Asymptotic properties of maximum likelihood estimator for the growth rate of an α-stable CIR process

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Outline

- α-stable CIR processes
  - asymptotics behaviour of the expectation
  - classification
    - subcritical, critical, supercritical
  - log-likelihood function
  - maximum likelihood estimator of the growth rate
- Asymptotic behaviour of the maximum likelihood estimator
  - subcritical case
  - supercritical case
  - critical case

α-stable Cox–Ingersoll–Ross (CIR) process

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \delta\sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, & t \geqslant 0, \\ Y_0 = y_0, \end{cases}$$

where $y_0 \geqslant 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$, $\delta > 0$, $\alpha \in (1, 2)$, $(W_t)_{t\geqslant 0}$ is a 1-dimensional standard Wiener process, and $(L_t)_{t\geqslant 0}$ is an independent spectrally positive strictly α-stable Lévy process with characteristic function

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta L_1}) = \exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta z} - 1 - \mathrm{i}\theta z)\frac{1}{\alpha\Gamma(-\alpha)z^{1+\alpha}}\,\mathrm{d}z\right\}, \qquad \theta \in \mathbb{R}.$$

1. There is a pathwise unique strong solution $(Y_t)_{t\geqslant 0}$ such that $\mathbb{P}(Y_0 = y_0) = 1$ and $\mathbb{P}(Y_t \geqslant 0 \text{ for all } t \geqslant 0) = 1$.
2. $(Y_t)_{t\geqslant 0}$ is a continuous state and continuous time branching process with immigration (CBI) process having branching and immigration mechanisms

$$R(z) = \frac{\sigma^2}{2}z^2 + \frac{\delta^\alpha}{\alpha}z^\alpha + bz, \qquad F(z) = az, \qquad z \geqslant 0.$$

Preliminaries

- PETER CARR and LIUREN WU (2004):
  Time-changed Lévy processes and option pricing. ($\sigma = 0$)
- ZONGFEI FU and ZENGHU LI (2010):
  Stochastic equations of non-negative processes with jumps.
  Existence and uniqueness of strong solutions.
- ZENGHU LI and CHUNHUA MA (2015): *Asymptotic properties of estimators in a stable Cox–Ingersoll–Ross model*.
  Conditional least squares estimators and the weighted conditional least squares estimators of the drift parameters based on low frequency observations $\{Y_0, Y_1, \dots, Y_n\}$ when $\sigma = 0$ and $b > 0$.
- YING JIAO, CHUNHUA MA and SIMONE SCOTTI (2017+):
  α-CIR model with branching processes in sovereign interest rate modelling.
- JUN PENG (2016): *A stable Cox–Ingersoll–Ross model with restart*.
- XU YANG (2017): *Maximum likelihood type estimation for discretely observed CIR model with small α-stable noises*.

Expectation

$$\mathbb{E}(Y_t) = y_0 \mathrm{e}^{-bt} + a \int_0^t \mathrm{e}^{-bx} = \begin{cases} y_0 \mathrm{e}^{-bt} + a\frac{1-\mathrm{e}^{-bt}}{b} & \text{if } b \neq 0, \\ y_0 + at & \text{if } b = 0. \end{cases}$$

Asymptotics of the expectation

$$\mathbb{E}(Y_t) = \begin{cases} \frac{a}{b} + \mathrm{O}(\mathrm{e}^{-bt}) & \text{if } b > 0, \\ at + \mathrm{O}(1) & \text{if } b = 0, \\ \left(y_0 - \frac{a}{b}\right) \mathrm{e}^{-bt} + \mathrm{O}(1) & \text{if } b < 0, \end{cases} \qquad \text{as } t \to \infty.$$

$$b = \text{growth rate}$$

Classification

$$\begin{cases} \text{subcritical} & \text{if } b > 0, \\ \text{critical} & \text{if } b = 0, \\ \text{supercritical} & \text{if } b < 0. \end{cases}$$

Stationarity in the subcritical and critical cases $b \geqslant 0$ (PINSKY 1972; KELLER-RESSEL and STEINER 2008; LI 2011)

