Large deviation principle for the maximal positions in the critical branching random walks with small drifts

Hongyan Sun
(China University of Geosciences, Beijing)

Beijing Normal University

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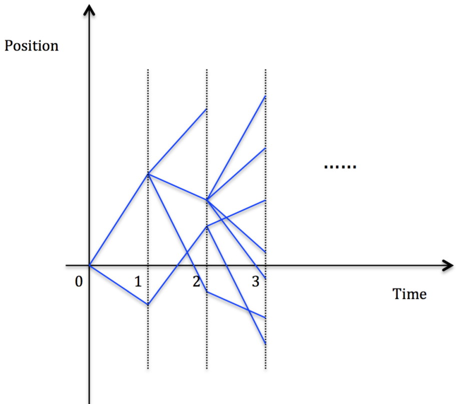
Branching random walks

A branching random walk: an initial particle, which forms the zeroth generation, is at the origin of R. It gives birth to offspring particles that form the first generation. Their displacements from their parent are described by a point process Θ.



Branching random walks

M_n : the maximal position of the particles in the n-th generation.

Supercritical case:

- Hammarsley (1974), Kingman (1975), Biggins (1976)
  $\frac{M_n}{n} \to \gamma$, a.s. on the non-extinction set.
- Addario and Reed (2009), Hu and Shi (2009), Hu (2012,2016)
  established the second or the third order of M_n.
- Biggins (1977) or Rouault (2000)
  give the large deviation principle of M_n.

Critical case: Suppose the displacements of particles from their parents are independently and identically distributed.

- Durrett (1991), Kesten (1995), Lalley and Shao (2015)
  (a) study the scaled limit distribution of M_n conditioned on that the system can survive to the n-th generation.
  (b) give the tail distribution of $M =: \sup_{n\geq 0} M_n$.

Branching random walks

- Zheng (2010)
  studies the maximal positions of a sequence of critical branching random walks with small drifts.

Subcritical case:

- Neuman and Zheng (2017)
  (a) study the scaled limit distribution of $\overline{M}_n =: \sup_{k \le n} M_k$.
  (b) give the tail distribution of $M =: \sup_{n \ge 0} M_n$.

Our problem

The large deviation principle of the maximal positions in the critical branching random walks with small drifts.

Critical BRWs with small drifts

For fixed $n \geq 1$, we assume $V^{(n)}$ is the branching random walk on $\mathbb{Z}_+$, which starts with one particle at the origin.
(A) At each time, particles produce offsprings as in a standard Galton-Waston process with mean 1 and finite variance.
(B) Each particle moves one time from its parent according to the transition probability $\mathbb{P}^{(\alpha,\beta,n)}$ below, where $\alpha > 0, \beta > 0$.

$$\begin{array}{ll} \mathbb{P}(x, x+1) = \frac{1}{2} - \frac{\beta}{n^\alpha} & \text{for } x \geq 1; \\ \mathbb{P}(x, x-1) = \frac{1}{2} + \frac{\beta}{n^\alpha} & \text{for } x \geq 1; \\ \mathbb{P}(0,1) = 1. & \end{array} \tag{1}$$

$V^{(n)}(x)$: the position of the particle x, $|x|$: x's generation.
$M_k^{(n)} =: \max_{|x|=k} V^{(n)}(x)$.
$Z_k^{(n)}$: the number of particles at the k-th generation in $V^{(n)}$.

Critical BRWs with small drifts

Zheng (2010)

When $\alpha = \frac{1}{2}$. For $\kappa > 1$, and every $\epsilon > 0$,

$$\lim_{n\to\infty} P\Big(\Big|\frac{M^{(n)}_{[n^\kappa]}}{\sqrt{n}\log n} - \frac{\kappa-1}{4\beta}\Big| > \epsilon| > |Z^{(n)}_{[n^\kappa]} > 0\Big) = 0.$$

For any α bigger than zero, the conditioned law of large number of $M^{(n)}_{[n^\kappa]}$ also holds with rate $\frac{\kappa-2\alpha}{4\beta}$, for $\kappa > 2\alpha$.

How about the large deviation principle of $M^{(n)}_{[n^\kappa]}$?

Main result

Assume $\{Z_m\}$ is a critical branching process with finite variance and offspring distribution $\{p_j\}_{j\geq 0}$.

