Irregular Birth-Death process: stationarity and quasi-stationarity

MAO Yong-Hua

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@
BNU

(based on joint works with W-J Gao and C Zhang)

CONTENTS

Stationarity or quasi-stationarity

- Markov process X_t on state space $(E, \mathscr{E})$ with life time ζ. We want to find probability μ such that

$$\mathbb{P}_\mu[X_t \in A | t < \zeta] = \mu(A), \quad A \in \mathscr{E}.$$

- If $\zeta = \infty$ a.e. then μ is called the stationary distribution.
- If $\zeta < \infty$ a.e. then μ is called quasi-stationary distribution, or conditional stationary distribution.
- Equivalently,

$$\sum_i u_i p_{ij}(t) = u_j e^{-\lambda t},$$

where

$$p_{ij}(t) = \mathbb{P}_i[X_t = j, t < \zeta].$$

Initial

- Consider a branching process whose branching mechanism ξ satisfying

$$m = \mathbb{E}\xi \leq 1 \quad \Rightarrow \quad \lim_{t\to 0} p_{i0}(t) = 1.$$

- Yaglom (1947) considered the following limit problem $\forall i, j \geq 1$

$$\lim_{t\to\infty} \mathbb{P}_i[X_t = j | \tau_0 > t] = \lim_{t\to\infty} \frac{\mathbb{P}_i[X_t = j, t < \tau_0]}{\mathbb{P}_i[t < \tau_0]}$$
$$= \lim_{t\to\infty} \frac{p_{ij}(t)}{\sum_{k\geq 1} p_{ik}(t)}.$$

- This is to consider the "ergodicity" of the process conditional on $\{1, 2, \cdots\}$? Or the limit behavior before the particles die out?

A.M. Yaglom (1947). Certain limit theorems of the theory of branching processes (in Russian). Doklady Akademii Nauk SSSR 56

Branching process

Theorem

Let $Z_0 = 1, Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k}$*. Denote*

$$f(s) = \mathbb{E}s^{\xi}, \quad g(s) = \lim_{n\to\infty} \mathbb{E}[s^{Z_n} | Z_n > 0].$$

If $m < 1$*, then* $g(s)$ *exists and satisfies*

$$g(f(s)) = mg(s) + 1 - m.$$

In particular,

$$\lim_n \mathbb{P}[Z_n = i | Z_n > 0] = b_i > 0, \quad \sum_{i\geq 1} b_i = 1.$$

Furthermore, there exists a family of QSD's:

$$\tilde{g}(s) = 1 - (1 - g(s))^{\alpha}, \quad \alpha \in (0, 1].$$

Finite Markov chain

- Darroch-Seneta (1965, 1967) considered a jump process on finite state space

$$\tilde{Q} = \begin{pmatrix} 0 & 0 \\ a & Q \end{pmatrix}.$$

- Perron-Frobenius theorem: there exist $\lambda > 0$ and positive vectors u, v such that

$$uQ = -\lambda u, \quad Qv = -\lambda v.$$

Thus

$$Q = -\lambda vu + O(\lambda'), \quad \lambda' > \lambda,$$

and

$$P(t) = e^{-t\lambda} vu + O(e^{-t\lambda'}).$$

J.N. Darroch and E. Seneta (1967). On quasi-stationary distributions in absorbing continuous-time finite Markov chains. J. Appl. Probab. 4,

QSD for finite Markov chain

- $\forall i, j \geq 1$,

$$\lim_{n\to\infty} \frac{p_{ij}(t)}{\sum_k p_{ik}(t)} = \lim_{t\to\infty} \frac{e^{-t\lambda} v_i u_j + o(e^{-t\lambda'})}{e^{-t\lambda} \sum_k v_i u_k + O(e^{-t\lambda'})} = \frac{u_j}{\sum_k u_k}$$

  independent of i.
- Assume $\sum_k u_k = 1$ then u is conditional limit distribution.
- On the other hand, from

$$uQ = -\lambda u. \quad \lambda \text{ invariant}$$

  we have

$$uP(t) = e^{-\lambda t} u, \quad \mathbb{P}_u[t < \zeta] = uP(t)1 = e^{-\lambda t}.$$

  Thus

$$\mathbb{P}_u[X_t = i | t < \zeta] = u.$$

Infinite state space

- Assume $\zeta = \tau_0 < \infty$.