$Y_t \xrightarrow{\mathcal{D}} \pi$ as $t \to \infty$, where π is the unique stationary distribution of $(Y_t)_{t \geqslant 0}$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{uy}\, \pi(\mathrm{d}y) = \exp\left\{ \int_u^0 \frac{F(x)}{R(x)}\, \mathrm{d}x \right\} = \exp\left\{ \int_u^0 \frac{ax}{\frac{\sigma^2}{2}x^2 + \frac{\delta\alpha}{\alpha}x^\alpha + bx}\, \mathrm{d}x \right\}$$

for $u < 0$. Moreover, the expectation of π is given by

$$\int_0^\infty y\, \pi(\mathrm{d}y) = \begin{cases} \frac{a}{b} & \text{in the subcritical case } b > 0, \\ \infty & \text{in the critical case } b = 0. \end{cases}$$

Ergodicity in the subcritical case $b > 0$ (LI and MA, 2015)

The process $(Y_t)_{t \in \mathbb{R}_+}$ is exponentially ergodic, and for all Borel measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with $\int_0^\infty |f(y)|\, \pi(\mathrm{d}y) < \infty$, we have

$$\frac{1}{T} \int_0^T f(Y_s)\, \mathrm{d}s \xrightarrow{\text{a.s.}} \int_0^\infty f(y)\, \pi(\mathrm{d}y) \qquad \text{as } T \to \infty.$$

$y_0, \ a, \ \sigma, \ \delta, \ \alpha$ are known, $b \in \mathbb{R}$ unknown parameter
$\mathbb{P}_b :=$ probability measure induced by $(Y_t)_{t\in\mathbb{R}_+}$ on $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$
For all $T \in (0, \infty)$, put $\mathbb{P}_{b,T} := \mathbb{P}_b|_{\mathcal{D}_T(\mathbb{R}_+,\mathbb{R})}$

Log-likelihood function

$$\log\left(\frac{\mathrm{d}\,\mathbb{P}_{b,T}}{\mathrm{d}\,\mathbb{P}_{\widetilde{b},T}}(\widetilde{Y})\right) = -\frac{b-\widetilde{b}}{\sigma^2}\left(\widetilde{Y}_T - y_0 - aT - \delta\int_0^T \sqrt[\alpha]{\widetilde{Y}_{u-}}\,\mathrm{d}L_u\right) - \frac{b^2-\widetilde{b}^2}{2\sigma^2}\int_0^T \widetilde{Y}_u\,\mathrm{d}u =: \Lambda_T(b, \widetilde{b}\,|\,\widetilde{Y})$$

Maximum likelihood estimator (MLE) of the growth rate

$$\widehat{b}_T := \arg\max_{b\in\mathbb{R}} \Lambda_T(b, \widetilde{b}\,|\,Y)$$

MLE of the growth rate

$$\widehat{b}_T = -\frac{Y_T - y_0 - aT - \delta\int_0^T \sqrt[\alpha]{Y_{u-}}\mathrm{d}L_u}{\int_0^T Y_s\,\mathrm{d}s}$$

Asymptotic behaviour of the MLE in the subcritical case $b > 0$

The MLE $\widehat{b}_T$ of b is strongly consistent and asymptotically normal, i.e., $\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \to \infty$, and

$$\sqrt{T}(\widehat{b}_T - b) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\sigma^2 b}{a}\right) \qquad \text{as } T \to \infty.$$

With a random scaling,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, \mathrm{d}s\right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as } T \to \infty.$$

$$\begin{aligned}
\widehat{b}_T - b &= -\frac{Y_T - y_0 - aT - \delta \int_0^T \sqrt[\alpha]{Y_{u-}} \, \mathrm{d}L_u + b \int_0^T Y_s \, \mathrm{d}s}{\int_0^T Y_s \, \mathrm{d}s} \\
&= -\sigma \frac{\int_0^T \sqrt{Y_s} \, \mathrm{d}W_s}{\int_0^T Y_s \, \mathrm{d}s} \xrightarrow{\text{a.s.}} 0 \qquad \text{as } T \to \infty,
\end{aligned}$$

by the strong law of large numbers for continuous local martingales, since $\int_0^T Y_s \, \mathrm{d}s \xrightarrow{\text{a.s.}} \infty$ as $T \to \infty$, which follows from the ergodicity $\frac{1}{T} \int_0^T Y_s \, \mathrm{d}s \xrightarrow{\text{a.s.}} \int_0^\infty y \, \pi(\mathrm{d}y) = \frac{a}{b} > 0$ as $T \to \infty$.

