Extinction time of a CSBP with competition in a Lévy random environment.

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Outline

CSBP with competition.

CSBP with competition in a Lévy random environment.

Brownian case

Logistic competition.

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Prototype example : logistic Feller diffusion

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where $B^{(b)} = (B_t^{(b)}; t \geq 0)$ is a standard Brownian motion.

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where $B^{(b)} = (B_t^{(b)}; t \geq 0)$ is a standard Brownian motion.

The logistic Feller diffusion can also be defined as scaling limits of Bienaymé-Galton-Watson processes with competition, i.e. a continuous time Markov chain with individuals behaving independently from one another and each giving birth to a (random) number of offspring (belonging to the next generation) but also considering competition pressure.

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More precisely, Lambert considered the following generalized Ornstein-Uhlenbeck process starting from $x > 0$,

$$\mathrm{d}R_t = \mathrm{d}X_t - cR_t\mathrm{d}t,$$

where $X = (X_t, t \geq 0)$ denotes a spectrally positive Lévy process whose law started from $x \in \mathbb{R}$ is denoted by $\mathbf{P}_x$.

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The law of X is completely characterized by its Laplace exponent ψ which is defined as $\psi(\lambda) = \log \mathbf{E}[e^{-\lambda X_1}]$ for $\lambda \geq 0$, and satisfies the so-called Lévy-Khintchine representation

$$\psi(u) = -bu + \gamma^2 u^2 + \int_{(0,\infty)} (e^{-ux} - 1 + ux\mathbf{1}_{\{x<1\}})\mu(\mathrm{d}x).$$

Lamperti-type transform

Let $T_0^R = \inf\{s : R_s = 0\}$ and we consider the clock

$$\eta_t = \int_0^{t\wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad \text{for} \quad t > 0.$$

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Let θ denotes the right-continuous inverse of the clock η. According to Lambert, the logistic branching process is defined as follows

$$Y_t = \begin{cases} R_{\theta_t} & \text{if } 0 \leq t < \eta_\infty \\ 0 & \text{if } \eta_\infty < \infty \text{ and } t \geq \eta_\infty. \end{cases}$$

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Actually, the logistic branching process satisfies the following SDE

$$Y_t = Y_0 + b\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - c\int_0^t Y_s^2 \mathrm{d}s,$$

where $B^{(b)}$ is a standard Brownian motion which is independent of the Poisson random measure $N^{(b)}$ which is defined on $\mathbb{R}^3_+$ with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ such that

$$\int_{(0,\infty)} (z \wedge z^2)\mu(\mathrm{d}z) < \infty, \tag{1.1}$$

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It is important to note that Ma (as well as Berestycki et al.) consider a more general competition mechanism g which is a non-decreasing continuous function on $[0,\infty)$ with $g(0)=0$.

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- If X is a subordinator and assume that either it has a drift δ, it is not a Compound Poisson or it is a Compound Poisson with mean strictly bigger than c. Then Y is positive recurrent in $(\delta/c, \infty)$ and converge in distribution to a size biased distribution of

$$\int_{(\delta/c,\infty)} \nu(\mathrm{d}r) e^{-\lambda r} = \exp\left\{ \int_0^{\lambda} \frac{\psi(s)}{cs} \mathrm{d}s \right\}$$

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- If X is a subordinator that doesn't fulfil the above conditions, then Y is null-recurrent in $(0, \infty)$ and converges to 0 in probability.

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If X is a not a subordinator then Y goes to 0, a.s. and it gets extinct at finite time a.s. if and only if

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The process Y comes down from ∞ and its entrance law can be computed explicitly.

