On the zero set of super-Brownian motion

Leonid Mytnik (Technion)
Joint work with C. Mueller and E. Perkins

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Stoshastic heat equation

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t),$$

where $\dot{W}$ is the Gaussian space-time white noise with

$$E\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(t-s)\delta(x-y).$$

$$\begin{aligned} X(t,x) &= \int p_t(x-y)X(0,y)dy \\ &\quad + \int_0^t \int p_{t-s}(x-y)\sigma(X(s,y))\, W(dy,ds). \end{aligned}$$

Uniqueness

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t).$$

- **Pathwise uniqueness (PU):**
  X^1, X^2 — two solutions, $X^1(0,\cdot) = X^2(0,\cdot)$
  $\Longrightarrow X^1(t,\cdot) = X^2(t,\cdot)\,, \forall t > 0.$

- **Uniqueness in law (weak):**
  X^1, X^2 — two solutions (even on different spaces),
  $X^1(0,\cdot) = X^2(0,\cdot) \Longrightarrow \{X^1(t,\cdot)\}_{t\geq 0} \stackrel{law}{=} \{X^2(t,\cdot)\}_{t\geq 0}\,.$

Uniqueness

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t).$$

If $\dot{W}$ is a space-time white noise, then function-valued solution exists if $d = 1$.
Uniqueness?
σ — Lipschitz $\Longrightarrow$ PU follows easily.
σ - non-Lipschitz ?

Super-Brownian motion

Branching Brownian motions in $\mathbf{R}^d$.
X^n:
$\sim n$ particles in $\mathbf{R}^d$ at time 0.
$\frac{1}{n}, \frac{2}{n}, \ldots$ — times of death or split,
$p_0 = p_2 = \frac{1}{2}$ — probabilities of death or split.
Critical branching: mean number of offspring $= 1$.
New particles move as independent Brownian motions.

$$X_t^n(A) = \frac{\#\text{ particles in } A \text{ at time } t}{n}, \quad A \subset \mathbf{R}^d.$$

$$X^n \Rightarrow X,$$

X is a super-Brownian motion — measure-valued process.

Characterization

Laplace transform:

$$E\left[e^{-\langle X_t,\phi\rangle}\right] = e^{-\langle X_0,u_t\rangle}, \quad \phi \geq 0.$$

where

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t - \frac{1}{2}u_t^2, \quad u_0 = \phi.$$

X is continuous (in time) measure-valued process.

Regularity properties?

Properties of SBM

- Singular measure if $d > 1$.

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- From now on

$$d = 1.$$

More properties

- $$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

  $X_t(x)$ is jointly continuous in (t, x).
  Hölder $1/2-$ in space, Hölder $1/4-$ in time.

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- Define:

$$\begin{aligned} BZ_t &\equiv \partial(\{x : X(t,x) = 0\}) \\ &= \{x : X(t,x) = 0, \forall\delta > 0\, X_t((x-\delta, x+\delta)) > 0\}. \end{aligned}$$

— the boundary of the zero set of X_t

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  — the boundary of the zero set of X_t
- Question:

  Properties of BZ_t?

Hausdorff dimension

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

In particular we are interested in Hausdorff dimension of BZ_t:

$$\dim(BZ_t) = ?$$

Motivation: Pathwise Uniqueness for SBM?

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

Weak uniqueness holds (by duality method)

Pathwise uniqueness (PU)?
$\sqrt{X}$ — non-Lipschitz.

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Is there a chance to get **PU**?

Pathwise uniqueness for SDEs

$$dX_t = \sigma(X_t)dB_t$$

B_t is a one-dimensional Brownian motion.

Theorem (Yamada, Watanabe (71))

If σ is Hölder continuous with exponent $1/2$, then PU holds.

Remark

There are counter examples for σ which is Hölder continuous with exponent less than $1/2$.

Back to SPDEs

Theorem (Perkins, M., 11)

Let $\sigma(x)$ be Hölder continuous with exponent γ.
*For any $\gamma > 3/4$, **PU** holds for*

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where $\dot{W}$ is the space-time white noise.

