The non-degenerate limit for supercritical superprocesses

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Outline

- Limit results for Galton-Watson branching processes
- $L \log L$ criteria for supercritical superprocesses
- The non-degenerate limit for supercritical superprocesses without $L \log L$

Galton-Watson branching process

The Galton-Watson branching process (G-W process) $\{Z_n, n \geq 0\}$ is a Markov chain on $\{0, 1, 2, \cdots\}$.

- at time 0, we have $Z_0 = 1$ individual.
- each individual i in the nth generation produces a random number $L_i^{(n)}$ with distribution $p_k = P(L_i^{(n)} = k), k \geq 0$.
- $L_1^{(n)}$, $L_2^{(n)}$, $\cdots$, $L_{Z_n}^{(n)}$, $n \geq 0$, are independent.

Let Z_n denote the number of individuals alive in the nth generation. Then

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_i^{(n)}, \quad n \geq 0.$$

Galton-Watson branching process

- We assume that the mean number of offspring

$$m = \mathsf{E} L_i^{(n)} = \sum_{i=1}^{\infty} i p_i < \infty.$$

- Define $q = P(Z_n = 0, \exists n > 0)$.
- The G-W process is said to be
  - **subcritical** if $m < 1$; $(q = 1)$
  - **critical** if $m = 1$; $(q = 1.)$
  - **supercritical** if $m > 1$. $(0 < q < 1)$

Strong law for G-W process ($m > 1$)

- Since $E(Z_{n+1}|Z_n) = mZ_n, \quad EZ_n = m^n$, we have

$$W_n := \frac{Z_n}{m^n} \quad \text{is a martingale}$$

  and

$$\lim_{n \longrightarrow \infty} W_n = W (< \infty) \quad \text{exists a.s.}$$

  Recall that

$$q = P(Z_n = 0, \exists n > 0).$$

$$P(W = 0) \geq q.$$

- A classical question is when $P(W = 0) = q$?

Strong law for G-W process

Kesten-Stigum Theorem (1966) Suppose $m > 1$.
The following are equivalent

- (i) $P(W = 0) = q$
- (ii) $E(W) = 1$
- (iii) $E(L_i^{(n)} \log^+ L_i^{(n)}) < \infty$ (the "$L \log L$ condition").
  ($\log^+ r = 0 \vee \log r$, for $r > 0$.)

Remark

- In 1995, Lyons, Pemantle and Peres gave a probabilistic proof or "conceptual proof" of the Kesten-Stigum.
- Later this method were extended to general processes (see Kurtz-Lyons-Pemantle- Peres(1997); Lyons(1997); Athreya (2000); Biggins-Kyprianou (2004)); Engländer and Kyprianou (2004).

Strong law for G-W process

- Liu-Ren-Song (2009) obtained "$L \log L$" condition for supercritical **superdiffusions** under some conditions.
- Liu-Ren-Song (2011) obtained "$L \log L$" condition for supercritical **branching Hunt processes** under some conditions.

Question: If "$L \log L$" condition fails, the above result says that $\lim_{n \longrightarrow \infty} m^{-n} Z_n = 0$ a.s. Is there another normalizing sequence c_n such that $c_n Z_n$ has non-degenerate limit?

Strong law for G-W process

- For G-W branching processes:
  - Seneta (1968): There is c_n such that $c_n Z_n$ convergence **in distribution** to non-degenerate W.
  - Heyde (1970): There is c_n such that $c_n Z_n$ convergence **a.s.** to non-degenerate W.

  Remark Later the problem of finding c_n such that $c_n Z_n$ convergence to non-degenerate limit is called the Seneta-Heyde norming problem.

- Hoppe (1976) generalized the result of Heyde (1970) to supercritical multitype branching processes.
- Herring (1978) obtained similar result for supercritical branching diffusions.

Related works

- Suppose $\{X_n, n \geq 1\}$ is a supercritical branching random walk, where the positions of children are given by a point process Z with intensity ν. Define $m(\theta) = \int e^{-\theta x}\nu(dx)$. Then $W_n(\theta) := \frac{\int e^{-\theta x} Z_n(dx)}{m^n(\theta)}$ is a martingale. When the "$L \log L$" condition fails, $W_n(\theta)$ has limit 0.
  - Biggins-Kyprianou (1997) proved that it is possible to find a renormalization γ_n such that $\gamma_n W_n(\theta)$ converges in probability to a finite nonzero limit when the process survives in the non-boundary case.
  - Aidekon-Shi (2014) studied the Seneta-Heyde norming problem for branching random walk in the boundary case.

- Our purpose is to consider the Seneta-Heyde norming for superprocesses.