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Such family of processes have been introduced recently by Palau and P. (see also He et al.)as the unique strong solution of the following SDE

$$
\begin{aligned}
Z_t = Z_0 + b\int_0^t Z_s \mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t Z_{s-}\mathrm{d}S_s \\
+ \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),
\end{aligned}
$$

where g is a non-decreasing continuous function on $[0,\infty)$ with $g(0) = 0$, $B^{(b)}$ and $N^{(b)}$ are defined as before but with the difference that the measure μ satisfies the following condition

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty,$$

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and S is a Lévy process independent of $B^{(b)}$ and $N^{(b)}$ which can be written as follows

$$S_t^{(e)} = \gamma t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} (e^z - 1) N^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)} (e^z - 1) \tilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z),$$

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with $\gamma \in \mathbb{R}, \sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}$ is a Poisson random measure taking values on $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ and with intensity $\mathrm{d}s\pi(\mathrm{d}z)$ satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

Some motivation (I hope !)

We consider that the branching and competition mechanisms are as follows

$$g(x) = kx^2 \qquad \text{and} \qquad \psi(\lambda) = b\lambda \qquad \text{for } x, \lambda \geq 0,$$

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In particular, it can be rewritten as follows

$$Z_t = \frac{Z_0 e^{K_t}}{1 + kZ_0 \displaystyle\int_0^t e^{K_s}\mathrm{d}s}, \qquad t \geq 0,$$

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ii) *If the process K oscillates, then $\liminf_{t\to\infty} Z_t = 0$ a.s.*

iii) *If the process K drifts to ∞, then Z has a stationary distribution whose density can be written in terms of the density of $I_\infty(-K) = \int_0^\infty e^{-K_s}\mathrm{d}s$.*
Moreover if K has finite mean, for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(Z_s)\mathrm{d}s = \mathbb{E}_x\left[f\left(\frac{1}{kI_\infty(-K)}\right)\right], \qquad a.s.$$

Time to extinction

We define the first hitting time to 0 of Z as follows

$$T_0^Z = \inf\{t \geq 0, Z_t = 0\},$$

with the convention that $\inf\{\emptyset\} = +\infty$. We denote by $\mathbb{P}_x$ for the law of Z starting from $x > 0$.

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In the sequel, we assume

$$\int_{(0,\infty)} (z \wedge z^2)\mu(\mathrm{d}z) + \int_{(-1,+\infty)} |z|\pi(\mathrm{d}z) < +\infty.$$

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We also assume tha that the branching mechanism ψ satisfies the so-called Grey's condition, i.e.

$$\int^{\infty} \frac{\mathrm{d}\lambda}{\psi(\lambda)} < \infty.$$

Important : Grey's condition is a necessary and sufficient condition for CB processes in random environment to be extinct (see He et al. (2016)).

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H1) There exists $\theta \geq 0$ such that for all $x, y \geq 0$,

$$g(z) - g(z+y) \leq (\theta - b)y.$$

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H1) There exists $\theta \geq 0$ such that for all $x, y \geq 0$,

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H2) There exists $a_0 > 0$ such that $by < g(y)$ for any $y \geq a_0$ and

$$\int_{a_0}^{+\infty} \frac{\mathrm{d}y}{g(y) - by} < +\infty.$$

Theorem

Assume that Grey's condition, **(H1)** *and* **(H2)** *hold, then*

$$\sup_{x\geq 0} \mathbb{E}_x\left[T_0^Z\right] < +\infty.$$

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A natural question : Does the process Z comes down from infinity ?

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Proposition

Let $W = (W_t, t \geq 0)$ be a standard Brownian motion independent of X and assume that g is a continuous function and non-decreasing on $[0, \infty)$ with $g(0) = 0$. For each $x > 0$, there is a unique strong solution to

$$\mathrm{d}R_t = \mathbf{1}_{\{R_{r-}>0: r\leq t\}}\mathrm{d}X_t - \mathbf{1}_{\{R_{r-}>0: r\leq t\}}\frac{g(R_t)}{R_t}\mathrm{d}t + \mathbf{1}_{\{R_{r-}>0: r\leq t\}}\sigma\sqrt{R_t}\mathrm{d}W_t.$$

Theorem

Let $R = (R_t, t \geq 0)$ *be as before and* $T_0^R = \sup\{s : R_s > 0\}$*. We also let* C *be the right-continuous inverse of* η*, where*

$$\eta_t = \int_0^{t \wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad t > 0.$$

Hence the process defined by

$$Z_t = \begin{cases} R_{C_t}, & \text{if } 0 \leq t < \eta_\infty \\ 0, & \text{if } \eta_\infty < \infty \text{ and } t \geq \eta_\infty, \end{cases}$$

is a CSBP with competition in a Brownian random environment.

