-biased offspring distribution $\hat{P}(A_u = k) = \frac{kp_k}{m}, k = 0, 1, 2, \cdots$
4. One of the A_u children is chosen uniformly at the fission time of u to continue the spine.
5. Each of the remaining $A_u - 1$ particles gives rise to an independent copy of a $\mathbf{P}$-branching BM started at its space-time point of creation.

Note that, $\hat{P}(A_u = 0) = 0$.

Many-to-one Theorem

Theorem

(Many-to-one theorem) Let $f(t) \in m\mathcal{G}_t$. In other words, $f(t)$ is $\mathcal{G}_t$-measurable. Suppose it has the representation

$$f(t) = \sum_{u \in N_t} f_u(t) 1_{\{u \in \xi\}},$$

where $f_u(t) \in m\mathcal{F}_t$, then

$$\mathbf{E}\left(\sum_{u \in N_t} f_u(t)\right) = \mathbf{Q}\left(\frac{1}{M_t}\sum_{u \in N_t} f_u(t)\right) = \widetilde{\mathbf{Q}}(e^{\beta_2|\xi_t| + \frac{1}{2}\beta_2^2 t} f(t)) = \widetilde{E}[e^{\beta_2 L_t} f(t)].$$

Spine decomposition

Theorem

Spine decomposition*:*

$$\widetilde{\mathbf{Q}}[M_t|\widetilde{\mathcal{G}}_\infty] = \exp\{-\beta_2|\xi_t| - \frac{1}{2}\beta_2^2 t\} + \sum_{u<\xi_{n_t}} (A_u - 1)\exp\{-\beta_2|\xi_{S_u}| - \frac{1}{2}\beta_2^2 S_u\}$$

where $\{S_u : u \in \xi\}$ *is the set of fission times along the spine.*

Main results

- Expected total population growth: $\beta_2 = \beta(m-1)$,

$$\mathbf{E}(|N_t|) = 2\Phi(\beta_2\sqrt{t})e^{\frac{\beta_2^2}{2}t} \sim 2e^{\frac{\beta_2^2}{2}t}, \ \ t \to \infty.$$

- Expected population growth rates:
  Let $N_t^{\lambda t} := \{u \in N_t : X_t^u > \lambda t\}$, $\lambda > 0$. Then

$$\mathbf{E}(|N_t^{\lambda t}|) = \Phi((\beta_2 - \lambda)\sqrt{t})e^{(\frac{\beta_2^2}{2} - \beta_2\lambda)t}.$$

- Almost sure total population growth rate:

$$\lim_{t\to\infty} \frac{\log |N_t|}{t} = \frac{1}{2}\beta_2^2 \qquad \mathbf{P}\{\cdot | M_\infty > 0\} - a.s.$$

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Main results

- Almost sure population growth rates: Let $\lambda > 0$.
  1. If $\lambda > \beta_2/2$, then
  $$\lim_{t\to\infty} |N_t^{\lambda t}| = 0, \qquad \mathbf{P}\text{-a.s.}$$
  2. If $\lambda < \beta_2/2$, then $\mathbf{P}\{\cdot | M_\infty > 0\}$-a.s.
  $$\lim_{t\to\infty} \frac{\log |N_t^{\lambda t}|}{t} = \Delta_\lambda = \begin{cases} \frac{1}{2}\beta_2^2 - \beta_2\lambda & \text{if } \lambda < \beta_2, \\ -\frac{1}{2}\lambda^2 & \text{if } \lambda \geq \beta_2. \end{cases}$$
- (Rightmost particle speed) Let $R_t := \sup_{u\in N_t} X_t^u$, $t \geq 0$. Then
  $$\lim_{t\to\infty} \frac{R_t}{t} = \frac{\beta_2}{2}, \quad \mathbf{P}\{\cdot | M_\infty > 0\}\text{-a.s.}$$

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Upper bound for the asymptotic behaviour of $|N_t^{\lambda t}|$

Proof. Note that $N_t^{\lambda t} := \{u \in N_t : X_t^u > \lambda t\}$. Thus for $t \in [n, n+1)$,

$$|N_t^{\lambda t}| \leq \sum_{u \in N_n} \sum_{v \in D_n(u)} 1_{\{\sup_{s \in [n,n+1)} X_s^v \geq \lambda n\}}$$

where $D_n(u)$ is the set of descendants of u that have ever exist in $[n, n+1)$.