Theorem 1

Assume $\kappa > 2\alpha$. If $\sum_{j\geq 0} j^2(\log j)p_j < \infty$, then
(a) for any $\lambda > \frac{\kappa-2\alpha}{4\beta}$, we have

$$\limsup_{n\to\infty} \frac{\log P(M^{(n)}_{[n^\kappa]} > \lambda n^\alpha \log n | Z^{(n)}_{[n^\kappa]} > 0)}{\log n} \leq \max\{\kappa - 2\alpha - 4\lambda\beta, 2\alpha - \kappa\},$$

(b) for any $\frac{\kappa-2\alpha}{4\beta} < \lambda < \frac{\kappa-2\alpha}{2\beta}$, we have

$$\liminf_{n\to\infty} \frac{\log P(M^{(n)}_{[n^\kappa]} > \lambda n^\alpha \log n | Z^{(n)}_{[n^\kappa]} > 0)}{\log n} \geq \kappa - 2\alpha - 4\lambda\beta.$$

Main result

Theorem 2

Under the assumption in Theorem 1, we have for any $0 < \lambda < \frac{\kappa - 2\alpha}{4\beta}$,

$$\limsup_{n\to\infty} \frac{\log P(M^{(n)}_{[n^\kappa]} \leq \lambda n^\alpha \log n | Z^{(n)}_{[n^\kappa]} > 0)}{\log n} \leq -(\kappa - 2\alpha - 4\beta\lambda).$$

Corollary 3

Under the assumptions in Theorem 1, we have that for all $\varepsilon > 0$, $M^{(n)}_{[n^\kappa]}$ *satisfies*

$$\lim_{n\to\infty} P\left(\left| \frac{M^{(n)}_{[n^\kappa]}}{n^\alpha \log n} - \frac{\kappa - 2\alpha}{4\beta} \right| \geq \varepsilon \middle| Z^{(n)}_{[n^\kappa]} > 0 \right) = 0.$$

Idea of the proofs

- Use the strategy of changing the conditional event, which is used in Zheng (2010).
- Divide the branching part and the walking part.

Preparations for the proofs

Lemma 4

Assume $\sum_{j\geq 0} j^2(\log j)p_j < \infty$ *holds. Let* $m_n \leq n$ *be integers and* $\epsilon_n > 0$ *be real values such that* $m_n/n \to 1$, $n - m_n \to \infty$ *and* $\epsilon_n \to 0$ *as* $n \to \infty$. *Let* $G_n = \{Z_n > 0\}$, $H_n = \{Z_{m_n} \geq n\epsilon_n\}$ *and* $K_n = \{Z_{m_n} > 0\}$. *Then*
(1) for all n *large enough,*

$$\frac{P(G_n \Delta K_n)}{P(K_n)} \leq \delta_2,$$

(2) for all n *large enough,*

$$\frac{P(G_n \Delta H_n)}{P(G_n)} \leq 4(\delta_1 + \delta_2),$$

where $\delta_1 = \frac{C\sigma^2 n\epsilon_n}{m_n}$, $\delta_2 = \frac{2}{1 - \frac{2}{\sigma^2} m_n \log\left(1 - \frac{\sigma^2}{n - m_n}\right)}$.

Preparations for the proofs

Let $\{S_k^{(n)}\}$ denote a random walk with transition probability $\mathbb{P}^{(\alpha,\beta,n)}$ given by (1).

Proposition 5

For any fixed $b > 0$, and any nonnegative sequences $\{l_n\}$, $\{k_n\}$ with $\limsup_{n\to\infty} l_n/n^\alpha < \infty$ and $\liminf_{n\to\infty} k_n/(n^{2\alpha}\log^2 n) > 0$, the random walks $\{S_k^{(n)}\}$ satisfy

$$\lim_{n\to\infty} P(S_{k_n}^{(n)} \geq bn^\alpha | S_0^{(n)} = l_n) = \exp(-4\beta b),$$

and

$$\lim_{n\to\infty} \frac{P(S_{k_n}^{(n)} \geq bn^\alpha \log n | S_0^{(n)} = l_n)}{n^{-4\beta b}} = 1.$$

Proof of Theorem 1 (a)

For all n large enough

$$\begin{aligned}
&|P(M^{(n)}_{[n^\kappa]} \geq \lambda n^\alpha \log n | Z^{(n)}_{[n^\kappa]} > 0) - P(M^{(n)}_{[n^\kappa]} \geq \lambda n^\alpha \log n | Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} > 0)| \\
&\leq 4n^{2\alpha-\kappa} \quad \text{(by Lemma 4).}
\end{aligned} \tag{2}$$