(ξ, ψ)-superprocesses

- $\xi = \{\xi_t, \Pi_x\}$: a Hunt process on E with semigroup $\{P_t, t \geq 0\}$ and generator L.
- **Branching mechanism** ψ:

For any $x \in E$, ψ is defined as

$$\psi(x, z) = -\beta(x)z + \frac{1}{2}\alpha(x)z^2 + \int_0^\infty \left(e^{-rz} - 1 + zr\right) n(x, \mathrm{d}r), \quad z > 0,$$

where $\alpha(x), \beta \in b\mathcal{B}(E)$ and $n(x, \mathrm{d}r)$ is kernel from E to $\mathbb{R} \setminus \{0\}$ such that $\sup_{x \in E} \int_0^\infty (r \wedge r^2) n(x, \mathrm{d}r) < \infty$.

(ξ, ψ)-superprocesses

- A superprocess is a limit of a sequence of branching Hunt processes under suitable scaling.
- A (ξ, ψ)-superprocess X is a measure-valued Markov process such that

$$\mathbb{P}_\mu \left(\exp\langle -f, X_t\rangle\right) = \exp\langle -u_f(t,\cdot), \mu\rangle, \text{ for any } f \in b\mathcal{B}_+(E) \tag{0.1}$$

  where u_f is the unique nonnegative mild solution to

$$\begin{cases} \dot{u}(s,x) = Lu - \psi(x, u(s,x)), & x \in E \\ u(0,x) = f(x). \end{cases} \tag{0.2}$$

- We also call X a superprocess corresponding to $L - \psi(u)$.

Assumptions

- There exists a family of continuous strictly positive functions $\{p(t,x,y) : t > 0\}$ on $E \times E$ s.t.

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy).$$

- Define $a_t(x) := \int_E p(t,x,y)^2\, m(dy)$ and $\widehat{a}_t(x) := \int_E p(t,y,x)^2\, m(dy)$.
- **Assumption 1** **(i)** *For any* $t > 0$, $\int_E p(t, {\color{red}x}, y)\, m({\color{red}dx}) \leq 1$.

  (ii) *For any* $t > 0$*, we have*

$$\int_E a_t(x)\, m(dx) = \int_E \widehat{a}_t(x)\, m(dx) < \infty. \qquad (0.3)$$

Moreover, $a_t(\cdot)$ *and* $\widehat{a}_t(\cdot)$ *are continuous on* E *.*

Assumptions

- Let $\{P_t^\beta\}_{t\geq 0}$ be the Feynman-Kac semigroup:

$$P_t^\beta f(x) = \Pi_x\left[\exp\left(\int_0^t \beta(\xi_s)\mathrm{d}s\right) f(\xi_t)\right], \quad x \in E.$$

$$\mathbb{P}_\mu\langle f, X_t\rangle = \langle P_t^\beta f, \mu\rangle.$$

- $\{\widehat{P}_t^\beta, t \geq 0\}$: the dual semigroup of $\{P_t^\beta, t \geq 0\}$.
- $L + \beta$: the generator of $\{P_t^\beta, t \geq 0\}$ in $L^2(E, m)$;
  $\widehat{L} + \beta$: the generator of $\{\widehat{P}_t^\beta, t \geq 0\}$ in $L^2(E, m)$.
- **Assumption 2** *(Supercritical assumption)*
  $\lambda_1 := \sup \mathrm{Re}(\sigma(L + \beta)) = \sup \mathrm{Re}(\sigma(\widehat{L} + \beta)) > 0.$

Assumptions

ϕ: positive eigenfunction of $L+\beta$ associated with λ_1.
$\widehat{\phi}$: positive eigenfunction of $\widehat{L}+\beta$ associated with λ_1.

$$\int_D \phi(x)\widehat{\phi}(x)dx = 1.$$

- **Assumption 3** **(i)** ϕ *is bounded.*

  (i) *(IU Property) The semigroups* $\{P_t^\beta\}_{t\geq 0}$ *and* $\{\widehat{P}_t^\beta\}_{t\geq 0}$ *are intrinsically ultracontractive, that is, for any* $t>0$*, there exists a constant* $c_t>0$ *such that*

$$p^\beta(t,x,y) \leq c_t\phi(x)\widehat{\phi}(y), \qquad \textit{for all } (x,y)\in E\times E.$$

Remark $\mathbb{P}_\mu\langle f, X_t\rangle = \langle P_t^\beta f, \mu\rangle \sim Ce^{\lambda_1 t}, \quad$ as $t \longrightarrow \infty$.

$L \log L$ condition

- Define $M_t(\phi) := e^{-\lambda_1 t}\frac{\langle \phi, X_t\rangle}{\langle \phi, \mu\rangle}$. Then $M_t(\phi), t \geq 0$ is a martingale and then

$$M_t(\phi) \longrightarrow M_\infty(\phi) < \infty, \quad \mathbb{P}_\mu\text{-a.s.}$$

- Define $l(y) := \int_1^\infty r \ln r n^\phi(y, \mathrm{d}r)$.