Take $\epsilon > 0$ and consider events

$$A_n = \left\{ \sum_{u \in N_n} \sum_{v \in D_n(u)} 1_{\{\sup_{s \in [n,n+1)} X_s^v \geq \lambda n\}} > e^{(\Delta_\lambda + \epsilon)n} \right\}.$$

If we can show that $\mathbf{P}(A_n)$ decays to 0 exponentially fast then by the Borel-Cantelli Lemma we would have $\mathbf{P}(A_n \text{ i.o.}) = 0$ and that would be sufficient to get the result.

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Another result about $|N_t^{\lambda t}|$

We can also say something about the rare events of $|N_t^{\lambda t}|$ being positive when we typically do not find particles with speeds $\lambda > \frac{\beta_2}{2}$.

Proposition

(Unusually fast particles) Assume that $\sum_{k=1}^{\infty}(k\ln k)p_k < \infty$. *For* $\lambda > \frac{\beta_2}{2}$,

$$\lim_{t\to\infty}\frac{\log \mathbf{P}(|N_t^{\lambda t}| \geq 1 | M_\infty > 0)}{t} = \triangle_\lambda.$$

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be some continuous compactly-supported function. If $\sum_{k=1}^{\infty}(k\ln k)p_k < \infty$, then

$$\lim_{t\to\infty} e^{-\frac{1}{2}\beta_2^2}\sum_{u\in N_t} f(X_t^u) = M_\infty \int_{\mathbb{R}} f(x)\beta_2 e^{-\beta_2|x|}dx, \quad \mathbf{P}\text{-a.s.}$$

where M_∞ is the almost sure limit of the $\mathbf{P}$-uniformly integrable additive martingale

$$M_t = \sum_{u\in N_t} \exp\{-\beta_2|X_t^u| - \frac{1}{2}\beta_2^2 t\}.$$

Key Proposition

It is sufficient to prove for functions $f(x) = e^{-\beta|x|}1_{\{x\in B\}}$. Let

$$U_t = e^{-\frac{1}{2}\beta_2^2 t}\sum_{u\in N_t} e^{-\beta_2|X_t^u|}1_{\{X_t^u\in B\}}.$$

Proposition

If $\sum_{k=1}^{\infty}(k\ln k)p_k < \infty$, then for any $m \in \mathbb{N}, \sigma > 0$,

$$\lim_{n\to\infty} |U_{(m+n)\sigma} - \mathbf{E}(U_{(m+n)\sigma}|\mathcal{F}_{n\sigma})| = 0, \ \mathbf{P}\text{-}a.s.$$

Idea of proof of the proposition

Let

$$\widetilde{U}_t = e^{-\frac{1}{2}\beta_2^2 t} e^{-\beta_2|\xi_t|} 1_{\{\xi_t \in B\}} + e^{-\frac{1}{2}\beta_2^2 t} \sum_{\nu < \xi_{n_t}} \sum_{j=1}^{A_\nu - 1} \langle e^{-\beta_2|\cdot|} 1_{\{\cdot \in B\}}, Y^j_{t-S_\nu} \rangle 1_{\{A_\nu \le e^{\frac{1}{2}\beta_2^2 (S_\nu + n\sigma/2)}\}}.$$

Lemma

For each $f \in \mathcal{B}_b^+(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\tilde{\mathbf{E}}[\widetilde{U}_t]^2 < \infty.$$