Remark 1

*Recently Yang, Zhou (2017) studied **PU** for SPDEs with Hölder coefficients driven by stable noise.*

Non-uniqueness

- Is $3/4$ sharp? Counter example: for $\gamma < 3/4$ try to construct non-triviual solution to

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial t}X(t,x) & = & \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma}\dot{W}(x,t), \\ X(0,\cdot) & = & 0. \end{array} \right. \tag{1}$$

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- Burdzy, Mueller, Perkins(2010); M., Mueller, Perkins(2012): If $0 < \gamma < 3/4$ there is solution $X(t,x)$ to (1) such that with positive probability, $X(t,x)$ is not identically zero.

All this was about any solution to the SPDE.
What happens if we restrict consideration to the class of non-negative solutions?

Non-uniqueness for non-negative solutions

- Burdzy, Mueller, Perkins(2010): If $0 < \gamma < 1/2$, $\psi \geq 0$, non-trivial, then **PU** fails for **non-negative** solutions to

$$\frac{\partial}{\partial t} X(t,x) = \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma} \dot{W}(x,t) + \psi, \quad (2)$$

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- Chen (2015): If $\psi \geq 0$, non-trivial, then **PU** fails for **non-negative** solutions to

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{1/2}\dot{W}(x,t) + \psi. \quad (3)$$

  This is super-Brownian motion with immigration ψ for which **weak uniqueness** holds!
  Presence of ψ is very important: whenever $\psi > 0$, boundary of the zero set of any solution has positive Lebesgue measure
  Heuristically, it is "easier" for two solutions to separate if the boundary of the "zero set" is "large".

Non-uniqueness for non-negative solutions

$\psi = 0$?

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The question whether **PU** holds for **non-negative** solutions to

$$\frac{\partial}{\partial t}X(t,x) \quad = \quad \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma}\dot{W}(x,t), \qquad (4)$$

for $\gamma < 3/4$ is still open.

As we mentioned the presence of ψ is very important: whenever $\psi > 0$, BZ_t of any solution has positive Lebesgue measure.

However if the set BZ_t is "small" then one may expect that PU holds also for some $\gamma < 3/4$.

This motivated our interest in the Hausdorff dimension of the boundary of the zero set of $X(t,\cdot)$ that solves (4).
At this point we can do it only in $\gamma = 1/2$ case: SBM without immigration.

Hausdorff dimension of BZ_t for SBM without immigration

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

Main question: Hausdorff dimension of BZ_t:

$$\dim(BZ_t) = ?$$

Main Theorems

Theorem 2

There exists $\eta \in (0,1)$, *such that,* $\forall t > 0, x \in R$

$$P(0 < X(t,x) \le \epsilon) \sim \epsilon^{\eta}, \quad \text{as } \epsilon \downarrow 0.$$

Theorem 3

For all $t > 0$

$$\dim(BZ_t) \le 1 - \eta, \quad P - a.s.$$

and with positive probability,

$$\dim(BZ_t) \ge 1 - \eta, \quad \text{on } \{X_t(R) > 0\},$$

where η *is from Theorem 2.*

Throughout the proofs we will get the vaue of η.

Proofs

By a Tauberian theorem

$$P(0 < X(t,x) \le \epsilon) \sim \epsilon^{\eta}, \text{ as } \epsilon \downarrow 0,$$

iff

$$E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \sim \lambda^{-\eta}, \text{ as } \lambda \uparrow \infty,$$

That is we need to study the assymptotic behavior of

$$\begin{aligned} & E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \\ = \;& E(e^{-\lambda X(t,x)}) - P(X(t,x) = 0), \text{ as } \lambda \uparrow \infty, \end{aligned}$$

Main question: assymptotic behavior of

$$E(e^{-\lambda X(t,x)}) - P(X(t,x) = 0), \ \ \text{as } \lambda \uparrow \infty.$$

Let V^λ be solution of log-Laplace equation with initial condition $V_0 = \lambda\delta_0$. That is

$$\frac{\partial V_t^\lambda}{\partial t} = \frac{1}{2}\Delta V_t^\lambda - \frac{1}{2}(V_t^\lambda)^2, \ \ V_0^\lambda = \lambda\delta_0.$$