$$\sqrt{T}(\widehat{b}_T - b) = -\sigma \frac{\frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s}\, \mathrm{d}W_s}{\frac{1}{T} \int_0^T Y_s\, \mathrm{d}s} \xrightarrow{\mathcal{D}} -\sigma \frac{\left(\int_0^\infty y\, \pi(\mathrm{d}y)\right)^{1/2} \mathcal{N}(0,1)}{\int_0^\infty y\, \pi(\mathrm{d}y)}$$

$$= \mathcal{N}\left(0, \frac{\sigma^2}{\int_0^\infty y\, \pi(\mathrm{d}y)}\right) \quad \text{as} \quad T \to \infty$$

by the central limit theorem for continuous local martingales and Slutsky's lemma.

$$\frac{1}{\sigma} \left(\int_0^T Y_s\, \mathrm{d}s \right)^{1/2} (\widehat{b}_T - b) = \frac{1}{\sigma} \left(\frac{1}{T} \int_0^T Y_s\, \mathrm{d}s \right)^{1/2} \sqrt{T}(\widehat{b}_T - b)$$

$$\xrightarrow{\mathcal{D}} \frac{1}{\sigma} \left(\int_0^\infty y\, \pi(\mathrm{d}y) \right)^{1/2} \mathcal{N}\left(0, \frac{\sigma^2}{\int_0^\infty y\, \pi(\mathrm{d}y)}\right) = \mathcal{N}(0,1) \quad \text{as} \quad T \to \infty$$

again by Slutsky's lemma.

Asymptotic behaviour of the MLE in the supercritical case $b < 0$

The MLE $\widehat{b}_T$ of b is strongly consistent and asymptotically mixed normal, i.e., $\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \to \infty$, and

$$\mathrm{e}^{-bT/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{D}} \sigma \left(\frac{-b}{V}\right)^{1/2} Z \qquad \text{as } T \to \infty,$$

where $V := \lim_{t\to\infty} \mathrm{e}^{bt} Y_t$, and Z is a standard normally distributed random variable, independent of V.
With a random scaling,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \,\mathrm{d}s\right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as } T \to \infty.$$

For all $s, t \in \mathbb{R}_+$ with $0 \leqslant s \leqslant t$,

$$\mathbb{E}(Y_t \mid \mathcal{F}_s^Y) = \mathbb{E}(Y_t \mid Y_s) = \mathrm{e}^{-b(t-s)} Y_s + a \int_0^{t-s} \mathrm{e}^{-bx} \,\mathrm{d}x$$

with $\mathcal{F}_s^Y := \sigma(Y_u, u \in [0,t])$, $t \in \mathbb{R}_+$, hence

$$\mathbb{E}(\mathrm{e}^{bt} Y_t \mid \mathcal{F}_s^Y) = \mathrm{e}^{bs} Y_s + a\mathrm{e}^{bt} \int_0^{t-s} \mathrm{e}^{-bx} \,\mathrm{d}x \geqslant \mathrm{e}^{bs} Y_s,$$

thus the process $(\mathrm{e}^{bt} Y_t)_{t\in\mathbb{R}_+}$ is a non-negative submartingale.