Idea of proof of the proposition

Note that we may always write

$$U_{(m+n)\sigma} = \sum_{i=1}^{N_{n\sigma}} e^{-\lambda_1 n\sigma} U_{m\sigma}^{(i)}$$

where given $\mathcal{F}_{n\sigma}$, the collection $\{U_{m\sigma}^{(i)} : i = 1, \ldots, N_{n\sigma}\}$ are mutually independent and equal in distribution to $U_{m\sigma}$ under $\mathbf{P}_{\delta_{Y_i}}$. Then we can write

$$\begin{aligned} U_{(m+n)\sigma} &= \sum_{i=1}^{N_{n\sigma}} e^{-\lambda_1 n\sigma} \widetilde{U}_{m\sigma}^{(i)} + \sum_{i=1}^{N_{n\sigma}} e^{-\lambda_1 n\sigma} \left(U_{m\sigma}^{(i)} - \widetilde{U}_{m\sigma}^{(i)} \right) \\ &:= U_{(m+n)\sigma}^{[1]} + U_{(m+n)\sigma}^{[2]}, \end{aligned} \tag{4}$$

where, under $\widetilde{\mathbf{P}}_{\delta_{Y_i}}$, $\{\widetilde{U}_{m\sigma}^{(i)} : i = 1, \ldots, N_{n\sigma}\}$ are equal in distribution to $\widetilde{U}_{m\sigma}$.

Key Proposition

Idea of proof: From the decomposition (4), we have

$$\begin{aligned} & U_{(m+n)\sigma} - \mathbf{E}(U_{(m+n)\sigma} | \mathcal{F}_{n\sigma}) \\ = \; & U_{(m+n)\sigma} - U^{[1]}_{(m+n)\sigma} + U^{[1]}_{(m+n)\sigma} - \tilde{\mathbf{E}}\left(U^{[1]}_{(m+n)\sigma} \middle| \mathcal{F}_{n\sigma} \right) \\ & - \tilde{\mathbf{E}}\left[\left(U_{(m+n)\sigma} - U^{[1]}_{(m+n)\sigma} \right) \middle| \mathcal{F}_{n\sigma} \right] \end{aligned}$$

Now the conclusion of this proposition follows immediately form the following three lemmas.

Key Lemma

Lemma

If $\sum_{k=1}^{\infty}(k\ln k)p_k < \infty$, *then*

$$\lim_{n\to\infty}|U_{(n+m)\sigma} - U^{[1]}_{(n+m)\sigma}| = 0, \quad \tilde{\mathbf{P}}{-}a.s.$$

$$\sum_{n=1}^{\infty}\tilde{\mathbf{E}}\left[U^{[1]}_{(m+n)\sigma} - \tilde{\mathbf{E}}(U^{[1]}_{(m+n)\sigma}|\mathcal{F}_{n\sigma})\right]^2 < \infty.$$

where $U^{[1]}_{(m+n)\sigma}$ *was defined in (4). In particular*

$$\lim_{n\to\infty}\left|U^{[1]}_{(m+n)\sigma} - \tilde{\mathbf{E}}(U^{[1]}_{(m+n)\sigma}|\mathcal{F}_{n\sigma})\right| = 0,\ \mathbf{P}_{\delta_x}\text{-a.s.}$$

Key Lemma

Lemma

If $\sum_{k=1}^{\infty}(k \ln k)p_k < \infty$, *then for any* $m \in \mathbb{N}, \sigma > 0$,

$$\sum_{n=0}^{\infty} \mathbf{E}\left[\left(U_{(m+n)\sigma} - U^{[1]}_{(m+n)\sigma}\right) \Big| \mathcal{F}_{n\sigma}\right] \text{ converges } \mathbf{P}\text{-a.s.}$$

Proof of strong law of large numbers

The proof will be finished in three parts.

Part I: This is finished by the above Key Proposition.

Part II:

$$\lim_{n\to\infty} \left|\mathbf{E}(U_{(m+n)\sigma}|\mathcal{F}_{n\sigma}) - \pi(B)M_\infty\right| = 0, \quad \mathbf{P} - a.s.$$

Part III: From lattice times to continuous-time limit.

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Thanks for all your attention!

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