Harmonic Moments and Large Deviations for Supercritical Branching Processes with Immigration

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Model: the GWI process

Definition Let $Z_0 = 1$, the Galton-Watson branching process with immigration (GWI process) is defined recursively as

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i^{(n)} + Y_n, \quad n \geq 1,$$

where

- $\{X_i^{(n)}\}$ are i.i.d. random variables with p.g.f. $f(s) = \sum_{i=0}^{\infty} p_i s^i$,
- $\{Y_n\}$ (immigration) are i.i.d. random variables with p.g.f. $h(s) = \sum_{i=0}^{\infty} h_i s^i$,
- Moreover, $\{X_i^{(n)}\}$ and $\{Y_n\}$ are independent.

Assumptions

- $m := EZ_1 = \sum_{i=0}^{\infty} i p_i > 1$ (supercritical)
- $p_0 = 0$ (without loss of generality)

Model

Some properties for supercritical GWI processes

- (Senata (1970)) $\exists$ a sequence of constant $\{C_n\}$ such that $V_n := \frac{Z_n}{C_n} \xrightarrow{a.s} V$ $(n \to \infty)$ (If $Y_n = 0$, Z_n degenerates to GW and $W_n := \frac{Z_n}{C_n} \xrightarrow{a.s} W$). Furthermore, $E \log Y_1 < \infty \Longrightarrow P\{V < \infty\} = 1$.

- Let $g_n(s)$ be the p.g.f. of $\{Z_n\}$. Then

$$g_n(s) = Es^{Z_n} = g_{n-1}\big(f(s)\big)h(s) = \prod_{i=0}^{n-1} h\big(f_i(s)\big) f_n(s), \quad n \geq 1.$$

- (J. Liu and M. Zhang (2016)) If $p_1 h_0 > 0$, then

$$\gamma_n(s) := \frac{g_n(s)}{(p_1 h_0)^n} = \Big(\frac{\prod_{i=0}^{n-1} h\big(f_i(s)\big)}{h_0^n}\Big)\Big(\frac{f_n(s)}{p_1^n}\Big) \nearrow \gamma(s),$$

uniformly for $s \in [0, 1)$ as $n \to \infty$.

Motivation

The harmonic moments of Z_n

$$E(Z_n^{-r} \mid Z_n > 0) \quad r > 0.$$

- For supercritical GW process, Ney and Vidyashankar (2003) studied the asymptotic behavior of $E(Z_n^{-r})$, there is a phase transition in converging rates depending on whether $p_1 m^r > 1, = 1$ or < 1.

Motivation

Theorem (Harmonic moments for supercritical GW processes, Ney and Vidyashankar, 2003)

Assume $1 < m < \infty$. *Let*

$$A_n(r) = \begin{cases} p_1^{-n}, & \text{if } p_1 m^r > 1, \\ p_1^{-n} \left(\sum_{k=0}^{n-1} p_1^{-k} C_k^{-r}\right)^{-1}, & \text{if } p_1 m^r = 1, \\ C_n^r, & \text{if } p_1 m^r < 1. \end{cases}$$

Then

$$\lim_{n\to\infty} A_n(r) E(Z_n^{-r}) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^\infty Q(e^{-u})\, u^{r-1}\, du, & \text{if } p_1 m^r > 1, \\ \frac{1}{\Gamma(r)} \int_1^m Q\big(\phi(u)\big)\, u^{r-1}\, du, & \text{if } p_1 m^r = 1, \\ \frac{1}{\Gamma(r)} \int_0^\infty \phi(u)\, u^{r-1}\, du, & \text{if } p_1 m^r < 1, \end{cases}$$

where $Q(s) = \lim_{n\to\infty} f_n(s)/p_1^n$ *and* $\phi(u) = Ee^{-uW}$.

Motivation

Question: For GWI process $\{Z_n\}$,

- the convergence rates of $E(Z_n^{-r})$?
- is there a phase transition? classified by?
- the precise convergence limits?