Theorem

Reciprocally, let Z CSBP with competition in a Brownian random environment with $Z_0 = x$ and let

$$C_t = \int_0^{t \wedge T_0^Z} Z_s \mathrm{d}s, \qquad t > 0.$$

If η denotes the right-continuous inverse of C, then the process defined by

$$R_t = \begin{cases} Z_{\eta_t}, & \textit{if } 0 \leq t < C_\infty \\ 0, & \textit{if } C_\infty < \infty \textit{ and } t \geq C_\infty, \end{cases}$$

satisfies the SDE from the previous definition.

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Bounded variation case

If X has bounded variation paths, then the process R satisfies

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In other words, R is a CBI process with branching mechanism

$$\psi_R(\theta) = \theta\left(c + \frac{\sigma^2\theta}{2}\right) \quad \text{and} \quad \phi_R(\theta) = b\theta + \int_{(0,\infty)} (1 - e^{\theta x})\mu(\mathrm{d}x),$$

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The process R is subcritical and according to Foucart & Uribe the only point that may be polar is 0. Actually 0 is polar or hit with positive probability accordingly as

$$\int_\theta^\infty \frac{\mathrm{d}z}{z\left(c + \frac{\sigma^2}{2}z\right)} \exp\left\{\int_\theta^z \frac{\phi_R(u)}{u\left(c + \frac{\sigma^2}{2}u\right)}\mathrm{d}u\right\} = \infty \quad \text{or} \quad < \infty,$$

for $\theta > 0$.

According to Duhalde et al., the CBI R is recurrent or transient provided that

$$\int_0^1 \frac{\mathrm{d}z}{z\left(c+\frac{\sigma^2}{2}z\right)} \exp\left\{-\int_z^1 \frac{\phi_R(u)}{u\left(c+\frac{\sigma^2}{2}u\right)}\mathrm{d}u\right\} = \infty \quad \text{or} \quad < \infty$$

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Moreover, if

$$\int_{(1,\infty)} \log(u)\mu(\mathrm{d}u) < \infty,$$

then R possesses an invariant probability distribution.

Let $\tau_a = \inf\{t \geq 0 : R_t \leq a\}$, for every $x \geq a$ and $\lambda \geq 0$ we have

$$\mathbf{E}_x\left[e^{-\lambda\tau_a}\right] = \frac{\int_0^\infty \frac{\mathrm{d}z}{z\left(c+\frac{\sigma^2}{2}z\right)} \exp\left\{-xz + \int_\theta^z \frac{\phi_R(u)+\lambda}{u\left(c+\frac{\sigma^2}{2}u\right)}\mathrm{d}u\right\}}{\int_\theta^\infty \frac{\mathrm{d}z}{z\left(c+\frac{\sigma^2}{2}z\right)} \exp\left\{-az + \int_\theta^z \frac{\phi_R(u)}{u\left(c+\frac{\sigma^2}{2}u\right)}\mathrm{d}u\right\}},$$

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and for all $x > a \geq 0$,

$$\mathbb{E}_x[\tau_a] = \int_0^\infty \frac{\mathrm{d}z}{z\left(c+\frac{\sigma^2}{2}z\right)}(e^{-az}-e^{-xz})\exp\left\{-\int_0^z \frac{\phi(u)}{u\left(c+\frac{\sigma^2}{2}u\right)}\mathrm{d}u\right\}.$$

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- whether the process Z hits 0 or not
- $\mathbb{P}_x(T_0^Z < \infty)$
- whether the process Z is transient or recurrent
- existence of an invariant distribution (under the log moment condition)
- the Laplace exponent of the total population

$$\int_0^{T_a^Z} Z_s \mathrm{d}s,$$

where $T_a^Z = \inf\{s : Z_s \leq a\}$

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Here we want to deduce an "explicit" expression for the law of T_0^Z, the first step to reach this result is to find an explicit formulation of the function

$$G_{q,x}(\lambda) = \int_0^{+\infty} e^{-qt}\mathbb{E}_x[e^{-\lambda Z_t}]dt.$$

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Indeed, we have that $\lim_{\lambda\to+\infty} qG_{q,x}(\lambda) = \mathbb{E}_x[e^{-qT_0}]$.