For simplicity, let $X_0 = \delta_0$. Then it is easy to check that

$$E_{\delta_0}(e^{-\lambda X(t,x)}) = e^{-V^\lambda(t,x)},$$

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$$\begin{aligned} E_{\delta_0}(e^{-\lambda X(t,x)}) &= e^{-V^\lambda(t,x)}, \\ P_{\delta_0}(X(t,x)=0) &= \lim_{\lambda\to\infty} E_{\delta_0}(e^{-\lambda X(t,x)}) \\ &= \lim_{\lambda\to\infty} e^{-V^\lambda(t,x)} \\ &=: e^{-V^\infty(t,x)}. \end{aligned}$$

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Thus

$$\begin{aligned} E_{\delta_0}(e^{-\lambda X(t,x)}) - P_{\delta_0}(X(t,x) = 0) &= e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} \\ &\sim V^\infty(t,x) - V^\lambda(t,x). \end{aligned}$$

The analysis of $P(0 < X(t,x) \le \epsilon) \sim \epsilon^{\eta}, \ \text{ as } \epsilon \downarrow 0$, boils down to to the analysis of behaviour of

$$V^{\infty}(t,x) - V^{\lambda}(t,x), \ \text{ as } \lambda \uparrow \infty.$$

Another simple reduction shows that in fact

$$V^{\infty}(t,x) - V^{\lambda}(t,x) \quad \sim \quad \lambda \frac{\partial V^{\lambda}(t,x)}{\partial \lambda} =: \lambda U^{\lambda}(t,x),$$

where U^{λ} solves the following equation

$$\frac{\partial U_t^{\lambda}}{\partial t} \quad = \quad \frac{1}{2} \Delta U_t^{\lambda} - V_t^{\lambda} U_t^{\lambda}, \ \ U_0^{\lambda} = \delta_0.$$

Therefore by Feynman-Kac and reversing the time we get

$$U^{\lambda}(t,x) \quad \approx \quad E_0\left(e^{-\int_0^t V^{\lambda}(s,W_s)ds}\right)$$

Analysis of behavior of U^λ

$$U^\lambda(t,x) \quad \sim \quad E_0(e^{-\int_0^t V^\lambda(s,W_s)ds})$$

Afer scaling and transformations

$$B_t \quad = \quad \lambda W_{\lambda^{-2}t}, \quad Y(t) = B(e^t - 1)e^{-t/2}$$

we get

$$\begin{aligned} U^\lambda(t,x) \quad &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{e^{s/2}}(1,Y_s)ds}) \\ &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{\infty}(1,Y_s)ds}) \end{aligned}$$

where Y is an Ornstein-Uhlenbeck process with generator

$$Lh(x) \quad = \quad \frac{1}{2}h''(x) - \frac{1}{2}xh'(x).$$

Analysis of behavior of U^λ

Y is an Ornstein-Uhlenbeck process with generator

$$Lh(x) = \frac{1}{2}h''(x) - \frac{1}{2}xh'(x).$$

Let

$$F(x) \equiv V^\infty(1,x).$$

$$U^\lambda(t,x) \sim E_0(e^{-\int_0^{\log(\lambda^2 t)} F(Y_s)ds}).$$

Then

$$\begin{aligned} U^\lambda(t,x) &\sim e^{-\nu_0(\log(\lambda^2 t))} \\ &= \lambda^{-2\nu_0} t^{-\nu_0}, \end{aligned}$$

where ν_0 is the smallest eigenvalue of

$$-L^F h \equiv -(Lh - Fh).$$

One can show: $1/2 < \nu_0 < 1$.

Finishing the proof of Theorem 2

$$U^{\lambda}(t,x) \quad \sim \quad \lambda^{-2\nu_0} t^{-\nu_0}, \ \text{ as } \lambda \uparrow \infty,$$

Recall that

$$E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \sim \lambda U^{\lambda}(t,x), \ \text{ as } \lambda \uparrow \infty.$$