Theorem

Equivalent:
(a) QSD u exists.
(b) u is $x(> 0)$-invariant probability ($uQ = -xu$) for Q such that $x = \sum_{i\geq 1} u_i q_{i0}$.

- Define decay rate

$$\lambda = \lim_{t\to\infty} -\frac{1}{t}\log p_{ij}(t).$$

Then $x \leq \lambda$.

D. Vere-Jones (1969). Some limit theorems for evanescent processes. Austral. J. Statist. 11

Birth-death process

- On $\{0, 1, 2, \dots\}$, birth rates $b_i > 0 (i \geq 0)$, death rates $a_0 \geq 0, a_i > 0 (i \geq 1)$:

$$Q = \begin{pmatrix} -(a_0+b_0) & b_0 & & & \\ a_1 & -(a_1+b_1) & b_1 & & \\ & a_2 & -(a_2+b_2) & b_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

- Define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 \cdots b_{i-1}}{a_1 \cdots a_i}, \quad i \geq 1; \quad \nu_i = \frac{1}{\mu_i b_i}, \quad i \geq 0;$$

$$R = \sum_{i=0}^{\infty} \nu_i \sum_{j=0}^{i} \mu_j, \quad S = \sum_{i=0}^{\infty} \nu_i \sum_{j=i+1}^{\infty} \mu_j.$$

Feller's 4 boundaties

- Natrual boundary: $R = S = \infty$;
- Entrance boundary: $R = \infty, S < \infty$;
- Exit boundary: $R < \infty, S = \infty$;
- Regular boundary: $R < \infty, S < \infty$.

W. Feller (1959)The birth and death processes as diffusion processes. J. Math. Pures Appl., 38

A diagram

- Done and to do

boundary	unique	$\lambda > 0$	$\sigma_{\text{ess}} = \emptyset$	$\alpha > 0$	QSD
Natural	V	$\delta < \infty$	$\delta_n \to 0$	X	$a_0 > 0, \lambda > 0$
Entrance	V	V	V	V	$a_0 > 0$
Exit-Min	X	V	V	V	?
Regular-Min	X	V	V	V	?
Regular-Max	X	V	V	?	?

- Min=minimal process, Max=maximal process, QSD=quasi-stationary distribution
- σ_{ess} = essential spectrum. Decay rate:

$$\lambda = -\lim_{t\to\infty} \frac{1}{t} \log |p_{ij}(t) - p_{ij}(\infty)|$$

and uniform decay rate:

$$\alpha = -\lim_{t\to\infty} \frac{1}{t} \log \sup_i \sum_j |p_{ij}(t) - p_{ij}(\infty)|.$$

Main theorem

Theorem

$$?=V.$$

How to find a QSD?

- Let us take as an example when $a_0 > 0, R = \infty$ and $S \leq \infty$.
- Consider
$$Qg(x) = -xg(x),$$
where $g(x) = (g_i(x), i \geq 0, g_0(x) = 1, g_i(0) = 1$ and $g_{-1}(x) = 0$.
Then
$$a_i g_{i-1} - (a_i + b_i)g_i + b_i g_{i+1} = -xg_i, \quad i \geq 0;$$
by letting $u_i(x) = \mu_i g_i(x)$, we obtain
$$uQ = -xu.$$
- The difficulty is to prove that
$$\sum_{i \geq 0} u_i(x) = \frac{a_0}{x}.$$

van Doorn's theorem (1991)

- Assume $a_0 > 0$ and $R = \infty$. Let

$$\lambda = -\lim_{t\to\infty} \frac{1}{t} \log p_{ij}(t).$$

Theorem

(1) If $S = \infty$ and $\lambda > 0$, then for every $0 < x \le \lambda$, there is a QSD.
(2) If $S < \infty$, then there is exactly one QSD at $x = \lambda$.

- The case when $S = \infty$ and $x = \lambda > 0$ is due to Good (1968).

- E.A. van Doorn (1991). Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. Adv. Appl. Probab. 23
- P. Good (1968). The limiting behavior of transient birth and death processes conditioned on survival. J. Austral. Math. Soc. 8

Our aim

- Extend the definition of QSD;
- Work out the QSD for the minimal process for exit or regular boundaries!

How

- Duality
- Spectral theory
- Boundary theory
- Fastest strong stationary time

Exit boundary

- First, we deal with the minimal process with exit boundary, that is $R < \infty, S = \infty$.