The submartingale convergence theorem applies, since for all $t \geqslant 0$,

$$\mathbb{E}(\mathrm{e}^{bt} Y_t) = y_0 + a\mathrm{e}^{bt} \int_0^t \mathrm{e}^{-bx}\,\mathrm{d}x = y_0 + a \int_0^t \mathrm{e}^{bx}\,\mathrm{d}x \leqslant y_0 + a \int_0^\infty \mathrm{e}^{bx}\,\mathrm{d}x < \infty.$$

Hence there exists a non-negative random variable V such that

$$\mathrm{e}^{bt} Y_t \xrightarrow{\text{a.s.}} V \qquad \text{as } t \to \infty.$$

The integral Toeplitz lemma yields

$$\mathrm{e}^{bT} \int_0^T Y_s\,\mathrm{d}s \xrightarrow{\text{a.s.}} \frac{V}{-b} \qquad \text{as } t \to \infty.$$

One can show $\mathbb{P}(V > 0) = 1$, thus we obtain $\int_0^T Y_s\,\mathrm{d}s \xrightarrow{\text{a.s.}} \infty$ as $T \to \infty$, and the strong law of large numbers for continuous local martingales implies

$$\widehat{b}_T - b = -\sigma \frac{\int_0^T \sqrt{Y_s}\,\mathrm{d}W_s}{\int_0^T Y_s\,\mathrm{d}s} \xrightarrow{\text{a.s.}} 0 \qquad \text{as } T \to \infty.$$

$$\mathrm{e}^{-bT/2}(\widehat{b}_T - b) = -\sigma \frac{\mathrm{e}^{bT/2} \int_0^T \sqrt{Y_s}\,\mathrm{d}W_s}{\mathrm{e}^{bT} \int_0^T Y_s\,\mathrm{d}s}$$

$$\xrightarrow{\mathcal{D}} -\sigma \frac{\left(-\frac{V}{b}\right)^{1/2} Z}{\frac{V}{-b}} = -\sigma \left(\frac{-b}{V}\right)^{1/2} Z$$

as $T \to \infty$ by the stable limit theorem and Slutsky's lemma.

$$\frac{1}{\sigma}\left(\int_0^T Y_s\,\mathrm{d}s\right)^{1/2}(\widehat{b}_T - b) = \frac{1}{\sigma}\left(\mathrm{e}^{bT}\int_0^T Y_s\,\mathrm{d}s\right)^{1/2}\mathrm{e}^{-bT/2}(\widehat{b}_T - b)$$

$$\xrightarrow{\mathcal{D}} \frac{1}{\sigma}\left(\frac{V}{-b}\right)^{1/2}\sigma\left(\frac{-b}{V}\right)^{1/2} Z \stackrel{\mathcal{D}}{=} \mathcal{N}(0,1) \qquad \text{as } T \to \infty$$

again by Slutsky's lemma.

We have $\mathrm{e}^{bt} Y_t \xrightarrow{\mathcal{D}} V$ as $t \to \infty$, hence

$$\mathbb{E}(u\mathrm{e}^{bt} Y_t) \to \mathbb{E}(\mathrm{e}^{uV}) \qquad \text{as } t \to \infty \text{ for all } u \leqslant 0.$$

By an idea of Clément Foucart (based on considerations of Zenghu Li for CB processes without immigration),

$$\mathbb{E}(\mathrm{e}^{uV}) = \exp\left\{ -y_0 f\Big(\frac{-u}{K(\lambda)}, \lambda\Big) + \int_0^{f\left(\frac{-u}{K(\lambda)}, \lambda\right)} \frac{F(z)}{R(z)} \, \mathrm{d}z \right\}, \qquad u \leqslant 0,$$

where $\lambda \in (0, \theta_0)$ with $\theta_0 := \inf\{z > 0 : R(z) \geqslant 0\} > 0$,

$$K(\lambda) := \lambda \exp\left\{ \int_0^\lambda \Big(\frac{b}{R(z)} - \frac{1}{z} \Big) \mathrm{d}z \right\},$$

and $f\left(\frac{-u}{K(\lambda)}, \lambda\right) \in [0, \theta_0)$ is such that $f(0, \lambda) := 0$ and

$$\int_\lambda^{f\left(\frac{-u}{K(\lambda)}, \lambda\right)} \frac{1}{R(z)} \, \mathrm{d}z = \frac{1}{b} \ln\left(\frac{-u}{K(\lambda)} \right), \qquad u \leqslant 0.$$

Asymptotic behaviour of the MLE in the critical case $b = 0$

The MLE $\widehat{b}_T$ of b is strongly consistent, i.e., $\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \to \infty$.