Main Results

Theorem 1 (Harmonic moments for supercritical GWI)

Assume that $\boldsymbol{E\log Y_1<\infty}$. *Define*

$$a_n(r)=\begin{cases}(p_1h_0)^{-n}, & \text{if } p_1h_0m^r>1,\\ (p_1h_0)^{-n}\left(\sum_{k=0}^{n-1}(p_1h_0)^{-k}C_k^{-r}\right)^{-1}, & \text{if } p_1h_0m^r=1,\\ C_n^r, & \text{if } p_1h_0m^r<1.\end{cases}$$

Then

$$\lim_{n\to\infty}a_n(r)E(Z_n^{-r})=\begin{cases}\frac{1}{\Gamma(r)}\int_0^\infty \gamma(e^{-u})\,u^{r-1}\,du, & \text{if } p_1h_0m^r>1,\\ \frac{1}{\Gamma(r)}\int_1^m \gamma\big(\phi(u)\big)\,u^{r-1}\,du, & \text{if } p_1h_0m^r=1,\\ \frac{1}{\Gamma(r)}\int_0^\infty \psi(u)\,u^{r-1}\,du, & \text{if } p_1h_0m^r<1,\end{cases}$$

where $\boldsymbol{\phi(u)=Ee^{-uW}}$, $\boldsymbol{\psi(u)=Ee^{-uV}}$.

Main Results

Theorem 2 (Large deviations for Z_{n+1}/Z_n)

Assume that $m > 1$ *and* $E \log Y_1 < \infty$. *For any* $\varepsilon > 0$, *define*

$$A(k, \varepsilon) = P(|\overline{X}_k + \frac{Y_1}{k} - m| > \varepsilon),$$

where $\overline{X}_k = \frac{1}{k} \sum_{i=1}^{k} X_i^{(1)}$. *If there exist* $r > 0$ *and* $C(\varepsilon, r) > 0$ *such that* $A(k, \varepsilon) \leq C(\varepsilon, r) k^{-r}$ *for all* $k \geq 1$, *then*

$$\limsup_{n \to \infty} a_n(r) P\left(\left| \frac{Z_{n+1}}{Z_n} - m \right| > \varepsilon \right) \leq C(\varepsilon, r) B,$$

where B *is a finite positive constant. Furthermore, if* $A(k, \varepsilon) \geq C(\varepsilon, r) k^{-r}$ *for all* $k \geq 1$, *then*

$$\liminf_{n \to \infty} a_n(r) P\left(\left| \frac{Z_{n+1}}{Z_n} - m \right| > \varepsilon \right) \geq C(\varepsilon, r) B.$$

Main Results

Remark (a sufficient condition)

If there exist $r \geq 1$ and $\delta > 0$ such that $E[X_1^{(1)}]^{2r+\delta} < \infty$ and $EY_1^r < \infty$, then $A(k,\varepsilon) \leq C(\varepsilon, r)k^{-r}$ holds.

More precisely, by virtue of Tao-Cheng's inequality of moments of weighted sums $M_r := \sup_k E[\sqrt{k}|\overline{X}_k - m|^{2r}] < \infty$. From this it follows, by triangle inequality and Markov's inequality,

$$\begin{aligned} A(k,\varepsilon) &:= P(|\overline{X}_k + \frac{Y_1}{k} - m| > \varepsilon) \\ &\leq P(|\overline{X}_k - m| > \frac{\varepsilon}{2}) + P(\frac{Y_1}{k} > \frac{\varepsilon}{2}) \\ &\leq \frac{4^r E|\sqrt{k}(\overline{X}_k - m)|^{2r}}{k^r \varepsilon^{2r}} + \frac{2^r E(Y_1^r)}{k^r \varepsilon^r} \\ &\leq C(\varepsilon, r)k^{-r}. \end{aligned}$$