Thus

$$E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \sim \lambda^{1-2\nu_0} t^{-\nu_0}, \quad \text{ as } \lambda \uparrow \infty,$$

and by the Tauberian theorem

$$P(0 < X(t,x) \leq \epsilon) \sim \epsilon^{\eta}, \ \text{ as } \epsilon \downarrow 0,$$

with

$$\eta = 2\nu_0 - 1.$$

$\dim(BZ_t)$

By Theorem 2

$$P(0 < X(t,x) \le \epsilon) \sim \epsilon^\eta, \quad \text{as } \epsilon \downarrow 0,$$
$$\text{with} \qquad \eta = 2\nu_0 - 1.$$

Theorem 2 is a corollary of Theorem 2, its proofs and known regularity of X on BZ_t and thus

$$\begin{aligned} \dim(BZ_t) &\le 1 - \eta \\ &= 2 - 2\nu_0, \quad \text{a.s.} \end{aligned}$$

and

$$\dim(BZ_t) \;=\; 2 - 2\nu_0, \quad \text{on } \{X_t(R) > 0\}$$

with positive probability. Note

$$\mathbf{0 < 2 - 2\nu_0 < 1}.$$

Numerics (Peiyuan, UBC):

$$\nu_0 \approx .8891 \quad \Rightarrow \dim(BZ_t) = 0.2218$$

Open Problems

- Proving sharp lower bound: P-a.s.

$$\dim(BZ_t) \;=\; 2-2\nu_0, \;\; \text{on } \{X_t(R)>0\}$$

- $$dim(BZ_t)?$$

  for $\gamma \neq 1/2$.
  Conjecture $\dim(BZ_t) \downarrow$ as $\gamma \uparrow$.

- Uniqueness/non-uniqueness of non-negative solutions to

$$\begin{cases} \frac{\partial}{\partial t}X(t,x) &= \frac{1}{2}\Delta X(t,x) + X(t,x)^{\gamma}\dot{W}(x,t), \\ X(0,\cdot) &\geq 0. \end{cases}$$

  for some $\gamma < 3/4$.

Other supeprocesses

$$\frac{\partial}{\partial t}X(t,x) = \Delta_\alpha X(t,x) + \sqrt{X(t,x)}\dot{W}(x,t),\ x \in R,\ t \geq 0,$$

where $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian, $\alpha \leq 2$.
Very different behavior! For $\alpha < 2$,

$$\text{supp}(X_t) = R,\ \text{on } X_t > 0,$$

$$BZ_t = Z_t := \text{ zero set of } X_t\,.$$

Based on the paper of Chen, Veron, Wang (15) one can conjecture:

$$Leb(Z_t) > 0,\ \ \text{if } \alpha < 2.$$

Other supeprocesses

Again assume

$$\alpha < 2.$$

Now consider

$$\frac{\partial}{\partial t} X(t,x) = \Delta_\alpha X(t,x) + X(t,x)^{\frac{1}{1+\beta}} \dot{L}(x,t), \; x \in R, \; t \geq 0,$$

where $\beta \in (0,1)$ and L is spectrally positive $(1+\beta)$-stable noise. Then again based on the paper of Chen, Veron, Wang (15) we can conjecture:

If $\beta \in (\frac{\alpha}{\alpha+1}, 1]$, then

$$Leb(Z_t) > 0,$$

However if $\beta < \frac{\alpha}{\alpha+1}$, then

$$Leb(Z_t) = 0,$$

Open problem: if $\beta < \frac{\alpha}{\alpha+1}$,

$$\dim(Z_t) = ?.$$

Thank You

Measure supported by the boundary of the zero set

Introduce random measures

$$
\begin{aligned}
L_t^\lambda(\phi) &= \lambda^{2\nu_0}\int \phi(x)X(t,x)e^{-\lambda X(t,x)}\,dx \\
(&\approx \lambda^{2\nu_0}\int \phi(x)1(0\le X(t,x)\le \lambda^{-1})\,dx,\ \text{as } \lambda\to\infty)
\end{aligned}
$$

Tnen (T. Hughes, UBC) there is a random finite non-trivial measure L_t on R such that for any bounded continuous ϕ,

$$L_t^\lambda(\phi)\to L_t(\phi) \text{ in } \mathbf{L}^2 \text{ as } \lambda\to\infty.$$

Conjecture: $L_t(1)>0$ a.s. on $\{X_t(1)>0\}$.

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Conjecture: $L_t(1)>0$ a.s. on $\{X_t(1)>0\}$.