Duality

- Assume that $a_0 = 0$ and $R < \infty, S = \infty$. Let

$$\tilde{a}_i = b_i, \quad \tilde{b}_i = a_{i+1}, \quad i \geq 0.$$

- There is a kind of dual relationship between Q and $\tilde{Q}$. Let

$$M = \begin{pmatrix} \nu_0 & \nu_1 & \nu_2 & \cdots \\ & \nu_1 & \nu_2 & \cdots \\ & & \nu_2 & \cdots \\ & & & \ddots \end{pmatrix}.$$

Then

$$M\tilde{Q} = QM.$$

- Thus $\tilde{Q}$ is such that $\tilde{a}_0 > 0$ and $\tilde{R} = \infty, \tilde{S} < \infty$. More Q and $\tilde{Q}$ has the same spectrum.

Boundedness of eigenfunction for $\tilde{Q}$

- From the above relation, we have

$$\sum_{i\geq 0}\mu_i g_i(\lambda) == \sum_{i\geq 0}[\tilde{g}_i(\lambda)-\tilde{g}_{i-1}(\lambda)] = \lim_{i\to\infty}\tilde{g}_i(\lambda).$$

- If we can prove that

$$\tilde{g}_\infty(\lambda) = \lim_{i\to\infty} g_i(\lambda) < \infty,$$

then this gives QSD.

Spectral representation for $\tilde{g}_i(\lambda)$

- Assume $\tilde{a}_0 > 0$ and $\tilde{R} = \infty, \tilde{S} < \infty$. Then

Lemma

Let $\tilde{X} = (\tilde{X}_t, t \geq 0)$ *be the* $\tilde{Q}$*-process and* $\tilde{\tau}_0 := \inf\{t > 0 : \tilde{X}_t = 0\}$ *be the hitting time of state* 0*. Then* $\tilde{g}_i(\lambda) = \mathbb{E}_i e^{\lambda \tilde{\tau}_0}$.

- Let $\tilde{\lambda}_{i,k}, k \geq 1$ be the eigenvalues of the process $\tilde{X}_t$ absorbed at i from right. The by Gong-M.-Zhang (2012), we have

Lemma

$\tilde{g}_\infty(\lambda) := \lim_{i\to\infty} \tilde{g}_i(\lambda) = \prod_{k=1}^\infty \frac{\tilde{\lambda}_{0,k}}{\tilde{\lambda}_{0,k} - \lambda} < \infty.$

Spectrum

- But $\lambda = \tilde{\lambda} = \tilde{\lambda}_{-1,1} < \tilde{\lambda}_{0,1}$, by

Lemma (Hart-Martínez-San Martín (2003))

If $\tilde{A} = \infty$ and $\tilde{\lambda}_{i,1} = \tilde{\lambda}_{j,1}$ for some $i > j \geq -1$, then $\tilde{\lambda}_{i,1} = \tilde{\lambda}_{i+1,1}$.

- But we can prove that $\tilde{\lambda}_{i,1} \to \infty$ as $i \to \infty$.

Regular boundary

- Next, we deal with the minimal process with regular boundary, that is $R < \infty, S < \infty$.
- NO duality: $\tilde{S} < \infty, \tilde{R} < \infty$.
- Use approximation!

Regular: minimal process

- Let $\delta_\nu, \nu \geq 1$ be the eigenvalues for minimal process, and $\delta_\nu^{(n)}, 1 \leq \nu \leq n$ be the eigenvalues for $-Q^{(n)} =$

$$\begin{pmatrix} -b_0 & b_0 & 0 & \cdots & 0 & 0 \\ a_1 & -(a_1+b_1) & b_1 & \cdots & 0 & 0 \\ 0 & a_2 & -(a_2+b_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & -(a_{n-1}+b_{n-1}) \end{pmatrix}.$$

  Then

$$\delta_\nu^{(n)} \downarrow \delta_\nu, \quad \forall \nu \geq 1.$$

- More, by letting τ_n be the hitting time to n,

$$\tau_n \uparrow \zeta \quad \text{(explosion time)}.$$

Approximation

- For $Q^{(n)}$-process $(p_{ij}^{(n)}(t))$, we have

$$\sum_{i=0}^{n-1} u_i^{(n)} p_{ij}^{(n)}(t) = e^{-\delta_1^{(n)} t} u_i^{(n)},$$

where

$$u_i^{(n)} = \frac{\mu_i g_i^{(n)}}{\sum_{i=0}^{n-1} \mu_i g_i^{(n)}} \quad \text{and} \quad g_i^{(n)} = \mathbb{E}_i e^{\delta_1^{(n)} \tau_{n-1}}.$$