Observe that the infinitesimal generator of Z satisfies, for any $f \in C_b^2(\mathbb{R}_+)$,

$$\mathcal{U}f(z) = z\mathcal{A}f(z) - cz^2 f'(z) + \frac{\sigma^2}{2} z^2 f''(z),$$

where $\mathcal{A}$ is the generator of the Lévy process X.

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By applying to the generator $\mathcal{U}$ to $e^{-\lambda z}$ we deduce the following Lemma.

Observe that the infinitesimal generator of Z satisfies, for any $f \in C_b^2(\mathbb{R}_+)$,

$$\mathcal{U}f(z) = z\mathcal{A}f(z) - cz^2 f'(z) + \frac{\sigma^2}{2} z^2 f''(z),$$

where $\mathcal{A}$ is the generator of the Lévy process X.

By applying to the generator $\mathcal{U}$ to $e^{-\lambda z}$ we deduce the following Lemma.

Lemma

For any $q \geq 0$, $x \geq 0$, $G_{q,x}$ is a positive solution to the differential equation

$$\omega(\lambda)y''(\lambda) - \varphi(\lambda)y'(\lambda) - qy(\lambda) = e^{-\lambda x},$$

where $\omega(\lambda) = c\lambda\left(1 + \frac{\sigma^2}{2c}\lambda\right)$.

Observe that $\omega(\lambda)$ is positive on $(0, +\infty)$. Hence, a classical application of Cauchy-Lipschitz's Theorem implies that any solution to the previous ODE is well defined on $(0, +\infty)$, and the space of solutions has dimension 2.

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$$m(\lambda) = \int_0^\lambda \frac{\psi(l)}{\omega(l)}dl \quad \text{and} \quad \theta(\lambda) = \int_0^\lambda e^{m(l)}dl.$$

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Note that θ is a positive increasing function with values in $(0, +\infty)$ and also that m is non-decreasing on $(\lambda_0, +\infty)$ and $\theta(\lambda)$ converges to $+\infty$ when λ tends to $+\infty$.

We denote the inverse function of θ by φ. A simple computation gives

$$\varphi'(\lambda) = \exp(-m \circ \varphi(\lambda)).$$

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$$h'(\lambda) = h^2(\lambda) - q\frac{\varphi'(\lambda)^2}{\omega(\varphi(\lambda))}.$$

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Lemma

For any $q > 0$, there exists a unique non-negative solution h_q to the equation

$$h' = h^2 - qr^2,$$

where $r(\lambda) = \frac{\varphi'(\lambda)}{\sqrt{\omega(\varphi(\lambda))}}$ such that it vanishes at $+\infty$. Moreover, h_q is positive on $(0, \infty)$, and for any λ sufficiently small or large, $h_q(\lambda) < \sqrt{q}r(\lambda)$. As a consequence, h_q is integrable at 0, and it decreases initially and ultimately.

Proposition

The extinction time T_0^Z of Z satisfies

$$\mathbb{E}_x[e^{-qT_0^Z}]$$
$$= 1 - \int_0^{+\infty} \int_l^{+\infty} qr(s)^2(1 - e^{-x\varphi(s)})e^{-\int_l^s h_q(u)\mathrm{d}u} e^{-\int_l^{+\infty} h_q(u)\mathrm{d}u} dsdl,$$

and

$$\mathbb{E}_x[T_0^Z] = \int_0^{+\infty} lr^2(l)(1 - e^{-x\varphi(l)})dl.$$