If true, it would help to prove sharp lower bound:

$$\dim(BZ_t) \;=\; 2-2\nu_0,\ \text{ on } \{X_t(R)>0\},\ P-\text{a.s.}$$

Very Singular Solution (VSS)

$$V^{\infty}(t,x) = \lim_{\lambda \to \infty} V^{\lambda}(t,x), \forall (t,x) \in R_+ \times R \setminus \{(0,0)\}.$$

V^{∞} is called *very singular solution (VSS)* to log-Laplace equation (Brezis, Peletier, Terman(86)).

One can easily check (BPT(86)) that $V = V^{\infty}$ is a $C^{1,2}$ (on $R_+ \times R \setminus \{(0,0)\}$) solution of

$$\begin{aligned} &(i) \quad \frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2}V^2 \\ &(ii) \quad V(0,x) = 0 \text{ for all } x \neq 0; \ \lim_{t \to 0} \int_R V(t,x)\,dx = \infty. \end{aligned} \tag{5}$$

Very Singular Solution (VSS)

If we define

$$F(x) = V^{\infty}(1, x),$$

Then it is known that

$$V^{\infty}(t, x) = t^{-1} F\left(\frac{x}{\sqrt{t}}\right).,$$

and F solves ode

$$\begin{cases} \frac{1}{2}F''(x) - \frac{1}{2}F^2(x) + \frac{1}{2}F'(x) + F(x) = 0 \\ F > 0 \\ F'(0) = 0, F(x) \sim c_0 y e^{-y^2/2}, asy \to \infty \end{cases} \tag{6}$$

Analysis of behavior of U^λ

$$U^\lambda(t,x) \quad \sim \quad E_0(e^{-\int_0^t V^\lambda(s,W_s)ds})$$

Scaling of V^λ:

$$V^\lambda(t,x) \quad = \quad \lambda^2 V^1(\lambda^2 t, \lambda x) \tag{7}$$

Define

$$B_t \quad = \quad \lambda W_{\lambda^{-2}t}, \quad Y(t) = B(e^t - 1)e^{-t/2}$$

Then

$$\begin{aligned} U^\lambda(t,x) \quad &\sim \quad E_0(e^{-\int_0^t \lambda^2 V^1(\lambda^2 s, \lambda W_s)ds}) \\ &= \quad E_0(e^{-\int_0^{\lambda^2 t} V^1(u,B_u)du}) \\ &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{e^{s/2}}(1,Y_s)ds}) \\ &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^\infty(1,Y_s)ds}) \end{aligned}$$

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$$\begin{cases} \frac{\partial}{\partial t} X(t,x) & = & \frac{1}{2}\Delta X(t,x) + X(t,x)^{\gamma} \dot{W}(x,t), \\ X(0,\cdot) & \geq & 0. \end{cases}$$

  for some $\gamma < 3/4$.

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

X^1, X^2 — two solutions, $\tilde{X} = X^1 - X^2$.

$$\frac{\partial \tilde{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \tilde{X}_t(x) + (\sigma(X_t^1(x)) - \sigma(X_t^2(x)))\dot{W}(t,x).$$

Clearly

$$|\sigma(X_t^1(x)) - \sigma(X_t^2(x))| \leq C|\tilde{X}_t(x)|^\gamma,$$

and thus one can show that it is enough to consider uniqueness of

$$\frac{\partial \bar{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \bar{X}_t(x) + |\bar{X}_t(x)|^\gamma \dot{W}(t,x),$$

with $\bar{X}_0 = 0$.

Regularity and uniqueness of $\bar{X}$

$$\frac{\partial \bar{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \bar{X}_t(x) + |\bar{X}_t(x)|^{\gamma} \dot{W}(t,x).$$

- $x \mapsto \bar{X}_t(x)$ is Hölder $1/2 - \epsilon$.
- For $x \in BZ_t$, roughly we have

  $x \mapsto \bar{X}_t(x)$ is Hölder with any exponent

$$\xi < \frac{1}{2(1-\gamma)}.$$

- We can show PU if

$$\gamma > \frac{1}{2} + \frac{1}{2\xi},$$

- Put it together: PU holds if

$$\gamma > 3/4.$$