Since $\left(\int_0^t Y_s \, \mathrm{d}s\right)_{t \in \geqslant 0}$ is monotone increasing almost surely, there exists an $[0, \infty]$-valued random variable ξ such that $\int_0^t Y_s \, \mathrm{d}s \xrightarrow{\text{a.s.}} \xi$ as $t \to \infty$. Consequently, $\int_0^t Y_s \, \mathrm{d}s \xrightarrow{\mathcal{D}} \xi$ as $t \to \infty$. Using an explicit formula for the Laplace transform

$$\mathbb{E}\left[\exp\left\{v \int_0^t Y_s \, \mathrm{d}s\right\}\right], \qquad v \leqslant 0,$$

one can show that $\mathbb{P}(\xi = \infty) = 1$, thus we obtain $\int_0^T Y_s \, \mathrm{d}s \xrightarrow{\text{a.s.}} \infty$ as $T \to \infty$, and the strong law of large numbers for continuous local martingales implies

$$\widehat{b}_T - b = -\sigma \frac{\int_0^T \sqrt{Y_s} \, \mathrm{d}W_s}{\int_0^T Y_s \, \mathrm{d}s} \xrightarrow{\text{a.s.}} 0 \qquad \text{as } T \to \infty.$$

Joint Laplace transform of Y_t and $\int_0^t Y_s \, \mathrm{d}s$

For all $u, v \leqslant 0$,

$$\mathbb{E}\left[\exp\left\{uY_t + v\int_0^t Y_s \, \mathrm{d}s\right\}\right] = \exp\left\{\psi_{u,v}(t)y_0 + a\int_0^t \psi_{u,v}(s) \, \mathrm{d}s\right\}$$

for $t \geqslant 0$, where $\psi_{u,v} : [0, \infty) \to (-\infty, 0]$ is the unique locally bounded solution to the Riccati-type DE

$$\psi'_{u,v}(t) = \frac{\sigma^2}{2}\psi_{u,v}(t)^2 + \frac{\delta^\alpha}{\alpha}(-\psi_{u,v}(t))^\alpha - b\psi_{u,v}(t) + v, \qquad t \geqslant 0,$$

with initial condition $\psi_{u,v}(0) = u$.

An open problem in the critical case $(b = 0)$:

Find the limit behaviour of $\frac{1}{\beta(t)} \int_0^t Y_s \, \mathrm{d}s$ as $t \to \infty$ with some appropriate scaling function $\beta : [0, \infty) \to (0, \infty)$.

In the language of DEs, find a function β satisfying

- $\lim_{t\to\infty} \beta(t) = \infty$,
- $\lim_{t\to\infty} \psi_{u, \frac{v}{\beta(t)}}(t) = 0$ for $u, v \leqslant 0$ (not to have depence on y_0 in the limit)
- for all $u, v \leqslant 0$, the limit

$$\lim_{t\to\infty} \int_0^t \psi_{u, \frac{v}{\beta(t)}}(s) \, \mathrm{d}s = \lim_{t\to\infty} \int_u^{\psi_{u, \frac{v}{\beta(t)}}(t)} \frac{x}{\frac{\sigma^2}{2} x^2 + \frac{\delta^\alpha}{\alpha}(-x)^\alpha + \frac{v}{\beta(t)}} \, \mathrm{d}x$$

exists and explicitly given, where, for all $t \geqslant 0$, the function $\psi_{u, \frac{v}{\beta(t)}} : [0, \infty) \to (-\infty, 0]$ is the solution to the DE

$$\psi'_{u, \frac{v}{\beta(t)}}(s) = \frac{\sigma^2}{2} \psi_{u, \frac{v}{\beta(t)}}(s)^2 + \frac{\delta^\alpha}{\alpha}(-\psi_{u, \frac{v}{\beta(t)}}(s))^\alpha + \frac{v}{\beta(t)}, \qquad s \geqslant 0,$$

with initial condition $\psi_{u, \frac{v}{\beta(t)}}(0) = u$.

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