Theorem (Ren-Song-Yang, preprint 2016)
$M_\infty(\phi)$ is non-degenerate under $\mathbb{P}_\mu$ for any nonzero finite measure on E if and only if

$$(L \log L \text{ condition }) \int_E \widehat{\phi}(x) l(x) \mathrm{d}x < \infty.$$

Remark See Liu-Ren-Song (2009) for superdiffusions.

When $L \log L$ holds

Theorem (Liu-Ren-Song 2011) Suppose X is a superdiffusion and the $(L \log L)$ condition holds. For every bounded $f \geq 0$ on E with compact support whose set of discontinuous points has zero Lebesgue measure, we have

$$\lim_{t\to\infty} \frac{\langle f, X_t\rangle(\omega)}{\mathbb{P}_\mu\langle f, X_t\rangle} = \frac{M_\infty(\phi)(\omega)}{\langle \phi, \mu\rangle}, \quad \mathbb{P}_\mu\text{−a.s.} \tag{0.4}$$

Remark Chen-Ren-Yang (2017) extended the above result to general superprocesses under a second moment condition.

When $L \log L$ fails

- Now our main objective is to find a Seneta-Heyde norming for the martingale $M_t(\phi)$.
- Define $q_t(x) := \mathbb{P}_{\delta_x}(\|X_t\| = 0)$.

Assumption 4 *There exists $t_0 > 0$ such that,*

$$\inf_{x \in E} q_{t_0}(x) > 0. \tag{0.5}$$

Lemma 5 *Define*

$$q(x) := \lim_{t \longrightarrow \infty} q_t(x).$$

For any $x \in E$,

$$q(x) < 1.$$

When $L \log L$ fails

- Define $v(x) := -\log q(x)$.
- Recall that

$$u_f(t,x) = -\log \mathbb{P}_{\delta_x}\left(\exp\langle -f, X_t\rangle\right).$$

We write $V_t f(x) := u_f(t,x)$.

- **Backward iterate:** $(\eta_t, t \geq 0) \in \mathcal{B}_+(E)$ is called a family of backward iterates if

$$\eta_t = V_s(\eta_{t+s}), \qquad t, s \geq 0.$$

We call a family (η_t) of backward iterates is **non-trivial** if, for some $t \geq 0$, neither $\eta_t = 0, a.e.$, nor $\eta_t = v, a.e.$

When $L \log L$ fails

Lemma 6 *There exists a non-trivial family of backward iterates* $(\eta_t, t \geq 0)$*.*

Lemma 7

$$\lim_{t \longrightarrow \infty} \|\eta_t\|_\infty = 0.$$

Moreover,

$$\lim_{t \longrightarrow \infty} \frac{\langle \eta_t, \widehat{\phi} \rangle_m}{\langle \eta_{t+s}, \widehat{\phi} \rangle_m} = e^{\lambda_1 s}. \qquad (0.6)$$

When $L \log L$ fails

Theorem 8 *(Ren-Song-Zhang, 2017)* *Let* (η_t) *be a non-trivial family of backward iterates define* $\gamma_t := \langle \eta_t, \widehat{\phi} \rangle_m$*. Then there is a non-degenerate random variable* W *such that for any* $\mu \in \mathcal{M}_F(E)$*,*

$$\lim_{t \longrightarrow \infty} \gamma_t \langle \phi, X_t \rangle = W, \qquad \text{a.s.-}\mathbb{P}_\mu$$

and

$$\mathbb{P}_\mu(W = 0) = e^{-\langle v, \mu \rangle}, \qquad \mathbb{P}_\mu(W < \infty) = 1, \qquad (0.7)$$

where $v(x) := -\log q(x)$*.*

Outline of the proof of Theorem 8

- By the definition of η_t,

$$\mathbb{P}_\mu(\exp\{-\eta_{t+s}, X_{t+s}\}|\mathcal{F}_t) = \mathbb{P}_{X_t}(\exp\{-\eta_{t+s}, X_s\}) = e^{-\langle \eta_t, X_t\rangle},$$

  which says that $\{\exp\{-\langle \eta_t, X_t\rangle\}, t \ge 0\}$ is a martingale. Hence $W := \lim_{t\longrightarrow\infty}\langle \eta_t, X_t\rangle \in [0,\infty]$ exists $\mathbb{P}_\mu$.

- It follows from Lemma 7 that

$$(1+\|h_t\|_\infty)^{-1}\langle \eta_t, X_t\rangle \le \gamma_t\langle \phi, X_t\rangle \le (1-\|h_t\|_\infty)^{-1}\langle \eta_t, X_t\rangle.$$

  Letting $t \longrightarrow \infty$, we get that

$$\lim_{t\longrightarrow\infty} \gamma_t\langle \phi, X_t\rangle = W, \qquad \text{a.s.-}\mathbb{P}_\mu.$$

- $\mathbb{P}_\mu(W=0) = e^{-\langle v,\mu\rangle}, \quad \mathbb{P}_\mu(W<\infty) = 1.$

Future work

- The Seneta-Heyde norming for general test function f.

END

Thank you!

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