We divide the probability space into two event. And can get that

$$\begin{aligned}
&P(M^{(n)}_{[n^\kappa]} \geq \lambda n^\alpha \log n, M^{(n)}_{[n^\kappa]-[n^{2\alpha}]} \geq (\lambda-\epsilon) n^\alpha \log n | Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} > 0) \\
&\leq E(Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} | Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} > 0) \times P(S^{(\alpha,\beta,n)}_{[n^\kappa]-[n^{2\alpha}]} > (\lambda-\epsilon) n^\alpha \log n) \times \rho_{[n^{2\alpha}]} \\
&\leq \sigma^2 (n^\kappa - n^{2\alpha}) \times 2n^{-4\beta(\lambda-\epsilon)} \times \frac{4}{\sigma^2 (n^{2\alpha})} \leq 8n^{\kappa-4\beta\lambda-2\alpha+4\beta\epsilon}.
\end{aligned}$$

Proof of Theorem 1 (a)

Let $M'^{(n)}_{n^\kappa}$ denote the rightmost position of the particles in $[n^\kappa]$-th generation whose ancestors in $([n^\kappa]-[n^{2\alpha}])$-th generation is in the left of $(\lambda-\epsilon)n^\alpha \log n$.

Since the branching random walk is critical, we get for all n large enough

$$\begin{aligned}
&P(M'^{(n)}_{[n^\kappa]} \geq \lambda n^\alpha \log n | Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} > 0) \\
&\leq E(Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} | Z^{(n)}_{[n^\kappa]-[n^{2\alpha}]} > 0) \\
&\qquad\qquad \cdot P(S_{[n^\kappa]} > \lambda n^\alpha \log n | S_{[n^\kappa]-[n^{2\alpha}]} < (\lambda-\epsilon)n^\alpha \log n) \\
&\leq \sigma^2(n^\kappa - n^{2\alpha}) \cdot C_1 n^{-b\epsilon^2 \log n} \leq C_1 \sigma^2 n^{-(b\epsilon^2 (\log n)^2 - \kappa)},
\end{aligned}$$

where C_1 and b are constants irrelevant to n.

Combining these two situations, we have

$$P(M^{(n)}_{n^\kappa} \geq \lambda n^\alpha \log n | Z^{(n)}_{n^\kappa - n^{2\alpha}} > 0) \leq 8n^{\kappa-4\beta\lambda-2\alpha+4\beta\epsilon} + C\sigma^2 n^{-(b\epsilon^2(\log n)^2-\kappa)}.$$

Proof of Theorem 1 (a)

Then by (2), we get that for all n large enough

$$\begin{aligned} P(M_{n^\kappa}^{(n)} \geq \lambda n^\alpha \log n | Z_{n^\kappa}^{(n)} > 0) \leq\ & 8n^{\kappa - 4\beta\lambda - 2\alpha + 4\beta\epsilon} + C\sigma^2 n^{-(b\epsilon^2(\log n)^2 - \kappa)} \\ & + 2n^{2\alpha - \kappa}. \end{aligned}$$

Therefore, we have that for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{\log P(M_{n^\kappa} \geq \lambda n^\alpha \log n | Z_{[n^\kappa]}^{(n)} > 0)}{\log n} \leq \max\{\kappa - 2\alpha - 4\beta\lambda + 4\beta\epsilon,\ 2\alpha - \kappa\}.$$

Reference

[1] Addario-Berry, L., Reed, B. (2009). Minima in branching random walks. *Ann. Prob.* **37** 1044-1079.

[2] Biggins, J. D. (1976). The first and last birth problems for a multitype age-dependent branching process. *Adv. Appl. Probab.* **8** 446-459.

[3] Durrett, R., Kesten, H., Waymire,E. (1991). On weighted heights of random trees. *J.Theoret.Prob.* **4** 223-237.

[4] Neuman, E. Zheng X.H. (2017). On the maximal displacement of subcritical branching random walks, *Probab. Related Fields.* **167** 1137-1164.

[5] Hu, Y. (2012). The almost sure limits of the minimal position and the additive martingale in a branching random walk. *J.Theor.Probab.* **28** 467-487.

[6] Hu, Y. Y. (2016). How big the minimum of a branching random walk. *Ann. Inst. H. Poincaré Probab. Statist.* **52** 233-260.

[7] Hu, Y., Shi, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann.Probab.* **37** 742-789.

Reference

[8] Kesten, H. (1995). Branching random walk with a critical branching part. *J. Theoret.Prob.* **8** 921-962.

[9] Lalley, S. P., Shao, Y. (2015). On the maximal displacement of critical branching random walk. *Probability Theory and Related Fields* **162** 71-96.

[10] Zheng, X. H. (2010). Critical branching random walks with small drift. *Stoch.Proc.Appl.* **120** 1821-1836.

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