- As

$$g_i^{(n)} = \frac{\mathbb{E}_0 e^{\delta_1^{(n)} \tau_{n-1}}}{\mathbb{E}_0 e^{\delta_1^{(n)} \tau_i}},$$

we have

$$u_i^{(n)} = \frac{\mu_i \left(\mathbb{E}_0 e^{\delta_1^{(n)} \tau_i}\right)^{-1}}{\sum_{i=0}^{n-1} \mu_i \left(\mathbb{E}_0 e^{\delta_1^{(n)} \tau_i}\right)^{-1}}.$$

FSST

- Let $T^{(n-1)}$ be the FSST for the process reflecting at $n-1$, then

$$\left(\mathbb{E}_0 e^{-sT^{(n-1)}}\right)^{-1} \sum_{i=0}^{n-1} \mu_i = \sum_{i=0}^{n-1} \mu_i \left(\mathbb{E}_0 e^{-s\tau_i}\right)^{-1}.$$

- This equality comes from the boundary theory:

$$\psi_{ij}^{(n)}(\lambda) = \phi_{ij}^{(n)}(\lambda) I_{\{j<n-1\}} + \frac{X_i^{(n)}(\lambda) X_j^{(n)}(\lambda)\mu_j}{\lambda \sum_{k=0}^{n-1} \mu_k X_k^{(n)}(\lambda)},$$

where $X_i^{(n)}(\lambda) = 1 - \lambda \mathbb{E}_i e^{-\lambda\tau_{n-1}}$.

- Thus

$$u_i^{(n)} = \frac{\mu_i \mathbb{E}_0 e^{\delta_1^{(n)} T^{(n-1)}}}{\mathbb{E}_0 e^{\delta_1^{(n)}\tau_i} \sum_{i=0}^{n-1} \mu_i}.$$

Spectral representation

- Let $0 = \lambda_0^{(n)}, \lambda_\nu^{(n)}, 1 \le \nu \le n-1$ be the eigenvalues for the process reflecting at $n-1$. Then

$$\mathbb{E}_0 e^{\delta_1^{(n)} T^{(n-1)}} = \prod_{\nu=1}^{n} \frac{\lambda_\nu^{(n)}}{\lambda_\nu^{(n)} - \delta_1^{(n)}}.$$

- If $\delta_1 = \lim_{n\to\infty} \delta_1^{(n)} < \lambda_1 = \lim_{n\to\infty} \lambda_1^{(n)}$, then by letting $n \to \infty$, we can get

$$u_i = \lim_{n\to\infty} u_i^{(n)} = \pi_i \prod_{\nu=1}^{\infty} \frac{\lambda_\nu}{\lambda_\nu - \delta_1} \Big/ \prod_{\nu=1}^{i} \frac{\lambda_\nu^{(i)}}{\lambda_\nu^{(i)} - \delta_1},$$

and

$$\sum_{i=0}^{\infty} u_i = 1.$$

Separation property for eigenvalues

- Assume $a_0 = 0$ and $R < \infty, S < \infty$. Let

$$\tilde{a}_i = b_i, \quad \tilde{b}_i = a_{i+1}, \quad i \geq 0.$$

- For

$$Qg(x) = -xg(x) \quad \text{and} \quad \tilde{Q}\tilde{g}(x) = -x\tilde{g}(x)$$

with usual convention, $g_i(x) \uparrow g_\infty(x)$ and $\tilde{g}_i(x) \uparrow \tilde{g}_\infty(x)$. Then the zeros of $g_\infty(x)$ and $\tilde{g}_\infty(x)$ have the separation property

$$0 < \xi_1 < \tilde{\xi}_1 < \xi_2 < \tilde{\xi}_2 < \dots.$$

- We can prove that

$$\delta_\nu = \xi_\nu \text{ and } \lambda_\nu = \tilde{\xi}_\nu, \quad \nu \geq 1$$

T.S. Chihara (1978). An Introduction to Orthogonal Polynomials. New York: Gordon and Breach.

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QED

THANKS !