Idea (Theorem 1)

For any nonnegative r.v. X, $0 < r < \infty$,

$$X^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-tX}\, t^{r-1}\, dt.$$

Let $X = Z_n$ and take expectations on both sides, we have

$$\begin{aligned} E(Z_n^{-r}) &= \frac{1}{\Gamma(r)} E\Big(\int_0^\infty e^{-tZ_n}\, t^{r-1}\, dt \Big) = \frac{1}{\Gamma(r)} \int_0^\infty g_n(e^{-t})\, t^{r-1}\, dt \\ &=: \frac{1}{\Gamma(r)} I_n(r), \end{aligned}$$

where

$$\begin{aligned} I_n(r) &= \int_0^{C_n^{-1}} g_n(e^{-t})\, t^{r-1}\, dt + \int_{C_n^{-1}}^1 g_n(e^{-t})\, t^{r-1}\, dt \\ &+ \int_1^\infty g_n(e^{-t})\, t^{r-1}\, dt =: H_n(1) + H_n(2) + H_n(3). \end{aligned}$$

Idea (Theorem 1)

Lemma 1 (First part)

(1) If $p_1 h_0 m^r > 1$, then $\lim_{n\to\infty} a_n(r) H_n(1) = 0$.

(2) If $p_1 h_0 m^r = 1$, then $\lim_{n\to\infty} a_n(r) H_n(1) = 0$.

(3) If $p_1 h_0 m^r < 1$, then $\lim_{n\to\infty} a_n(r) H_n(1) = \int_0^1 \psi(u)\, u^{r-1}\, du$,

where $\psi(u) = E e^{-uV}$.

Lemma 2 (Last part)

(1) If $p_1 h_0 m^r > 1$, then $\lim\limits_{n\to\infty} a_n(r) H_n(3) = \int_1^\infty \gamma(e^{-u})\, u^{r-1}\, du$,

where $\gamma(s) := \lim_{n\to\infty} g_n(s)/(p_1 h_0)^n$.

(2) If $p_1 h_0 m^r = 1$, then $\lim_{n\to\infty} a_n(r) H_n(3) = 0$.

(3) If $p_1 h_0 m^r < 1$, then $\lim_{n\to\infty} a_n(r) H_n(3) = 0$.

Idea (Theorem 1)

Lemma 3 (Second part)

$$\lim_{n\to\infty} a_n(r)H_n(2) = \begin{cases} \int_0^1 \gamma(e^{-u})\, u^{r-1}\, du, & \text{if } p_1 h_0 m^r > 1, \\ \int_1^m \gamma\big(\phi(u)\big)\, u^{r-1}\, du, & \text{if } p_1 h_0 m^r = 1, \\ \int_1^\infty \psi(u)\, u^{r-1}\, du, & \text{if } p_1 h_0 m^r < 1, \end{cases}$$

where $\phi(u) = Ee^{-uW}$.

Idea (Theorem 2)

Recall the definition $A(k,\varepsilon) = P(|\overline{X}_k + \frac{Y_1}{k} - m| > \varepsilon)$, where $\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i^{(1)}$.

- By the branching property,

$$\begin{aligned} P\Big(\Big|\frac{Z_{n+1}}{Z_n} - m\Big| > \varepsilon\Big) &= \sum_{k=1}^{\infty} P\Big(\Big|\frac{Z_{n+1}}{Z_n} - m\Big| > \varepsilon \,|\, Z_n = k\Big) P(Z_n = k) \\ &= \sum_{k=1}^{\infty} A(k,\varepsilon) P(Z_n = k) \\ &\leq C(\varepsilon, r) E(Z_n^{-r}) \end{aligned}$$

(assumption: $\exists \ r > 0$ and $C(\varepsilon, r) > 0$ s.t. $A(k,\varepsilon) \leq C(\varepsilon, r) k^{-r}$)

- Apply Theorem 1

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Thanks!